The Adjacent Vertex Distinguishing Total Chromatic Number of Some Families of Snarks

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Abstract

The adjacent vertex distinguishing total chromatic number $\chi_{at}(G)$ of a graph G is the smallest integer k for which G admits a proper k-total coloring such that no pair of adjacent vertices are incident to the same set of colors. Snarks are connected bridgeless cubic graphs with chromatic index 4. In this paper, we show that $\chi_{at}(G) = 5$ for two infinite subfamilies of snarks, i.e., the Loupekhine snark and Blanusa snark of first and second kind. In addition, we give an adjacent vertex distinguishing total coloring using 5 colors for Watkins snark and Szekeres snark, respectively.

Keywords: Adjacent vertex distinguishing total coloring, Loupekhine snark, Blanusa snark

1 Introduction

Let $G = (V, E)$ be a simple graph. The maximum degree of G is denoted by Δ. A proper k-total coloring of G is a mapping f from $V \cup E$ to the set of colors $\{1, 2, \dots, k\}$ such that $f(x) \neq f(y)$ for every pair of adjacent or incident elements $x, y \in V \cup E$. The total chromatic number of G, $\chi_t(G)$, is the least k for which G has a k-total coloring. For each vertex $v \in V$, we set $C_f(v) = \{f(v)\} \cup \{f(uv) | uv \in E(G)\}$ and $\bar{C}_f(v) = \{1, 2, \dots, k\} \setminus C_f(v)$.

If it is clear from the context, we simply use $C(v)$ and $\overline{C}(v)$ instead of $C_f(v)$ and $\overline{C}_f(v)$, respectively. The coloring f is called an adjacent vertex distinguishing total coloring (avd-total coloring) if $C(u) \neq C(v)$ for any pair of adjacent vertices u and v. The adjacent vertex distinguishing total chromatic number $\chi_{at}(G)$ of G is the least k such that G admits a k-avd-total coloring.

Zhang et al. [10] introduced the concept of avd-total coloring. Base on the argument for some special classes of graphs, they proposed the following conjecture.

Conjecture 1.1 *[10] Let G be a connected graph with at least two vertices. Then* $\chi_{at}(G) \leq \Delta + 3$.

Chen [2] and Wang [9], independently, confirmed this conjecture for case $\Delta \leq 3$. Later, Hulgan [5] presented a concise proof for this result. For the lower bound of $\chi_{at}(G)$, it is evident that $\chi_{at}(G) \geq \Delta + 1$. Moreover, Zhang et al. gave the following result.

Lemma 1.1 *[10] If G is a graph with two adjacent vertices of maximum degree, then* $\chi_{at}(G) \geq \Delta + 2$.

Thus, the adjacent vertex distinguishing total chromatic number of a cubic (3-regular) graph is either 5 or 6. A snark is a bridgeless non-3-edge colorable cubic graph. The Petersen graph is the smallest and earliest known snark, which has 10 vertices. There is no snark of order 12, 14 or 16 (see for example [3, 4]). Preissmann [7] shown that there are just two nontrivial snarks of order 18, the Blanusa snarks.

Two infinite families of snarks, Flower and Goldberg snarks have had their adjacent vertex distinguishing total chromatic number been determined in [1]. In this paper, we give the adjacent vertex distinguishing total chromatic number for the Loupekhine snark and Blanusa snark of first and second kind. Graphs in these families share a common property that they can be built from a suitable glueing of some subgraphs of the Petersen

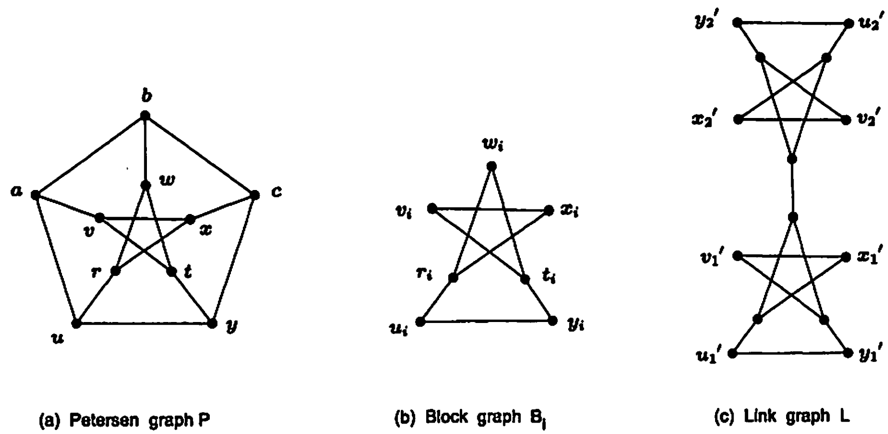


Fig. 1: Construction of a block and a link graph.

graph. In addition, we give a 5-avd-total coloring for Watkins snark and Szekeres snark, respectively.

2 The Loupekhine Snarks

F. Loupekhine created a method for constructing new snarks from already known ones. Loupekhine's construction presented here was introduced by Issacs [6]. Sasaki et al. [8] proved that the total chromatic number of Loupekhine family is 4 . Let P be the Petersen graph. We define the block graph B_i as the subgraph of P by removing a path of three vertices. Due to the high symmetry of P, B_i is unique (see Fig.1(b)). A link graph L is defined as the junction of two block graphs, where the eight degree two vertices are called border vertices (see Fig.1(c)).

The first Loupekhine snark FL_3 of first kind is formed by B_1, B_2, B_3 and edges $\{v_ix_{i+1}, u_iy_{i+1}|1 \leq i \leq 2\} \cup \{ww_i|1 \leq i \leq 3\} \cup \{v_3x_1, u_3y_1\}$, where w is a new vertex (see Fig.2(a)). For each odd $r \geq 5$, FL_r is recursively con-

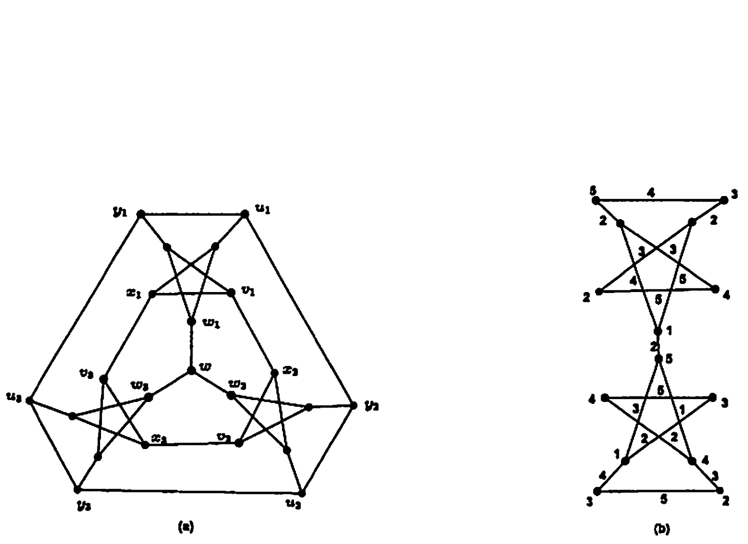


Fig. 2: FL_3 and a 5-avd-total coloring for L_3.

structed by adding the link graph to the snark FL_{r-2}. More precisely, if we define $E_{r-2}^{out} = \{v_1x_2, u_1y_2, v_{r-2}x_1, u_{r-2}y_1\}$ and $E_r^{in} = \{u_{r-2}y_2', v_{r-2}x_2', u_2'y_1, v_2'x_1; v_1x_1', u_1y_1', v_1'x_2, u_1'y_2\}$, then $V(FL_r) = V(FL_{r-2}) \cup V(L)$ and $E(FL_r) = (E(FL_{r-2}) \setminus E_{r-2}^{out}) \cup E(L) \cup E_r^{in}$. So each FL_r, r odd, is formed by appropriate connecting r block graphs $B_1, B_2, \ldots, B_r$ along a cycle. Finally, we rename the index of the r blocks of FL_r in anticlockwise direction.

The Loupekhine family of second kind SL_r, r odd, is obtained from FL_r, by replacing the edges $x_{\frac{r+3}{2}}v_{\frac{r+1}{2}}$ and $y_{\frac{r+3}{2}}u_{\frac{r+1}{2}}$ with edges $x_{\frac{r+3}{2}}u_{\frac{r+1}{2}}$ and $y_{\frac{r+3}{2}}v_{\frac{r+1}{2}}$, respectively.

Theorem 2.1 *Let L_r ($r \geq 3, r$ is odd) be a Loupekhine snark of first or second kind. Then $\chi_{at}(L_r) = 5$.*

Proof. By Lemma 1.1, it suffices to show that each L_r admits a 5-avd-total coloring such that all edges of E_r^{out} receive the same color 1. We proceed by induction based on the recursive procedure described above. A 5-avd-total coloring of L, FL_3 and SL_3 have been depicted as in Figs. 2(b) and 3. Note that the edges of E_r^{out} in FL_3 and SL_3 have the same color 1.

Assume that r is odd and $r \geq 5$. By induction hypothesis, L_{r-2} has a 5-avd-total coloring such that edges of E_{r-2}^{out} have the same color 1. We obtain a coloring f of L_r as follows. Assign color 1 to the edges of E_r^{in}. Each

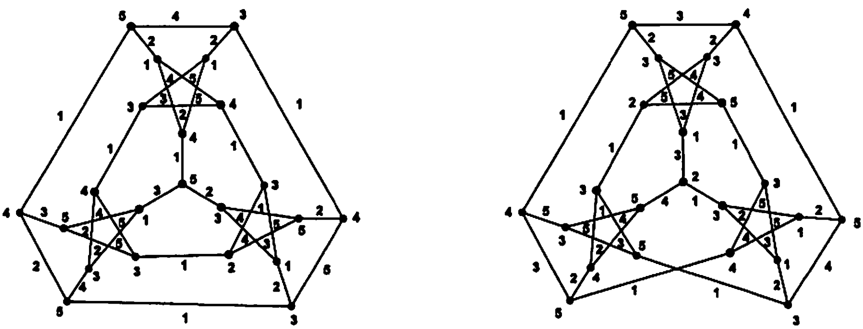


Fig. 3: 5-avd-total colorings for graphs FL_3 and SL_3.

element of $(V(L_r) \cup E(L_r)) \setminus E_r^{in}$ has the same color as their corresponding part in L_{r-2} and L.

We complete the proof by showing that f is a proper 5-avd-total coloring for L_r. First, note that the ends of edges in E_r^{in} get different colors, and no edge incident with border vertices in L has color 1. So f is a 5-total coloring. In what follows, we show that the ends of edges in E_r^{in} have distinct color sets. Since both kinds of Loupekhine snark share the same coloring of the link graph, we have:

$C(x_1') = \{4\}$, $\overline{C}(y_1') = \{4\}$, $C(v_1') = \{3\}$, $C(u_1') = \{2\}$,

$C(x_2') = \{4\}$, $C(y_2') = \{3\}$, $C(v_2') = \{2\}$, $C(u_2') = \{5\}$.

Next, we distinguish between two cases.

Case 1 $L_r = FL_r$. Observe that, in B_1, we have: $C(x_1) = \{5\}$, $C(y_1) = \{3\}$, $C(v_1) = \{3\}$ and $C(u_1) = \{5\}$.

When $r = 5$, we can check that in FL_{r-2}, $C(x_2) = \{2\}$, $C(y_2) = \{3\}$, $C(v_{r-2}) = \{3\}$ and $C(u_{r-2}) = \{5\}$; when $r \geq 7$, then in FL_{r-2}, $C(x_2) = \{4\}$, $C(y_2) = \{4\}$, $C(v_{r-2}) = \{2\}$ and $C(u_{r-2}) = \{5\}$.

Case 2 $L_r = SL_r$. Now in B_1, we have: $C(x_1) = \{4\}$, $C(y_1) = \{4\}$, $C(v_1) = \{2\}$ and $C(u_1) = \{5\}$.

When $r = 5$, we can check that in SL_{r-2}, $C(x_2) = \{4\}$, $C(y_2) = \{3\}$, $C(v_{r-2}) = \{2\}$ and $C(u_{r-2}) = \{2\}$; when $r \geq 7$, then in SL_{r-2}, $C(x_2) =$

$\{4\}$, $C(y_2) = \{4\}$, $\overline{C}(v_{r-2}) = \{2\}$ and $\overline{C}(u_{r-2}) = \{5\}$.

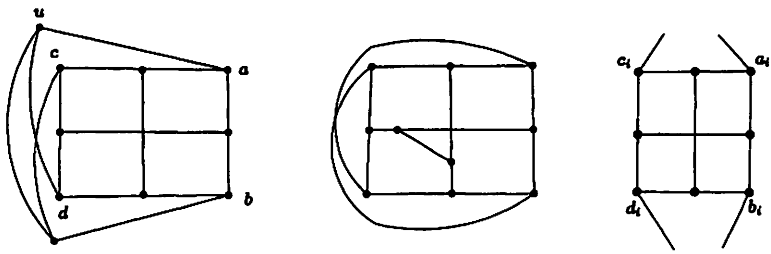
By the above analysis, it is easy to see that f is adjacent vertex distinguishing.

3 The Blanusa Snarks

Besides the Petersen graph, two Blanusa snarks of order 18 are the two smallest known snarks, which yields the two families of Blanusa snarks [7]. The subsequent members of these two families are formed by all possible applications of "dot product" to the previous snark and the Petersen graph. In this section, we only consider two subfamilies of Blanusa snarks. To explore the structure of these graphs, it is helpful to display the Petersen graph as in Figs. 4 (a) and (b). Define the block graph B_i as the subgraph of P_1 by cutting edges au, du, bv and cv (see Fig.4(c)). The first members of both Blanusa subfamilies are shown in Fig.5. The subfamilies are defined recursively as follows: the k-th member of Blanusa snark subfamilies, denoted by $B_i(k)$, ($k \geq 2$, $i \in \{1,2\}$), is obtained by inserting the block graph B_k into $B_i(k-1)$, where b_k and d_k are the two vertices adjacent to c_1 and a_1 in the bottom copy of the block graph in $B_i(k-1)$. So in each $B_i(k)$, there is a vertical chain of k copies of the block graph. Fig.6 shows the first two members of $B_1(k)$.

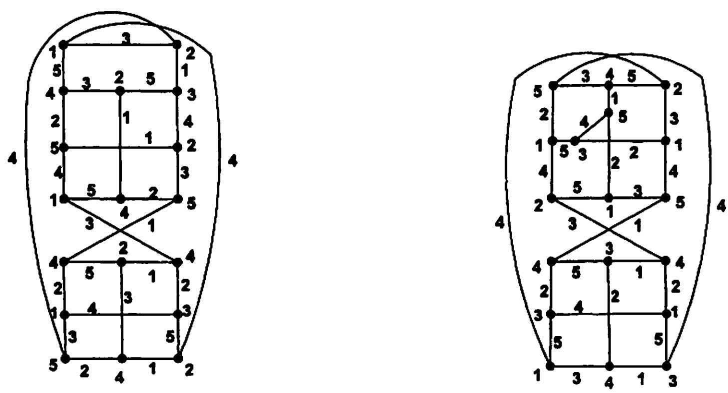
Theorem 3.1 *Let $B_i(k)$, ($k \geq 2$, $i \in \{1,2\}$) be a member of the subfamilies of Blanusa snark. Then $\chi_{at}(B_i(k)) = 5$.*

Proof. The proof is similar to the previous one. We also present an approprise 5-avd-total colorings for each $B_i(k)$ in the two subfamilies of Blanusa snark. Unlike the case of Loupekhine family and its link graph, here, we use two 5-avd-total colorings for the block graph, as depicted in Fig.7. First, we construct a 5-avd-total coloring for $B_1(1)$ and $B_2(1)$, respectively (see Fig.5). For each $k \geq 2$, a 5-avd-total coloring f for $B_i(k)$ is obtained in the following way: the subgraph induced by the top ten and



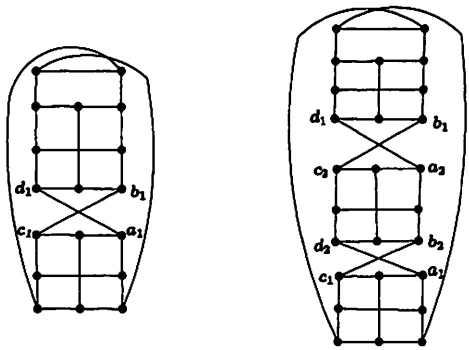
(a) Petersen graph P_1 (b) Petersen graph P_2 (c) Block graph B_i

Fig. 4: Drawings of the Petersen graph and block graph.



(a) $B_1(1)$ (b) $B_2(1)$

Fig. 5: 5-avd-total colorings for graphs $B_1(1)$ and $B_2(1)$.



(a) $B_1(1)$ (b) $B_1(2)$

Fig. 6: The first two Blanusa snarks of the first kind.

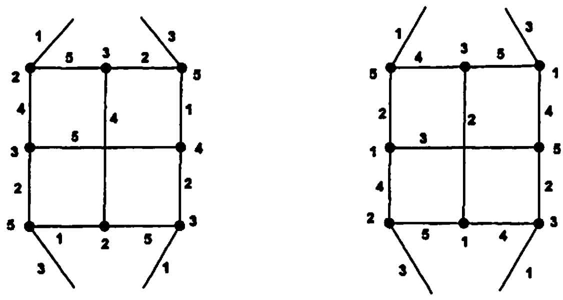
the bottom eight vertices is colored as in $B_i(1)$, the $k-1$ copies of B_i between them are colored using ϕ and ϕ' alternately, beginning by ϕ for the copy on the top of the chain.

To conclude the proof, we show that the coloring f is 5-avd-total colorings for $B_i(k)$. For each $k \geq 2$, we define $E_{k-1}^{out} = \{a_1 d_{k-1}, c_1 b_{k-1}\}$ and $E_k^{in} = \{a_k d_{k-1}, c_k b_{k-1}, c_1 b_k, a_1 d_k\}$. It is easy to check that the ends of E_k^{in} have different colors. Considering the color sets of the ends of E_k^{in}.

If $k \geq 3$ is odd, then in both subfamilies of Blanusa snark, we have: $C(a_k) = \{2\}$, $C(c_k) = \{3\}$, $C(b_k) = \{5\}$, $C(d_k) = \{1\}$, and $C(d_{k-1}) = \{4\}$, $C(b_{k-1}) = \{4\}$, $C(c_1) = \{3\}$, $C(a_1) = \{5\}$.

If $k \geq 2$ is even, then $C(a_k) = \{4\}$, $C(c_k) = \{3\}$, $C(b_k) = \{4\}$, $C(d_k) = \{4\}$, $C(c_1) = \{3\}$, $C(a_1) = \{5\}$; when $k = 2$, then in $B_1(1)$, $C(d_{k-1}) = \{2\}$, $C(b_{k-1}) = \{4\}$, and in $B_2(1)$, $C(d_{k-1}) = \{1\}$, $C(b_{k-1}) = \{2\}$; when $k \geq 4$, then $C(d_{k-1}) = \{1\}$, $C(b_{k-1}) = \{5\}$.

By the above analysis, it is easy to see that f is adjacent vertex distinguishing. This ends the proof.



For B_i, i is even *For B_i, i is odd*

Fig. 7: Two 5-avd-total colorings ϕ and ϕ' of block graph.

4 Watkins Snark and Szekeres Snark

In this section, we give a 5-avd-total coloring for Watkins and Szekeres snark, respectively, as shown in Fig.8. The underlined labels denote the colors on edges. So the adjacent vertex distinguishing total chromatic number of these two snarks are both 5. In order to check that the total coloring is adjacent vertex distinguishing, we define a function $g : v \to \overline{C}(v)$, $v \in V(G)$, as shown in Fig.9. Then it is routine to check that $g(u) \neq g(v)$ for $uv \in E(G)$ in both graphs.

5 Conclusion

In this work, we present a 5-avd-total coloring for the infinite subfamilies of Loupekhine and Blanusa family. Moreover, we show a 5-avd-total coloring for Watkins snark and Szekeres snark, respectively. Our results contribute as one more evidence that all snarks are 5-avd-total colorable. Therefore, we propose the following problem.

Problem 5.1 *Is there any snark G with $\chi_{at}(G) = 6$.*

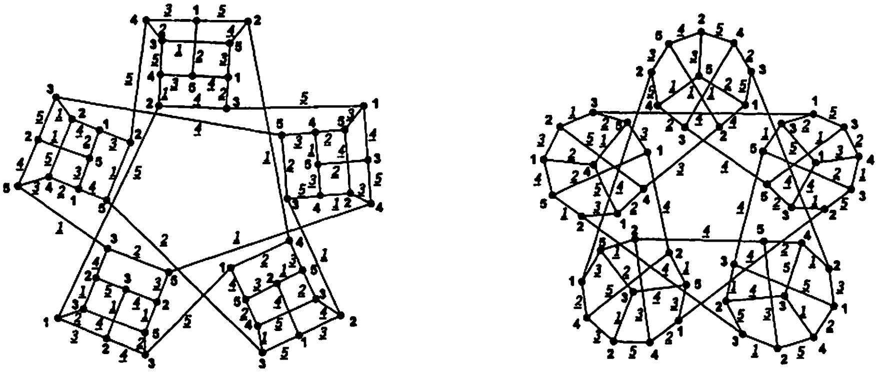


Fig. 8: 5-avd-total colorings for Watkins Snark and Szekeres snark.

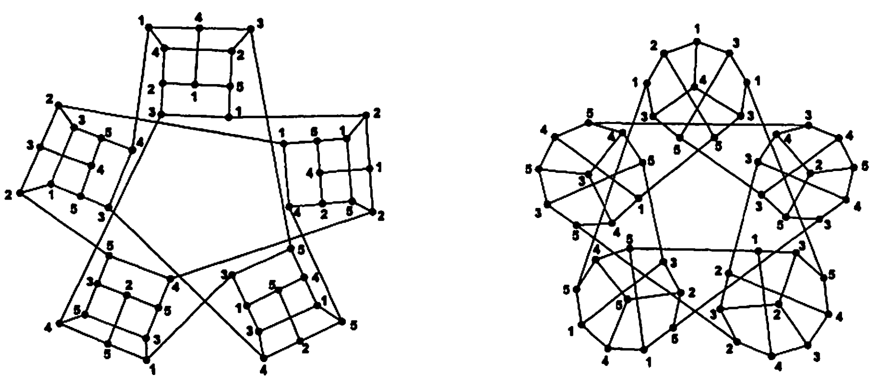


Fig. 9: The function g for Watkins Snark and Szekeres snark.

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