On the signless Laplacian spectral radius of tricyclic graphs with n vertices and diameter d

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Abstract: Let G be a tricyclic graph. Tricyclic graphs are connected graphs in which the number of edges equals the number of vertices plus two. In this paper, we determine graphs with the largest signless Laplacian spectral radius among all the tricyclic graphs with n vertices and diameter d.

Keywords: Signless Laplacian spectral radius; Tricyclic graph; Diameter.

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1 Introduction

In this paper all graphs are undirected finite graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ the degree of the graph G, $N(v_i)$ the set of vertices which are adjacent to vertex v_i. Let $A(G)$ be the adjacency matrix and $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of the graph G, where $D(G) = diag(d(v_1), d(v_2), \cdots, d(v_n))$ denotes the diagonal matrix of vertex degrees of G. The characteristic polynomial

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$\Psi(G,x)$ of G is defined as $\Psi(G,x) = det(xI - A(G))$. The signless Laplacian characteristic polynomial $\Phi(G,x)$ of G is defined as $\Phi(G,x) = det(xI - Q(G))$. The spectrum of $Q(G)$ is also called the signless Laplacian spectrum of G.

The matrix Q is real symmetric and positive semidefinite, the eigenvalues of Q can be arranged as

$$q_1(Q) \geq q_2(Q) \geq \cdots \geq q_n(Q) \geq 0$$

where the largest eigenvalue $q_1(Q)$ is called Q- index of graph G. When G is connected, $Q(G)$ is irreducible and by the Perron-Frobenius Theorem, the signless Laplacian spectral radius is simple and there is a unique positive unit eigenvector corresponding to $q_1(G)$. We shall refer to such an eigenvector as the Perron vector of G.

A tricyclic graph is a connected graph with the number of edges equaling the number of vertices plus two. Recently, the problem concerning graphs with maximal spectral radius or the Laplacian spectral radius of a given class of graphs has been studied by many authors. Guo [3] determined the spectral radius of trees with fixed diameter. Tan [4] determined the largest eigenvalue of signless Laplacian matrix of a graph. Geng and Li [5] determined the graph with the largest spectral radius among all the tricyclic graphs with n vertices and diameter d. Guo [6] determined the Laplacian spectral radius of trees with fixed diameter. He and Li [19] identified graphs with the maximal signless Laplacian spectral radius among all the unicyclic graphs with n vertices and diameter d. In this paper, we determine the unique graph with the largest signless Laplacian spectral radius among all the tricyclic graphs with n vertices and diameter d.

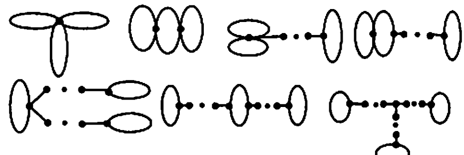

Fig.1 The base of three cycles in Γ_n^3

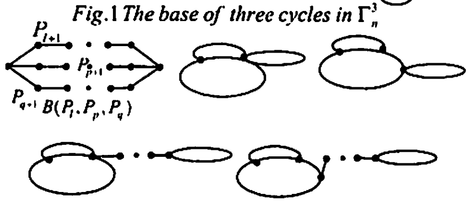

Fig.2 The base of four cycles in Γ_n^4

Let C_n and P_n the cycle and the path, on n vertices, respectively. Let $G-u$ or $G-uv$ denote the graph obtained from G by deleting the vertex $u \in V(G)$ or the edge $uv \in E(G)$. Similarly, $G + uv$ is a graph obtained from G by adding an edge uv, where $u, v \in V(G)$ and $uv \notin E(G)$. For two vertices u and v ($u \neq v$), the distance between u and v is the number of edges in a shortest path joining u and v. The diameter of a graph is the maximum distance between any two vertices of G. For a real number x, we use $\lfloor x \rfloor$ to represent the largest integer not greater than x and $\lceil x \rceil$ to represent the smallest integer not less than x. By [7], it is easy to know that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there does not exist 5 cycles in G. Then let $\Gamma_n = \Gamma_n^3 \bigcup \Gamma_n^4 \bigcup \Gamma_n^6 \bigcup \Gamma_n^7$, where Γ_n^i denotes the set of tricyclic graphs in Γ_n with exactly i cycles for $i = 3, 4, 6, 7$. [see Fig. 1-Fig. 4]

Fig.3 The base of six cycles in Γ_n^6

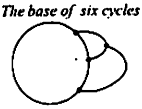

Fig.4 The base of seven cycles in Γ_n^7

An internal path of a graph G is a sequence of vertices $v_1, v_2, \cdots, v_m$ with $m \geq 2$ such that:
(1) The vertices in the sequences are distinct (except possibly $v_1 = v_m$);
(2) v_i is adjacent to v_{i+1}, ($i = 1, 2, \cdots, m-1$);
(3) The vertex degrees $d(v_i)$ satisfy $d(v_1) \geq 3$, $d(v_2) = \cdots = d(v_{m-1}) = 2$ (unless $m = 2$) and $d(v_m) \geq 3$.

2 Preliminaries

In this section, we give the following lemmas which will be used to prove our main results.

Lemma 1.1 ([1]). Let G be a connected graph, and u, v be two vertices of G. Suppose that $v_1, v_2, \ldots, v_s \in N(v) \backslash N(u)$ ($1 \leq s \leq d(v)$) and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i ($1 \leq i \leq n$). Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i (1 \leq i \leq s)$. If $x_u \geq x_v$, then $q(G) < q(G^*)$.

The coalescence of G and H with respect to $v_1 \in V(G)$ and $v_2 \in V(H)$ is formed by identifying v_1 and v_2, and is denoted by $G \cdot H$. In other words, $V(G \cdot H) = V(G) \cup V(H) \cup \{v^*\} - \{v_1, v_2\}$, with two vertices in

$G \cdot H$ adjacent if they are adjacent in G or H, or if one is v^* and the other is adjacent to v_1 or v_2 in G or H.[see [2]]

Lemma 1.2([2]). Let $G \cdot H$ be the graph obtained from disjoint graphs G and H by coalescing the vertex u of G with the vertex v of H. Then

$$\Psi(x, G \cdot H) = \Psi(x, G)\Psi(x, H - v) + \Psi(x, G - u)\Psi(x, H)$$

$$-x\Psi(x, G - u)\Psi(x, H - v).$$

By similar reasoning as that of Theorem 3.1 of [6] and Lemmas 2 and 7 of [15], we have the following result.

Lemma 1.3([15]). Let $P : v_1 v_2 \dots v_k (k \geq 2)$ be an internal path of a connected graph G. Let G' be a graph obtained from G by subdividing some edge of P. Then we have $q_1(G') < q_1(G)$.

Let $S(G)$ be a graph obtained by subdividing every edge of G.

Lemma 1.4([2,12]). Let G be a graph on n vertices and m edges. Then

$$\Psi_{S(G)} = x^{m-n}\Phi_G(x^2).$$

Lemma 1.5([13]). Let u be a vertex of a connected graph G and $d(u) \geq 2$. Let $G_{k,l}(k, l \geq 0)$ be the graph obtained from G by attaching two pendant paths of lengths k and l at u, respectively. If $k \geq l \geq 1$ then

$$q_1(G_{k,l}) > q_1(G_{k+1,l-1}).$$

Lemma 1.6([1]). Let G be a simple graph on n vertices which has at least one edge. Then

$$\Delta(G) + 1 \leq q_1(G) \leq 2\Delta(G),$$

where $\Delta(G)$ is the largest degree of G. Moreover, if G is connected, then the first equality holds if and only if G is the star $K_{1,n-1}$; and the second equality holds if and only if G is a regular graph.

Lemma 1.7([14]). Let e be an edge of the graph G. Then

$$q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \cdots \geq q_n(G) \geq q_n(G - e) \geq 0.$$

Lemma 1.8([4]). Let e be an edge of a graph G, and let $E_G(e)$ denote the set of all edges (containing no e) adjacent to e in G and $J_G(e)$ the set of all distinct line graph cycles containing e in G. Then the signless Laplacian characteristic polynomial of G satisfies that

$$\Phi(G, x) = \frac{x-2}{x}\Phi(G-e, x) - \sum_{\bar{e} \in E_G(e)} \frac{\Phi(G - e - \bar{e}, x)}{x^2} - 2\sum_{Z \in J_G(e)} \frac{\Phi(G - E(Z), x)}{x^{|E(Z)|}},$$

where $E(Z)$ is the set of edges in a subgraph (or an edge sequence) Z, $|E(Z)|$ is the cardinality of $E(Z)$.

Let G be a connected graph, and $uv \in E(G)$. The graph G_{uv} is obtained from G by subdividing the edge uv, i.e., adding a new vertex w and edges wu, wv in $G - uv$.

Lemma 1.9([17]). Let G_{uv} be the graph obtained from a connected graph G by subdividing its edge uv. Then the following holds:
(i) if uv belongs to an internal path then $q_1(G_{uv}) < q_1(G)$;
(ii) if $G \not\cong C_n$ for some $n \geq 3$, and if uv is not on the internal path then $q_1(G_{uv}) > q_1(G)$. Otherwise, if $G \cong C_n$ then $q_1(G_{uv}) = q_1(G) = 4$.

Lemma 1.10([18]). Let u be a vertex of a graph G, let $\varphi(u)$ be the collection of all cycles containing u. Then the signless Laplacian characteristic polynomial $\Phi(G)$ satisfies

$$\Phi(G,x) = (x-d(u))\Phi(G-u,x) - \sum_{v \in N(u)} \Phi(G-u-v,x) - 2\sum_{Z \in \varphi(u)} \Phi(G-V(Z),x),$$

where the first summation extends over those vertices v adjacent to u, and the second summation extends over all $Z \in \varphi(u)$.

Lemma 1.11. Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \bigcap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a pendant edge to $u(= v)$. Then $q_1(G) < q_1(G^*)$.

Proof. Let x_u and x_v denote the components of the Perron vector of G corresponding to u and v, respectively. Suppose that $N(u) = \{v, v_1, \ldots, v_s\}$ and $N(v) = \{u, u_1, \ldots, u_t\}$. Because $e = uv$ is a non-pendant edge of G, it is easy to see that $s, t \geq 1$. If $x_u \geq x_v$, let

$$G' = G - \{vu_1, \ldots, vu_t\} + \{uu_1, \ldots, uu_t\}$$

If $x_u < x_v$, let

$$G'' = G - \{uv_1, \ldots, uv_s\} + \{vv_1, \ldots, vv_s\}$$

Obviously, $G^* = G' = G''$. By lemma 1.1, we have

$$q_1(G) < q_1(G^*).$$

The proof is completed.□

Lemma 1.12. Let G, G', G'' be three connected graphs disjoint in pairs. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $q_1(G_1) < q_1(G_2)$ or $q_1(G_1) < q_1(G_3)$.

3 Main results

In this section, we will determine the graph with the largest signless Laplacian spectral radius among all tricyclic graphs with n vertices and diameter d. Let $\Gamma_{n,d} = \Gamma^3_{n,d} \bigcup \Gamma^4_{n,d} \bigcup \Gamma^6_{n,d} \bigcup \Gamma^7_{n,d}$, where $\Gamma_{n,d}$ denotes the set of tricyclic graphs with n vertices and diameter d, and $\Gamma^i_{n,d}$ denotes the set of tricyclic graphs in $\Gamma_{n,d}$ with exactly i cycles for $i = 3, 4, 6, 7$.

Suppose we have two graphs G and H with $u \in V(G)$ and $v \in V(H)$, the coalescence of G and H with respect to u and v is formed by identifying u and v, and is denoted by $G_u \cdot H_v$.

Let $Q_v(G)$ be the principal submatrix of $Q(G)$ formed by deleting the row and column corresponding to the vertex v.

Theorem 2.1([18]) Let $G_u \cdot H_v$ be the graph obtained from disjoint graphs G and H by coalescing the vertex u of G with the vertex v of H. Then

$$\Phi(G_u \cdot H_v, x) = \Phi(G, x)\Phi(Q_v(H), x) + \Phi(Q_u(G), x)\Phi(H, x)$$

$$-x\Phi(Q_u(G), x)\Phi(Q_v(H), x). \tag{3.1}$$

Theorem 2.2. Let G and H be two graphs.
(i)([20]) If $\Phi(H; x) > \Phi(G; x)$ for $x \geq q_1(H)$, then $q_1(G) > q_1(H)$;
(ii)([19]) If H is a proper subgraph of G and G is a connected graph, then $q_1(G) > q_1(H)$;
(iii)([4]) If H is a proper subgraph of G and G is a connected graph, then $\Phi(H; x) > \Phi(G; x)$, for $x \geq q_1(G)$.

Theorem 2.3. If $d \geq 3$, let $T^{6(1)}_{d+1,i}, T^{6(2)}_{d+1,i}, T^{6(3)}_{d+1,i}, T^{6(4)}_{d+1,i} \in \Gamma^{d,6}_n$, we have

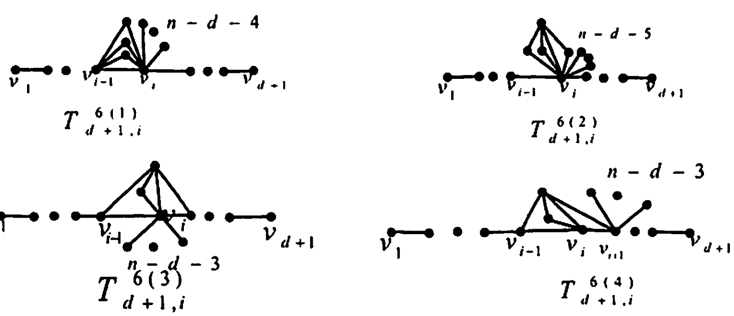

Fig.5 $T^{6(1)}_{d+1,i}, T^{6(2)}_{d+1,i}, T^{6(3)}_{d+1,i}, T^{6(4)}_{d+1,i}$

(i) $q_1(T^{6(1)}_{d+1,i}) \geq q_1(T^{6(2)}_{d+1,i})$, where $2 \leq i \leq d$, and the equality holds if and only if $i = 2$.
(ii) $q_1(T^{6(3)}_{d+1,i}) > q_1(T^{6(4)}_{d+1,i})$, where $2 \leq i \leq d - 1$.

Proof. Since $T^{6(1)}_{d+1,i}, T^{6(2)}_{d+1,i} \in \Gamma^{d,6}_n$, for $2 \le i \le d$, and note that $T^{6(1)}_{d+1,2} \cong T^{6(2)}_{d+1,2}$, obviously we have

$$q_1(T^{6(1)}_{d+1,2}) = q_1(T^{6(2)}_{d+1,2}),$$

when $3 \le i \le d$. Suppose the new path $P_{i-1} : v_1, v_2, \ldots, v_{i-1}$ in $T^{6(1)}_{d+1,i}$ (respectively, $T^{6(2)}_{d+1,i}$), and let $B = T^{6(1)}_{d+1,i} - \{v_1, v_2, \ldots, v_{i-2}\}$. Then let u be the vertex of degree 4 in the graph B, in which we also choose a pendent vertex, say v, such that $v \ne v_{d+1}$, we have $T^{6(1)}_{d+1,i} = (P_{i-1})_{v_{i-1}} \cdot B_u$, and $T^{6(2)}_{d+1,i} = (P_{i-1})_{v_{i-1}} \cdot B_v$. By using the Theorem 2.1, we have

$$\begin{aligned}
&\Phi(T^{6(1)}_{d+1,i}, x) - \Phi(T^{6(2)}_{d+1,i}, x) = \Phi(P_{i-1} \cdot B_u, x) - \Phi(P_{i-1} \cdot B_v, x) \\
&= \Phi(P_{i-1}, x)\Phi(B_u, x) + \Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B, x) - x\Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B_u, x) \\
&\quad -\Phi(P_{i-1}, x)\Phi(B_v, x) - \Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B, x) + x\Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B_v, x) \\
&= \Phi(P_{i-1}, x)\Phi(B_u, x) - \Phi(P_{i-1}, x)\Phi(B_v, x) + x\Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B_v, x) \\
&\quad -x\Phi((P_{i-1})_{v_{i-1}}, x)\Phi(B_u, x) \\
&= (\Phi(B_u, x) - \Phi(B_v, x))(\Phi(P_{i-1}, x) - x\Phi((P_{i-1})_{v_{i-1}}, x))
\end{aligned}$$

$$= (\Phi(B_u, x) - \Phi(B_v), x)(\Phi(P_{i-1}, x) - x\Phi(P_{i-2}, x)) \tag{3.2}$$

Since $B-u$ is a proper subgraph of $B-v$, by Theorem 2.2, for $x \ge q_1(B-v)$, we have

$$\Phi(B - u, x) - \Phi(B - v, x) > 0 \tag{3.3}$$

By using Theorem 2.2, for $x \ge q_1(P_{i-1})$, we have

$$\Phi(P_{i-1}, x) - x\Phi(P_{i-2}, x) < 0 \tag{3.4}$$

It is easy to see that $B - v$ and P_{i-2} are the proper subgraph of $T^{6(2)}_{d+1,i}$, by applying Theorem 2.2, we have

$$q_1(T^{6(2)}_{d+1,i}) > \{q_1(B - v), q_1(P_{i-2})\}.$$

From (3.2),(3.3) and (3.4), we have

$$\Phi(T^{6(1)}_{d+1,i}, x) < \Phi(T^{6(2)}_{d+1,i}, x),$$

for $x > q_1(T^{6(2)}_{d+1,i})$. So using Theorem 2.2, we have that

$$q_1(T^{6(1)}_{d+1,i}) > q_1(T^{6(2)}_{d+1,i}),$$

for $3 \leq i \leq d$.

(ii) Denote $n-d-3$ pendent vertices in $T^{6(3)}_{d+1,i}$ (or $T^{6(4)}_{d+1,i}$) by $v_{d+4}, \ldots, v_n$. Let H be the induced subgraph with vertex set $\{v_i, v_{d+4}, \ldots, v_n\}$ and edge set $\{v_iv_{d+4}, v_iv_{d+5}, \ldots, v_iv_n\}$ in $T^{6(3)}_{d+1,i}$. Let $D = T^{6(3)}_{d+1,i} \backslash \{v_{d+4}, \ldots, v_n\}$, and let u be the vertex of degree 4 of D, and choose one vertex of degree 3 in D, we have

$$T^{6(3),x}_{d+1,i} = H_{v_i} \cdot D_u, \qquad T^{6(4)}_{d+1,i} = H_{v_{i-1}} \cdot D_v.$$

By using Theorem 2.1, it is easy to see that

$$\Phi(T^{6(3)}_{d+1,i}, x) - \Phi(T^{6(4)}_{d+1,i}, x) = (\Phi(D_u, x) - \Phi(D_v, x))(\Phi(H, x) - x^{n-d-2}) \quad (3.5)$$

From Theorem 2.2, for $x \geq q_1(D - v)$, then

$$\Phi(D_u, x) - \Phi(D_v, x) > 0 \quad (3.6)$$

We have from the Theorem 2.2 that for $x \geq q_1(D - v)$

$$\Phi(H, x) - x^{n-d-2} < 0 \quad (3.7)$$

Since $D - v$ is a proper subgraph of $T^{6(4)}_{d+1,i}$, by using Theorem 2.2, we have

$$q_1(T^{6(4)}_{d+1,i}) > q_1(D_v) > 0.$$

Combining (3.5), (3.6) and (3.7), we have

$$\Phi(T^{6(3)}_{d+1,i}, x) < \Phi(T^{6(4)}_{d+1,i}, x),$$

for $x \geq q_1(T^{6(4)}_{d+1,i})$.

By applying Theorem 2.2 once more, it is easy to see that

$$q_1(T^{6(3)}_{d+1,i}) > q_1(T^{6(4)}_{d+1,i}),$$

for $2 \leq i \leq d$. The proof is completed.□

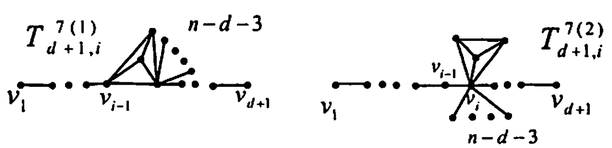

Fig.6 $T^{7(1)}_{d+1,i}$, $T^{7(2)}_{d+1,i}$

Using similar arguments to the proof of Theorem 2.3, we can prove the following Theorem 2.4 and Theorem 2.5.

Theorem 2.4. If $d \geq 3$, let $T^{7(1)}_{d+1,i}, T^{7(2)}_{d+1,i} \in \Gamma^{d,7}_n$, we have

$$q_1(T^{7(1)}_{d+1,i}) \geq q_1(T^{7(2)}_{d+1,i}),$$

where $2 \leq i \leq d$, and the equality holds if and only if $i = 2$.

Theorem 2.5. If $n \geq d+3$, we have $q_1(T^{6(1)}_{d+1,i}) > q_1(T^{6(3)}_{d+1,i})$, where $4 \leq i \leq d-1$.

Theorem 2.6. If $n \geq d+4$, $d \geq 4$ and d is even, we have

$$q_1(T^{6(1)}_{d+1,\frac{d+4}{2}}) < q_1(T^{6(1)}_{d+1,\frac{d+2}{2}}).$$

Proof. Let $\gamma = \frac{d-2}{2}$. Then $T^{6(1)}_{d+1,\frac{d+2}{2}} \triangleq G(\gamma+1,\gamma)$, $T^{6(1)}_{d+1,\frac{d+4}{2}} \triangleq G(\gamma,\gamma+1)$. Using Lemma 1.10, we have

$$\begin{aligned}
&\Phi(T^{6(1)}_{d+1,\frac{d+4}{2}}) - \Phi(T^{6(1)}_{d+1,\frac{d+2}{2}})\\
&= \Phi(G(\gamma+1,\gamma)) - \Phi(G(\gamma,\gamma+1))\\
&= (x-d(u))\Phi(G(\gamma,\gamma)) - \Phi(G(\gamma-1,\gamma)) - (x-d(u))\Phi(G(\gamma,\gamma)) + \Phi(G(\gamma,\gamma-1))\\
&= \Phi(G(\gamma,\gamma-1)) - \Phi(G(\gamma-1,\gamma))\\
&= \cdots\\
&= \Phi(G(1,0)) - \Phi(G(0,1))\\
&= (x-d(u))\Phi(G(0,0)) - \Phi(K_{1,n-d-1}) - (x-d(u))\Phi(G(0,0))\\
&\qquad + (x-d(u))^{n-d-4}\Phi(K_{1,3}) > 0
\end{aligned}$$

for $\forall x \geq q_1(T^{6(1)}_{d+1,\frac{d+4}{2}})$. According to Theorem 2.2, we have that

$$q_1(T^{6(1)}_{d+1,\frac{d+4}{2}}) < q_1(T^{6(1)}_{d+1,\frac{d+2}{2}}).$$

The proof is completed.□

Using similar arguments to the proof of Theorem 2.6, we can prove the following Theorem 2.7.

Theorem 2.7. If $n \geq d+4$, $d \geq 4$ and d is even, we have

$$q_1(T^{7(1)}_{d+1,\frac{d+4}{2}}) < q_1(T^{7(1)}_{d+1,\frac{d+2}{2}}).$$

Theorem 2.8. Let $G \in \Gamma_{n,d}$, for $n \geq d+4$ and $d \geq 3$. Then

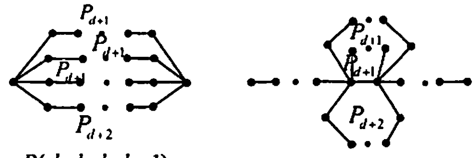

Fig.7 $B(d,d,d,d+1)$ *and* H

(i) If $d \geq 4$, we have

$$q_1(G) \leq \{q_1(T^{6(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}), q_1(T^{7(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor})\},$$

and the equality holds if and only if $G \cong T^{6(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}$ (or $G \cong T^{7(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}$).
(ii) If $d = 3$, then $q_1(G) \leq q_1(T^{7(1)}_{4,3})$, and the equality holds if and only if $G \cong T^{7(1)}_{4,3}$.

Proof. Choose $G \in \Gamma^d_n$ such that the Q-index of G is as large as possible. Suppose G has vertices $v_1, v_2, \ldots, v_n$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. There are five cases.

Case 1. There does not exist the internal path with length greater than 1 unless this internal path is in a cycle with length 3.

Assume to the contrary, we consider two possible cases.

Case 1.1. $G \cong B(d,d,d,d+1)$. Let $B(d,d,d,d+1)$ be the graph as shown in Fig. 7.

It is easy to see that $B(d,d,d,d+1) \in \Gamma^d_n$. By Lemma 1.11 and Theorem 2.2, we get that $q_1(G) < q_1(H)$, where H is shown in Fig. 7. Moreover, using Lemma 1.11 and Theorem 2.2 several times we have $q_1(H) < q_1(T^{6(1)}_{d+1,i})$, which implies that $q_1(G) < q_1(T^{6(1)}_{d+1,i})$, a contradiction.

Case 1.2. $G \ncong B(d,d,d,d+1)$.

Suppose $P_{k+1} = v_1v_2\ldots v_{k+1}$ is an internal path of G with length $k \geq 2$ such that P_{k+1} does not exist on a cycle of length 3 in G. Let $G' = G-\{v_1v_2, v_2v_3\}+\{v_1v_3\}$. If the diameter of $G'-v_2$ is d, it is easy to see that $\exists v_0 \in \{G'-v_2\}$ such that $G^* = G'+\{v_0v_2\} \in \Gamma^d_n$. If the diameter of $G'-v_2$ is $d-1$. Let $G^* = G'+\{vv_i\}$ or $G^* = G'+\{vv_{i+1}\}$, then $G^* \in \Gamma^d_n$. According to Lemma 1.11 and Theorem 2.2, we have $q_1(G^*) > q_1(G)$, a contradiction. From the proof as above, we know that $G \in \Gamma^{d,3}_n \bigcup \Gamma^{d,4}_n \bigcup \Gamma^{d,6}_n \bigcup \Gamma^{d,7}_n$.

Case 2. $G \notin \Gamma^{d,3}_n$.

Assume to the contrary, $G \in \Gamma^{d,3}_n$. The base of three cycles, say C_p, C_q, C_l, in G has seven possible cases [see Fig. 1]. We first prove that $|V(P_{d+1}) \bigcap V(C_p)| \geq 1$ or $|V(P_{d+1}) \bigcap V(C_q)| \geq 1$ or $|V(P_{d+1}) \bigcap V(C_l)| \geq$

1. Otherwise, $|V(P_{d+1})\bigcap V(C_p)| = 0$, $|V(P_{d+1})\bigcap V(C_q)| = 0$, $|V(P_{d+1})\bigcap V(C_l)| = 0$. Suppose $P_k = u_1, u_2, \ldots, u_k$ is a shortest path such that $u_1 \in V(P_{d+1})$ and $u_k \in V(C_p)\bigcup V(C_q)\bigcup V(C_l)$. Then $k \geq 2$. Using Lemma 1.11, we have a graph $G^* \in \Gamma_n^{d,3}$ with $q_1(G^*) > q_1(G)$, a contradiction. Hence, $|V(P_{d+1})\bigcap V(C_p)| \geq 1$ or $|V(P_{d+1})\bigcap V(C_q)| \geq 1$ or $|V(P_{d+1})\bigcap V(C_l)| \geq 1$.

Let $V' = V(P_{d+1})\bigcap V(C_p \bigcup C_q \bigcup C_l)$ and $G' = G(V')$ be the induced subgraph of G. Then G' is obtained from G by attaching some trees. It is easy to see that all these attached trees are stars. Furthermore, applying Lemma 1.12, we get that all these pendent edges are attached at the same vertex of G'.

From Case 1, we know that $p = q = l = 3$. Without loss of generality, we choose v_i and v_j in $V(C_p)$ and $V(C_q)$ respectively, such that v_i and v_j are not adjacent and $v_iv_j \notin P_{d+1}$. If $x_i \geq x_j$, let $G^* = G - v_tv_j + v_tv_i$, where $v_t \in N(v_j)\backslash N(v_i)$ and if $x_i < x_j$, let $G^* = G - v_kv_i + v_kv_j$, where $v_k \in N(v_i)\backslash N(v_j)$. Obviously, $G^* = G^* \in \Gamma_n^{d,4}$. By Lemma 1.1, we have $q_1(G^*) > q_1(G)$ a contradiction. So $G \notin \Gamma_n^{d,3}$.

Case 3. $G \notin \Gamma_n^{d,4}$.

Assume to the contrary, $G \in \Gamma_n^{d,4}$. Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex-disjoint paths, where $l, p, q \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of $P_{l+1}, P_{p+1}, P_{q+1}$, respectively, the resulting graph [see Fig. 2], denoted by $P(l,p,q)$, is called a θ-graph. Suppose C_m be a cycle. Connect C_m and $P(l,p,q)$ by a path and denote the resulting graph by G'. Then G' has four types [see Fig. 2].

Similar to the proof of Case 2, we can verify that $|V(P_{d+1})\bigcap V(G')| \geq 1$. Let $V'' = V(P_{d+1})\bigcup V(G')$, and $G'' = G[V'']$ be the induced subgraph of G. Furthermore, similar to the proof of Case 2, we can show that G'' is obtained from G by attaching some pendent edges to one vertex.

It is easy to see that $l = 1, p = q = 2$. By Case 1, we have $m = 3$. Choose v_k and v_t in $V(P(1,1,2))$ and C_3, respectively, such that v_k and v_t are not adjacent. By using Lemma 1.1, we have that $G^* \in \Gamma_n^{d,6}$. Then $q_1(G^*) > q_1(G)$, a contradiction. So, $G \notin \Gamma_n^{d,4}$.

Case 4. If $G \in \Gamma_n^{d,6}$, then $G \cong T^{6(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}$.

Denote the subgraph, say G', induced by the six cycles contained in G. Similar to the proof of Case 2, we can verify that $|V(P_{d+1})\bigcap V(G')| \geq 1$. Let $V'' = V(P_{d+1})\bigcup V(G')$, and $G'' = G[V'']$ be the induced subgraph of G. Furthermore, similar to the proof of Case 2, we can show that G'' is obtained from G by attaching some pendent edges to one vertex.

Now we show that the three cases (1), (2) and (3) (Fig. 3) of G' are H_1, H_2, H_3 (shown in Fig.8), respectively. That is, $(1) = H_1, (2) = H_2, (3) = H_3$. We first show that $(1) = H_1$. Denote (1) by $P(l,p,q,m)$. We have that $l, p, q, m \geq 1$ and at most one of them is 1. Without loss

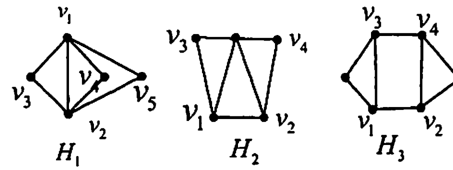

Fig.8 H_1, H_2, H_3

of generality, we may assume that $l \le p \le q \le m$. We claim that $l = 1$ and $p = q = m = 2$. Indeed, by case 1, $l = 1, p \le q \le m \le 3$ and if $m = 3$, then $p = q = 2$. If $m = 3$, denote $P_{m+1} = u_1uvu_2$. By Case 1, we have $d(u) > 2$ and $d(v) > 2$. In the case, when neither u_1u nor vu_2 lies on P_{d+1}, applying Lemma 1.11 to vu_2, we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. So we assume that u_1u lies on P_{d+1}. If neither u_1u_2 nor uv lies on P_{d+1}, applying Lemma 1.11 to vu_2, we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. Then we assume that u_1u_2 lies on P_{d+1}, and then $d(v) > 2$. By Lemma 1.1 to u and v, we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. Therefore, $l = 1, p = q = m = 2$. That is $(1) = H_1$. Similar the proof as above, we can prove that $(2) = H_2, (3) = H_3$.

Next, we prove that the G' is neither H_2 nor H_3. Assume that $G' = H_2$. If $v_1v_3 \in P_{d+1}$ or $v_2v_4 \in P_{d+1}$, then by symmetry we only consider $v_1v_3 \in P_{d+1}$. If $x_1 \ge x_2$, let $G^* = G - v_2v_4 + v_1v_4$, then $G^* \in \Gamma_n^{d,6}$. If $x_1 < x_2$, let $G_1^* = G - v_1v_3 + v_2v_3$, then applying Lemma 1.11 to v_1v_m ($v_1v_m \in P_{d+1}$ and $v_m \notin G'$), we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G_1^*)$. By Lemma 1.1, there is a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction.

If $v_2v_1v_3 \in P_{d+1}$ or $v_1v_2v_4 \in P_{d+1}$, similar to the proof as above, there is a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction.

If neither v_1v_3 nor v_2v_4 exists on P_{d+1}, then applying Lemma 1.1 to v_1 and v_2, we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. Therefore, we obtained $G' \ne H_2$.

Now, we show that $G' \ne H_3$. If $G' = H_3$, then applying Lemma 1.11 to v_1v_2 (if $v_1v_2 \notin P_{d+1}$, otherwise, applying Lemma 1.11 to v_3v_4), we obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction.

Thus, H_1 in Fig.8 is just the graph induced by the six cycles contained in G.

If $|V(P_{d+1}) \bigcap V(H_1)| = 1$, then without loss of generality we assume that $v_2 \in V(P_{d+1})$ or $v_3 \in V(P_{d+1})$. If $v_3 \in V(P_{d+1})$, using Lemma 1.1 to v_2 and v_3, we can obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. Hence, $v_2 \in V(P_{d+1})$ applying Lemma 1.12, we have

$G \cong T^{6(2)}_{d+1,i}$, which is a contradiction by Theorem 2.3.

If $|V(P_{d+1}) \bigcap V(H_1)| = 2$, then without loss of generality we assume that $v_1v_2 \in V(P_{d+1})$ or $v_2v_3 \in V(P_{d+1})$. let $v_2v_3 \in V(P_{d+1})$, using Lemma 1.1 to v_1 and v_3, we can obtain a graph $G^* \in \Gamma_n^{d,6}$ such that $q_1(G^*) > q_1(G)$, a contradiction. Let $v_2v_3 \in V(P_{d+1})$, then by Case 1, Lemmas 1.1 and 1.12, we can prove that all the pendent edges, not lying on P_{d+1}, of G must be attached to just one of v_1 and v_2. That is to say that $G \cong T^{6(1)}_{d+1,i}$. Furthermore, by Lemma 1.5 and Theorem 2.6, we obtain $G \cong T^{6(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}$.

If $|V(P_{d+1}) \bigcap V(H_1)| = 3$, then without loss of generality we assume that $v_2v_3v_4 \in V(P_{d+1})$. Applying to Case 1 and Lemmas 1.1 and 1.12, we can similarly prove that all the pendant edges, not lying on P_{d+1}, of G must be attached to just one of v_2 and v_3. That is to say that $G \cong T^{6(3)}_{d+1,i}$ or $G \cong T^{6(4)}_{d+1,i}$.

If $d \geq 3$, by Lemma 1.5, we have $2 \leq i \leq d-1$. By Theorem 2.5, $q_1(T^{6(4)}_{d+1,i}) < q_1(T^{6(3)}_{d+1,i})$. When $d \geq 5$, by Lemma 1.5, we have $4 \leq i \leq d-1$. By Theorem 2.3, $q_1(T^{6(3)}_{d+1,i}) < q_1(T^{6(4)}_{d+1,i})$. If $d = 4$, by Lemmas 1.5 and 1.10, we have $q_1(T^{6(3)}_{4,3}) > q_1(T^{6(2)}_{4,3}) > q_1(T^{6(1)}_{4,3})$. That is to say that for $d \geq 3$, $q_1(T^{6(3)}_{d+1,i}) < q_1(T^{6(1)}_{d+1,i})$. Furthermore, we have that for $G \in \Gamma_n^{d,6}$,

$$G \cong T^{6(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}. \tag{3.8}$$

Case 5. $G \in \Gamma_n^{d,7}$, $G \cong T^{7(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}$.

The base of seven cycles contained in G has only one case [see Fig. 4]. By Case 1, Lemmas 1.1 and 1.12, the graph induced by the seven cycles contained in G is K_4. Similar to the proof of Case 4, we get that $G \cong T^{7(1)}_{d+1,i}$ or $G \cong T^{7(2)}_{d+1,i}$. Furthermore, by Theorems 2.4 and 2.7, we have

$$G \cong T^{7(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}. \tag{3.9}$$

From Cases 2-5, and by (3.8) and (3.9), Theorem 2.8 (i) follows.

For $d = 3$, applying to Lemmas 1.5 and 1.10, we have

$$q_1(T^{7(1)}_{4,3}) > q_1(T^{6(1)}_{4,3}). \tag{3.10}$$

From Cases 2-5, and by (3.8), (3.9) and (3.10), Theorem 2.8 (ii) follows. The proof is completed.□

From Theorem 2.8, it is easy to get the following two corollaries.

Corollary 2.9. For $d \geq 3$ and $n \leq d+3$, let G be a tricyclic graph with n vertices and diameter d. Then

$$q_1(G) \leq q_1(T^{7(1)}_{d+1,\lfloor\frac{d+3}{2}\rfloor}),$$

and the equality holds if and only if $G \cong T^{7(1)}_{d+1,\lfloor \frac{d+3}{2} \rfloor}$.

Proof. It is easy to check that there does not exist $G \in \Gamma_{n,d}$ such that $n \leq d+2$. Hence, we only need to consider the case $G \in \Gamma_{n,d}, n = d+3$ with $d \geq 3$. We know that the largest signless Laplacian spectral radius must be in $\{T^{6(3)}_{d+1,i}, T^{6(4)}_{d+1,i}, T^{7(1)}_{d+1,i}\}$. By Theorem 2.3, $q_1(T^{6(3)}_{d+1,i}) > q_1(T^{6(4)}_{d+1,i})$. Now we are to show that $q_1(T^{7(1)}_{d+1,i}) > q_1(T^{6(3)}_{d+1,i})$. Similarly to the proof of Theorem 2.3, we can prove that

$$q_1(G) \leq q_1(T^{7(1)}_{d+1,\lfloor \frac{d+3}{2} \rfloor}).$$

The proof is completed.□

Corollary 2.10. Among all the tricyclic graphs with n vertices, the largest and the second largest signless Laplacian spectral radius are, respectively, of the form $T^{7(1)}_{3,2}$ and $T^{6(1)}_{3,2}$.

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