# Recounting the number of peaks and valleys in compositions of integers

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**Abstract**

In this note, we provide a combinatorial proof of a recent formula for the total number of peaks and valleys (either strict or weak) within the set of all compositions of a positive integer into a fixed number of parts.

## 1 Introduction

A *composition* of a positive integer $n$ is an ordered collection of one or more positive integers, called *parts*, whose sum is $n$. If $n \geqslant m \geqslant 1$, let $C(n,m)$ denote the set of compositions of $n$ having exactly $m$ parts. Recall that $|C(n,m)| = \binom{n-1}{m-1}$ [1, p. 54].

If $n \geqslant 1$, then let $[n] = \{1, 2, \ldots, n\}$, with $[0] = \varnothing$. Let $[n]^m$ denote the set of words of length $m$ in the alphabet $[n]$. If $\sigma \in [n]^m$, let $\text{red}(\sigma)$ denote the member of $[n]^m$ obtained by replacing all occurrences of the smallest letter of $\sigma$ by 1, replacing all occurrences of the second smallest letter of $\sigma$ by 2, and so on. For example, if $\sigma = 46323 \in [6]^5$, then $\text{red}(\sigma) = 34212$. A member $\tau \in [a]^b$ for some positive integers $a$ and $b$ that contains each element of $[a]$ at least once is called a *subword* or *subword pattern.*

We will say that a word $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ *contains* an occurrence of the subword $\tau$ if $\text{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+b-1}) = \tau$ for some index $i$, where $i \leqslant m-b+1$. The number of occurrences of $\tau$ in $\sigma$ is the number of indices $i$ such that $\text{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+b-1}) = \tau$. In what follows, we will regard compositions as words and consider the occurrences of various subword patterns within these words, as has been done, for example, in [4] (see also [5] and the references contained therein).

For example, within the composition $\lambda = 3+2+3+4+1+4+2+2 \in C(21,8)$, there are two occurrences of the pattern $\tau = 212$ (corresponding to 323 and 414) and one occurrence of the pattern $\tau = 211$ (corresponding to 422).

We now recall some terminology used in [3]. A *strict peak* within a composition is an ascent followed directly by a descent (i.e., an occurrence of any one of the subword patterns 121, 231, or 132). By a *strict valley*, we mean a descent followed directly by an ascent (i.e., an occurrence of either 212, 213, or 312). A *weak peak* is an occurrence of 221, whereas a *weak valley* is an occurrence of 211. For example, the composition $\lambda$ above has two strict peaks, two strict valleys, no weak peaks, and one weak valley.

As part of an attempt to find connections between compositions and integer partitions (see [2, 3]), an explicit formula was determined for the generating function $F(x,y,q)$ which counts the compositions of $n$ having $m$ parts according to the combined number of occurrences of peaks and valleys (either strict or weak). The following formula for the total number of peaks and valleys within all of the members of $C(n,m)$ was found in [3] by differentiating the aforementioned generating function with respect to $q$, and setting $q=1$.

**Theorem 1.1.** *If $n \geqslant m \geqslant 2$, then the total number of peaks and valleys (either strict or weak) within all of the members of*

*$C(n,m)$ is given by*

$$2(m-2)\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \binom{n-3k}{m-2}.$$

Here, we provide a direct combinatorial proof of this result by defining a suitable bijection between a set of ordered triples whose cardinality is given by the above formula and the set of all peaks and valleys within the members of $C(n,m)$. Modifying our proof yields comparable formulas for various subclasses of compositions.

# 2 Proof of Theorem 1.1

We start with the following definition.

**Definition 1.** *By a $k$-occurrence of a strict or weak peak (or valley) at index $i$ within a composition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we mean an occurrence of one of these patterns in which the first 1 in the pattern corresponds to the part $\lambda_i$ of $\lambda$ with $\lambda_i = k$.*

The same terminology will be applied to particular subword patterns. For example, within $\lambda = 5+6+4+3+3 \in C(21,5)$, there is a 4-occurrence of the pattern 231 at index 3 and a 3-occurrence of the pattern 211 at index 4.

We now proceed with the proof of Theorem 1.1.

*Proof.* Let $n \geqslant m \geqslant 2$. Given $1 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$, let $S_k$ denote the set of ordered triples $\rho = (\lambda, i, j)$, where $\lambda \in C(n+1-3k, m-1)$, $i \in [m-2]$, and $j \in [2]$. Note that $|S_k| = 2(m-2)\binom{n-3k}{m-2}$. For each $k = 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor$, we will define a bijection between $S_k$ and the set $T_k$ consisting of all $k$-occurrences of peaks or valleys (either strict or weak) within the members of $C(n,m)$. Putting together these bijections then shows that $S = \cup_k S_k$ has the same cardinality as the set of all peaks and valleys (strict or weak) within the members of $C(n,m)$, as desired.

To define our bijection between $S_k$ and $T_k$, let $\rho = (\lambda, i, j) \in S_k$, where $i$ is given and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$. First suppose $\lambda_i \geqslant \lambda_{i+1}$. If $j = 1$, then insert the part $k$ between the parts $\lambda_i$ and $\lambda_{i+1}$ of $\lambda$, add $k$ to the part $\lambda_i$, and add $k-1$ to the part $\lambda_{i+1}$ to obtain the composition

$$\alpha = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + k, k, \lambda_{i+1} + k - 1, \lambda_{i+2}, \ldots, \lambda_{m-1}),$$

which belongs to $C(n, m)$. Note that $\lambda_i \geqslant \lambda_{i+1}$ implies that $\alpha$ has a $k$-occurrence at index $i+1$ of either 312 or 211. This correspondence is then seen to define a bijection between the members of $S_k$ with $i$ given, $j = 1$, and $\lambda_i \geqslant \lambda_{i+1}$ and the set of $k$-occurrences of 312 or 211 at index $i+1$ within all the members of $C(n, m)$. Considering all possible $i$, we see that the number of members $\rho$ of $S_k$ for which $\lambda_i \geqslant \lambda_{i+1}$ in $\lambda$ and $j = 1$ equals the number of $k$-occurrences of 312 or 211 within all members of $C(n, m)$.

For example, suppose $n = 21$, $m = 6$ and $k = 2$. Then $\rho = (\lambda, 2, 1)$, where $\lambda = 5 + 6 + 3 + 1 + 1$, corresponds to the 2-occurrence of 312 at index 3 in $\alpha = 5{+}8{+}2{+}4{+}1{+}1 \in C(21, 6)$, while $\rho = (\lambda, 4, 1)$ would correspond to the 2-occurrence of 211 at index 5 in $\alpha = 5 + 6 + 3 + 3 + 2 + 2 \in C(21, 6)$.

If $j = 2$, then we insert the part $k$ directly before the part $\lambda_i$ of $\lambda$, add $k$ to $\lambda_i$, and add $k-1$ to $\lambda_{i+1}$ to obtain the composition

$$\beta = (\lambda_1, \ldots, \lambda_{i-1}, k, \lambda_i + k, \lambda_{i+1} + k - 1, \lambda_{i+2}, \ldots, \lambda_{m-1})$$

belonging to $C(n, m)$. Note that $\lambda_i \geqslant \lambda_{i+1}$ implies $\beta$ has a $k$-occurrence at index $i$ of either 132 or 121. Considering all possible $i$, we see that the number of members of $S_k$ for which $\lambda_i \geqslant \lambda_{i+1}$ and $j = 2$ equals the number of $k$-occurrences of 132 or 121 within the members of $C(n, m)$.

Now suppose $\lambda_i < \lambda_{i+1}$ in $\lambda$. If $j = 1$, then proceeding as before in this case shows that members of $S_k$ for which $\lambda_i < \lambda_{i+1}$ and $j = 1$ equals the number of $k$-occurrences of 213 or 212 within all members of $C(n, m)$. If $j = 2$, then insert the part $k$

directly to the right of $\lambda_{i+1}$, add $k$ to $\lambda_i$, and add $k-1$ to $\lambda_{i+1}$ to obtain the composition

$$\gamma = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + k, \lambda_{i+1} + k - 1, k, \lambda_{i+2}, \ldots, \lambda_{m-1})$$

in $C(n, m)$. Considering all possible $i$, we see that the number of members of $S_k$ for which $\lambda_i < \lambda_{i+1}$ and $j = 2$ equals the number of $k$-occurrences of 231 or 221 within all of the members of $C(n, m)$.

Putting together the mappings in each of the four cases above yields the desired bijection between $S_k$ and $T_k$ for all $k$, which completes the proof. □

# 3 Further remarks

It is possible to extend the proof in the prior section and find comparable formulas for certain classes of compositions. For example, modifying the proof above yields the following result concerning the subclass of $C(n, m)$ whose members contain only parts greater than or equal a fixed positive integer $r$, which we'll denote by $C_r(n, m)$.

**Proposition 3.1.** *If $n \geqslant m \geqslant 2$ and $r \geqslant 1$, then the total number of peaks and valleys within all of the members of $C_r(n, m)$ is given by*

$$2(m-2) \sum_{k=r}^{\lfloor \frac{n-r(m-3)}{3} \rfloor} \binom{n-(r-1)(m-3)-3k}{m-2}.$$

Summing the formula in Theorem 1.1 above over $m$, and interchanging summation, yields a simple closed formula for the total number of peaks and valleys within all of the compositions of $n$, though it depends on the value of $n$ mod 3. A comparable formula may be obtained for compositions of $n$ whose parts are at least $r$, using Proposition 3.1.

We also have the following result concerning the subclass of $C(n,m)$ whose members contain only odd parts, which we'll denote by $C_o(n,m)$.

**Proposition 3.2.** *If $n \geqslant m \geqslant 2$, then the total number of peaks and valleys within all of the members of $C_o(n,m)$ is given by*

$$2(m-2)\sum_{k=1}^{\lfloor \frac{n+2}{6} \rfloor} \binom{\frac{n+m}{2}-3k}{m-2},$$

*where $n$ and $m$ have the same parity.*

*Proof.* First recall that the number of compositions of a positive integer $a$ into $b$ odd parts is

$$\binom{\frac{a+b}{2}-1}{b-1}, \qquad 1 \leqslant b \leqslant a,$$

where $a$ and $b$ are of the same parity. Thus, there are $\binom{\frac{n+m}{2}-3k}{m-2}$ members of $C_o(n+3-6k, m-1)$, where $1 \leqslant k \leqslant \lfloor \frac{n+2}{6} \rfloor$. Now proceed as in the proof of Theorem 1.1 above. Given $\lambda \in C_o(n+3-6k, m-1)$, $i \in [m-2]$, and $j \in [2]$, we insert the part $2k-1$ at some place within $\lambda$ and then add $2k$ to one of the parts of $\lambda$ and add $2k-2$ to another part. For example, if $\lambda_i \geqslant \lambda_{i+1}$ in $\lambda$ and $j=1$, then we insert the part $2k-1$ between $\lambda_i$ and $\lambda_{i+1}$, add $2k$ to $\lambda_i$, and add $2k-2$ to $\lambda_{i+1}$ to obtain

$$\alpha = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i+2k, 2k-1, \lambda_{i+1}+2k-2, \lambda_{i+2}, \ldots, \lambda_{m-1}),$$

which is seen to belong to $C_o(n,m)$. This shows that the ordered triples $(\lambda, i, j)$ for which $\lambda_i \geqslant \lambda_{i+1}$ and $j=1$ are equinumerous with the $(2k-1)$-occurrences of 312 or 211 within members of $C_o(n,m)$. The bijection in the remaining three cases is defined in a comparable manner, which completes the proof. □

One can also give an analogous formula for the total number of peaks and valleys in palindromic compositions of $n$ having $m$

parts, though it is more complicated. Finally, we remark that it is possible to give formulas for other finite discrete structures, such as $k$-ary words or set partitions (represented canonically as restricted growth functions). We leave the proof of the following result as an exercise for the interested reader.

**Proposition 3.3.** *If $n \geqslant 2$ and $k \geqslant 1$, then the total number of peaks and valleys within all of the words of length $n$ in the alphabet $[k]$ is given by*

$$\frac{2(n-2)(k^2-1)k^{n-2}}{3}.$$

# References


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2010 *Mathematics Subject Classification:* Primary 05A19; Secondary 05A05.