Graham's pebbling conjecture on the middle graphs of even cycles*

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Abstract: A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The pebbling number of a graph G, denoted by $f(G)$, is the least integer n such that, however n pebbles are located on the vertices of G, we can move one pebble to any vertex by a sequence of pebbling moves. For any connected graphs G and H, Graham conjectured that $f(G \times H) \leq f(G)f(H)$. In this paper, we give the pebbling number of some graphs and prove that Graham's conjecture holds for the middle graphs of some even cycles.

Keywords: Graham's conjecture, even cycles, middle graphs, pebbling number.

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1 Introduction

Pebbling in graphs was first introduced by Chung [2]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A peb-

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bling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. The pebbling number of a vertex v, the target vertex, in a graph G is the smallest number $f(G,v)$ with the property that, from every placement of $f(G,v)$ pebbles on G, it is possible to move one pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum of $f(G,v)$ over all the vertices of G.

There are some known results regarding the pebbling number (see [2–5, 7]). If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if u is at a distance d from v, and $2^d - 1$ pebbles are placed on u, then no pebble can be moved to v. So it is clear that $f(G) \geq \max\{|V(G)|, 2^D\}$, where D is the diameter of graph G. Furthermore, we know that $f(K_n) = n$ and $f(P_n) = 2^n - 1$ (see [2]), where K_n is the complete graph and P_n is the path, respectively on n vertices.

The *middle graph* of a graph G, denoted by $M(G)$, is obtained from G by inserting a new vertex into each edge of G, and joining the new vertices by an edge if the two edges they inserted share the same vertex of G.

Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian product of them is denoted by $G_1 \times G_2$. It has vertex set $V_1 \times V_2 = \{(u_i, v_j) | u_i \in V_1, v_j \in V_2\}$, where (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or $(u_1, u_2) \in E_1$ and $v_1 = v_2$. Clearly, we have that $G_1 \times G_2 \cong G_2 \times G_1$. One may view $G_1 \times G_2$ as the graph obtained from G_2 by replacing each of its vertices with a copy of G_1, and each of its edges with $|V_1|$ edges joining corresponding vertices of G_1 in the two copies. Let $u \in G, v \in H$, then uH and vG are subgraphs of $G \times H$ with $V(uH) = \{(u,v) | v \in V(H)\}$, $E(uH) = \{(u,v)(u,v') | vv' \in E(H)\}$ and $V(vG) = \{(u,v) | u \in V(G)\}$, $E(vG) = \{(u,v)(u',v) | uu' \in E(G)\}$. It is clear that $uH \cong H$ and $vG \cong G$.

The following conjecture (see [2]), by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture (Graham): The pebbling number of $G \times H$ satisfies $f(G \times H) \leq f(G)f(H)$.

Ye *et al.* (see [6]) proved that $f(M(C_{2n+1}) \times M(C_{2m+1})) \leq f(M(C_{2n+1}))$ $f(M(C_{2m+1}))$ and $f(M(C_{2n}) \times M(C_{2m+1})) \leq f(M(C_{2n}))f(M(C_{2m+1}))$. In this paper, we will prove that $f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n}))f(M(C_{2m}))$

for $m, n \geq 5$ and $|n - m| \geq 2$.

Throughout this paper, G will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. P_n and C_n will denote a path and a cycle with n vertices, respectively. Given a distribution of pebbles on the vertices of G, define $p(K)$ to be the number of pebbles on a subgraph K of G and $p(v)$ to be the number of pebbles on a vertex v of G. Moreover, we let $\tilde{p}(K)$ and $\tilde{p}(v)$ denote the numbers of pebbles on K and v after some sequence of pebbling moves, respectively.

2 Main results

Definition 2.1. (see [5]) *Let $P_n = v_1v_2\cdots v_n$ be a path. We say that P_n has weight $\sum_{i=1}^{n-1} 2^{i-1}p(v_i)$ with respect to v_n and this is written as $\omega_{P_n}(v_n)$.*

Proposition 2.2. (see [5]) *Let $P_n = v_1v_2\cdots v_n$ be a path. If $\omega_{P_n}(v_n) \geq k2^{n-1}$, then at least k pebbles can be moved from $P_n\backslash v_n$ to v_n.*

Corollary 2.3. *Let $P_n = v_1v_2\cdots v_n$ be a path. Let $\omega_{P_n}(v_k) = \sum_{i=1}^{k-1} 2^{i-1}p(v_i) + \sum_{j=k+1}^{n} 2^{n-j}p(v_j)$ for $2 \leq k \leq n-1$. If $\omega_{P_n}(v_k) \geq t2^{k-1} + 2^{n-k} - 1$ for $\frac{n+1}{2} \leq k \leq n$, $\omega_{P_n}(v_k) \geq 2^{k-1} + t2^{n-k} - 1$ for $1 \leq k < \frac{n+1}{2}$, then at least t pebbles can be moved from $P_n\backslash v_k$ to v_k.*

Proof. Without loss of generality, we assume that $\frac{n+1}{2} \leq k \leq n$.

If $k = n$, it follows from Proposition 2.2.

If $\frac{n+1}{2} \leq k \leq n-1$, let $L_1 = v_1v_2\cdots v_k$, $L_2 = v_kv_{k+1}\cdots v_n$ be two subpaths of P_n.

.Suppose $\omega_{P_n}(v_k) \geq t2^{k-1} + 2^{n-k} - 1$, then either $\sum_{i=1}^{k-1} 2^{i-1}p(v_i) \geq t2^{k-1}$ or $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) \geq 2^{n-k}$ holds.

Case 1. $\sum_{i=1}^{k-1} 2^{i-1}p(v_i) \geq t2^{k-1}$, by Proposition 2.2, we can move t pebbles from $L_1\backslash v_k$ to v_k.

Case 2. $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) \geq 2^{n-k}$, we may assume that $\sum_{j=k+1}^{n} 2^{n-j}p(v_j) = s2^{n-k} + h$, where s and h are integers satisfying $s \geq 1$ and $0 \leq h < 2^{n-k}$.

With $p(v_j)$ pebbles on v_j $(k+1 \le j \le n)$, we can move s pebbles from $L_2 \backslash v_k$ to v_k.

Note that $2^{k-1} \ge 2^{n-k}$ for $k \ge \frac{n+1}{2}$, we have

$$\begin{aligned}\sum_{i=1}^{k-1} 2^{i-1} p(v_i) =& \omega_{P_n}(v_k) - \sum_{j=k+1}^{n} 2^{n-j} p(v_j) \\ \ge& t2^{k-1} + 2^{n-k} - 1 - (s2^{n-k} + h) \\ =& (t2^{k-1} - s2^{n-k}) + (2^{n-k} - h) - 1 \\ \ge& (t-s)2^{k-1}.\end{aligned}$$

So we can move $t-s$ pebbles from $L_1 \backslash v_k$ to v_k with $p(v_i)$ pebbles on v_i $(1 \le i \le k-1)$. That is to say we can move $s + (t-s) = t$ pebbles to v_k. ∎

Corollary 2.4. *Let $P_n = v_1 v_2 \cdots v_n$ be a path. Then $f(M(P_n) - \{v_1, v_n\}) = 2^{n-2} + n - 2$.*

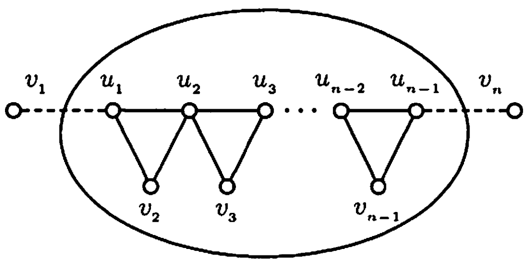

Figure 1: The graph $M(P_n) - \{v_1, v_n\}$ in Corollary 2.4.

Proof. To get $M(P_n)$, we insert one new vertex u_i into the edge $v_i v_{i+1}$ and add the edge $u_i u_{i+1}$ for each $i \in \{1, 2, \ldots, n-2\}$. Let $U = u_1 u_2 \cdots u_{n-1}$ be a subpath of $M(P_n) - \{v_1, v_n\}$.

It is clear that $f(M(P_n) - \{v_1, v_n\}) \ge 2^{n-2} + n - 2$. If we place one pebble on each of vertices $v_2, \ldots, v_{n-1}$, and place $2^{n-2} - 1$ pebbles on the vertex u_{n-1}, then we can not move one pebble to u_1. So $f(M(P_n) - \{v_1, v_n\}) \ge 2^{n-2} + n - 2$.

Now, assume that $2^{n-2} + n - 2$ pebbles are located at $V(M(P_n) - \{v_1, v_n\})$.

First, we prove that one pebble can be moved to u_k $(1 \le k \le n-1)$.

For $m \le k$, we move $\lfloor p(v_m)/2 \rfloor$ pebbles from v_m to u_m. For $m > k$, we move $\lfloor p(v_m)/2 \rfloor$ pebbles from v_m to u_{m-1}. Then we have

$$\begin{aligned}\omega_U(u_k) \ge& 2^{n-2} + n - 2 - \sum_{t=2}^{n-1} p(v_t) + 2\sum_{t=2}^{n-1} \lfloor p(v_t)/2 \rfloor \\ \ge& 2^{n-2}.\end{aligned}$$

It is clear that $2^{n-2} \ge 2^{k-1} + 2^{n-k-1} - 1$ for $1 \le k \le n-1$. By Corollary 2.3, we can move one pebble from the vertices of $U \backslash u_k$ to the vertex u_k for $1 \le k \le n-1$.

Second, we prove that one pebble can be moved to v_k, for $2 \le k \le n-1$. Without loss of generality, we assume that $k \ge \frac{n+1}{2}$.

If $m < k$, we can move $\lfloor p(v_m)/2 \rfloor$ pebbles from v_m to u_m. If $m > k$, we can move $\lfloor p(v_m)/2 \rfloor$ pebbles from v_m to u_{m-1}.

We will prove that after a sequence of pebbling moves above, two pebbles can be moved from U to u_{k-1}, so that one pebble can be moved from u_{k-1} to v_k.

We consider the worst case: $p(u_{k-1}) = 0$.

$$\begin{aligned}\omega_U(u_{k-1}) \ge& 2^{n-2} + n - 2 - \sum_{\substack{j=2 \\ j \ne k}}^{n-1} p(v_j) + 2\sum_{\substack{j=2 \\ j \ne k}}^{n-1} \lfloor p(v_j)/2 \rfloor \\ \ge& 2^{n-2} + 1.\end{aligned}$$

It is clear that $2^{n-2} + 1 \ge 2 \times 2^{(k-1)-1} + 2^{n-(k-1)-1} - 1$ for $\frac{n-1}{2} \le k-1 \le n-2$. By Corollary 2.3, we can move two pebbles from $U \backslash u_{k-1}$ to u_{k-1} ($\frac{n-1}{2} \le k-1 \le n-2$). So we can move one pebble to v_k ($\frac{n+1}{2} \le k \le n-1$), and we are done. ∎

Definition 2.5. (see [5]) *The t-pebbling number of a graph G is the smallest number $f_t(G)$ with the property that from every placement of $f_t(G)$ pebbles on G, it is possible to move t pebbles to any vertex v by a sequence of pebbling moves.*

Lemma 2.6. (see [6]) *If $n \ge 2$, then $f(M(C_{2n})) = 2^{n+1} + 2n - 2$.*

Corollary 2.7. *If $n \ge 2$, then $f_t(M(C_{2n})) \le t2^{n+1} + 2n - 2$.*

Proof. Let $C_{2n} = v_0v_1 \cdots v_{2n-1}v_0$, $M(C_{2n})$ is obtained from C_{2n} by inserting u_i into $v_iv_{(i+1)mod(2n)}$, and connecting $u_iu_{(i+1)mod(2n)}$ for $0 \le i \le 2n-1$.

Without loss of generality, we may assume that our target vertex is u_0 or v_0.

Case 1. The target vertex is u_0. In this case, we prove the result by using induction on t.

The result is obvious for $t = 1$ from Lemma 2.6.

Now suppose that $t2^{n+1} + 2n - 2$ pebbles are located at the vertices of $M(C_{2n})$.

We consider the worst case: $p(u_0) = 0$.

Let $A = \{u_0, v_1, u_1, \ldots, v_n, u_n\}$, $B = \{u_n, v_{n+1}, \ldots, v_{2n-1}, u_{2n-1}, v_0, u_0\}$ and $G = M(C_{2n})$. Then we know that either A or B contains more than $2^n + n$ pebbles.

Note that $G[A] \cong G[B] \cong M(P_{n+2}) - \{v_1, v_{n+2}\}$, according to Corollary 2.4, with $2^n + n$ pebbles on A or B, one pebble can be moved to u_0.

Note that $2^n + n \leq 2^{n+1}$, the number of remaining pebbles is more than $(t-1)2^{n+1} + 2n - 2$. So we can move $t - 1$ pebbles to u_0 with the remaining pebbles by the induction hypothesis, and we are done.

Case 2. The target vertex is v_0.

Let $A' = \{u_0, v_1, \ldots, v_{n-1}, u_{n-1}\}$, $B' = \{u_{2n-1}, v_{2n-1}, \ldots, v_{n+1}, u_n\}$.

Suppose that $t2^{n+1} + 2n - 2$ pebbles are located at the vertices of $M(C_{2n})$.

We consider the worst case, that is $p(v_0) = 0$.

By proposition 2.2, if $p(v_n) \geq t2^{n+1}$, then t pebbles can be moved to v_0.

Now suppose that $t2^{n+1} - h$ pebbles are located at v_n, without loss of generality, we assume that $p(A') \geq p(B')$, namely $p(A') \geq n - 1 + \lceil h/2 \rceil$.

Let $L = v_0 u_0 u_1 \cdots u_{n-1} v_n$ be a subpath of G with length $n + 1$ and $q = \sum\limits_{i=0}^{n-1} p(u_i)$.

If $q \geq \lceil h/2 \rceil$, then $\omega_L(v_0) = p(v_n) + \sum\limits_{i=0}^{n-1} 2^{n-i} p(u_i) \geq t2^{n+1} - h + 2q \geq t2^{n+1}$.

By Proposition 2.2, t pebbles can be moved from the vertices of $L \backslash v_0$ to v_0.

If $q < \lceil h/2 \rceil$, then $\sum\limits_{j=1}^{n-1} p(v_j) \geq n - 1 + \lceil h/2 \rceil - q$. So we can move at least $\lfloor\frac{1}{2}(\lceil\frac{h}{2}\rceil+1-q)\rfloor$ pebbles to the vertices of the set $\{u_0,u_1,\ldots,u_{n-2}\}$. Then we have

$$\omega_L(v_0)=p(v_n)+\sum_{i=0}^{n-1}2^{n-i}\tilde{p}(u_i)\geq t2^{n+1}-h+2q+4\times\frac{1}{2}(\frac{h}{2}-q)\geq t2^{n+1}.$$

By Proposition 2.2, t pebbles can be moved from the vertices of $L\backslash v_0$ to v_0. The result follows. ∎

Theorem 2.8. *If $m,n\geq 5$ and $|n-m|\geq 2$, then*

$$f(M(C_{2n})\times M(C_{2m}))\leq f(M(C_{2n}))f(M(C_{2m})).$$

Proof. Without loss of generality, we assume that $n\geq m+2$ $(m\geq 5)$. Let $V(M(C_{2n}))=\{u_1,u_2,\ldots,u_{4n}\}$, $V(M(C_{2m}))=\{v_1,v_2,\ldots,v_{4m}\}$. For simplicity, let $G=M(C_{2n})\times M(C_{2m})$.

Now we assume that $(2^{n+1}+2n-2)(2^{m+1}+2m-2)$ pebbles have been distributed arbitrarily on the vertices of G. Suppose the target vertex is (u_i,v_j). Note that the vertex (u_i,v_j) belongs to both $V(u_iM(C_{2m}))$ and $V(v_jM(C_{2n}))$. If $p(u_iM(C_{2m}))\geq 2^{m+1}+2m-2$ or $p(v_jM(C_{2n}))\geq 2^{n+1}+2n-2$, then we can move one pebble to (u_i,v_j) by Lemma 2.6.

Suppose that $p(u_iM(C_{2m}))\leq 2^{m+1}+2m-3$ and $p(v_jM(C_{2n}))\leq 2^{n+1}+2n-3$.

We will prove that if we move as many as possible pebbles from the vertices of $u_lM(C_{2m})$ to (u_l,v_j) which belongs to $v_jM(C_{2n})$ $(1\leq l\leq 4n)$, then one pebble can be moved from $v_jM(C_{2n})$ to (u_i,v_j).

We may assume that

$$p_k=p(u_k(M(C_{2m})))\leq 2^{m+1}+2m-3\ (1\leq k\leq s)$$

and

$$p_k=p(u_k(M(C_{2m})))\geq 2^{m+1}+2m-2\ (s+1\leq k\leq 4n).$$

Now we consider the worst case scenario (i.e. the most wasteful distribution of pebbles possible). Therefore we may assume that

$$p_k=\begin{cases}2^{m+1}+2m-3, & \text{if } 1\leq k\leq s,\\ t_k2^{m+1}+2m-2+(2^{m+1}-1), & \text{if } s+1\leq k\leq 4n-1,\\ t_k2^{m+1}+2m-2+R, & \text{if } k=4n,\end{cases}$$

where $0\leq R\leq 2^{m+1}-1$ and t_k is a positive integer. According to Corollary 2.7, we can move at least $\sum_{k=s+1}^{4n}t_k$ pebbles to $v_j(M(C_{2n}))$.

Let

$$\begin{aligned}\Delta &= (2^{n+1}+2n-2)(2^{m+1}+2m-2)-s(2^{m+1}+2m-3)\\ &\quad -(4n-s-1)(2^{m+1}-1)-(4n-s)(2m-2)\\ &= (2^{n+1}-2n-2)(2^{m+1}+2m-2)+2^{m+1}+4n-1.\end{aligned}$$

Therefore,

$$\frac{\Delta}{2^{m+1}} = 2^{n+1}-2n-1+\frac{1}{2^{m+1}}\left[(2^{n+1}-2n-2)(2m-2)+4n-1\right].$$

Note that $\Delta = \left(\sum_{k=s+1}^{4n} t_k\right)2^{m+1}+R$, so $\sum_{k=s+1}^{4n} t_k > \frac{\Delta}{2^{m+1}}-1$. It follows that

$$\begin{aligned}p(v_jM(C_{2n})) \geq \sum_{k=s+1}^{4n} t_k &> 2^{n+1}-2n-2\\ &+\frac{1}{2^{m+1}}\left[(2^{n+1}-2n-2)(2m-2)+4n-1\right].\end{aligned}$$

To the end, we only need to prove that we can move one pebble from $v_j(M(C_{2n}))$ to (u_i,v_j) with $2^{n+1}-2n-2+\frac{\lfloor(2^{n+1}-2n-2)(2m-2)+4n-1\rfloor}{2^{m+1}}$ pebbles.

So we only need to prove that

$$2^{n+1}-2n-2+\frac{1}{2^{m+1}}\left[(2^{n+1}-2n-2)(2m-2)+4n-1\right] \geq 2^{n+1}+2n-2. \qquad (*)$$

After some direct simplifications and calculations, we reduce the inequality of $(*)$ to its equivalent form as follows:

$$2^{m+1} \leq \frac{m-1}{n}(2^n-1)-m+2-\frac{1}{4n}. \qquad (**)$$

It is clear that the right side of the inequality $(**)$ is an increasing function of n, for $7 \leq m+2 \leq n$. So we only need to show that $(**)$ holds when $n=m+2$. Substituting $n=m+2$ into $(**)$, we have

$$2^{m+1} \leq \frac{m-1}{m+2}(2^{m+2}-1)-m+2-\frac{1}{4(m+2)},$$

namely,

$$(m-4)2^{m+1}-m^2-m+\frac{19}{4} \geq 0. \qquad (***)$$

The left side of (***) is an increasing function of m if $m \geq 5$. Clearly, (***) holds for $m = 5$. This completes the proof. ∎

In this paper, we have shown that if $m, n \geq 5$ and $|m - n| \geq 2$, then $f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n}))f(M(C_{2m}))$. However, the remaining question is open.

Problem 2.9. *$f(M(C_{2n}) \times M(C_{2m})) \leq f(M(C_{2n}))f(M(C_{2m}))$, for $m = n$ or $m = n - 1$.*

3 Remark

Let $C_{2n} = v_0 v_1 \cdots v_{2n-1} v_0$, $M(C_{2n})$ is obtained from C_{2n} by inserting u_i into $v_i v_{(i+1)mod(2n)}$, and connecting $u_i u_{(i+1)mod(2n)}$ for $0 \leq i \leq 2n - 1$. For any vertex $u \in V(M(C_{2n}))$, we say $u \notin V(C_{2n})$ means that $u \in \{u_0, u_1, \ldots, u_{2n-1}\}$, similarly, for any vertex $(u, v) \in V(M(C_{2n}) \times M(C_{2m}))$, we say $(u, v) \notin V(C_{2n} \times C_{2m})$ means that $u \notin V(C_{2n})$ or $v \notin V(C_{2m})$.

Then, by a similar argument as the proof of Corollary 2.7, we can prove that

Corollary 3.1. *For any vertex $u \in V(M(C_{2n}))$, we have that if $u \notin V(C_{2n})$, then $f_t(M(C_{2n}), u) \leq 2^{n+1} + 2n - 2 + (t - 1)(2^n + n)$.*

Moreover, we can prove the following theorem.

Theorem 3.2. *For any vertex $(u, v) \in V(M(C_{2n}) \times M(C_{2m}))$, we have that if $(u, v) \notin V(C_{2n} \times C_{2m})$, then*

$$f(M(C_{2n}) \times M(C_{2m}), (u, v)) \leq f(M(C_{2n}))f(M(C_{2m})),$$

where m, $n \geq 5$.

Proof. If $(u, v) \notin V(C_{2n} \times C_{2m})$, then we can get $u \notin V(C_{2n})$ or $v \notin V(C_{2m})$. Without loss of generality, we assume that $u \notin V(C_{2n})$.

Let $V(uM(C_{2m})) = \{v_1, v_2, \ldots, v_{4m}\}$.

If we move as many as possible pebbles from $v_j M(C_{2n})$ to the vertex $(u, v_j) \in V(uM(C_{2m}))$, for $1 \leq j \leq 4m$, then by a similar process as in the proof of Theorem 2.8, if $(2^{n+1} + 2n - 2)(2^{m+1} + 2m - 2)$ pebbles have been distributed arbitrarily on the vertices of $M(C_{2n}) \times M(C_{2m})$, then at least $2^{m+1} + 2m - 2$ pebbles can be moved to the vertices of $uM(C_{2m})$, and therefore at least one pebble can be moved from $uM(C_{2m})$ to (u, v) with these pebbles. ∎

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