The orientable genus of the generalized Petersen graph $P(km, m)$

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Abstract

In the paper we show that the orientable genus of the generalized Petersen graph $P(km, m)$ is at least $\frac{km}{4} - \frac{m}{2} - \frac{km}{4m-4} + 1$ if $m \geq 4$ and $k \geq 3$. We determine the orientable genera of $P(3m, m)$, $P(4k, 4)$, $P(4m, m)$ if $m \geq 4$, $P(6m, m)$ if $m \equiv 0 \pmod 2$ and $m \geq 6$, and so on.

Key Words: generalized Petersen graph, orientable surface, orientable genus.

AMS Subject Classification(2000) : 05C10

1 Introduction

The *generalized Petersen graph* $P(n, m)$ is a cubic graph with the vertex set $\{v_i, u_i | i = 0, 1, \ldots, n-1\}$ and the edge set $\{v_i v_{i+1}, u_i u_{i+m}, v_i u_i | i = 0, 1, \ldots, n-1\}$, where $m < \frac{n}{2}$, and the index is read modulo n.

Generalized Petersen graphs are an important class of cubic graphs. Many properties of the generalized Petersen graph have been investigated, such as Hamiltonian problem(see [1],[2]), edge-coloring (see [5]),crossing number(see [6],[8],[10]), etc. But the orientable genus of the generalized Petersen graph has little been explored. In general, it is hard to determine the orientable

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genus of a graph, even if it is a cubic graph. Thomassen[11] showed that the orientable genus problem for cubic graphs is NP-complete.

In the paper we will study the orientable genus of $P(km, m)$. By the definition of the generalized Petersen graph, k is at least 3 if $n = km$. Since $P(km, m)$ is a planar graph if $m = 1$ or 2, we suppose that $m \geq 3$ in the paper.

The paper is arranged as follows. In Section 2, we will give an upper bound of the orientable genus of $P(km, m)$. In Section 3, we will give a lower bound of the orientable genus of $P(km, m)$ if $m \geq 4$, and we will determine orientable genera of some graphs such as $P(4k, 4)$, $P(4m, m)$ if $m \geq 4$, $P(6m, m)$ if $m \equiv 0 \pmod 2$ and $m \geq 6$, and so on. In Section 4, we will determine the genus of $P(3m, m)$. The rest of the section is contributed for other terminologies. The undefined terms can be found in [4] or [9].

A surface is a connected compact 2-dimensional manifold without boundary. Surfaces contain two classes: The orientable surfaces and nonorientable surfaces. In the paper a surface is always an orientable surface. The orientable surface $S_g (g \geq 0)$ can be obtained from the sphere with g handles attached, where g is called the *genus* of S_g.

A graph G is able to embed in a surface S if it can be drawn in the surface such that any edge does not pass through any vertex and any two edges do not cross each other. An embedding Π of a connected graph in a surface S is called *2-cell embedding*, if any connected component of S-Π, called *a face*, is homeomorphic to an open disc. In a 2-cell embedding of a connected graph G, the boundary of a face is a closed walk of G, which is called the *facial walk*. If a facial walk is a cycle, then it is called a facial cycle. The length of a facial walk is the number of its edges (if an edge appears twice then it is counted twice).

The *orientable genus* of a connected graph G, denoted by $\gamma(G)$, is the smallest nonnegative integer g such that G can be embedded in the surface S_g. Any embedding of a connected graph in the surface $S_{\gamma(G)}$ is a 2-cell embedding (see [12]).

By contracting a subgraph G' of a graph G to a vertex w, we mean that all edges in G' are deleted and all vertices in G' are identified with w and any edge incident with any vertex of G' whose two ends are not in G' is incident with w. A graph H' is a *minor* of a graph H if H' can be obtained from a subgraph of H by contracting edges.

2 An upper bound of the orientable genus of $P(km, m)$

In the section we will give an upper bound of $\gamma(P(km, m))$. We observe that the induced subgraph of $P(km, m)$ by the vertices $v_0, v_1, \ldots, v_{km-1}$ is a cycle, which is called the *principal cycle*. For $i = 0, 1, \ldots, k-1$, we observe that the induced subgraph of $P(km, m)$ by the vertices $u_i, u_{i+m}, \ldots, u_{i+(k-1)m}$ is also a cycle, which is denoted by C_i. Also, we call v_iu_i a *spoke* of $P(km, m)$.

We now give a drawing of $P(km, m)$ in the plane (or the sphere). For $i = 0, 1, \ldots, k-1$, let $P_i = v_{im}v_{im+1}\cdots v_{im+m-1}$. Now, $P_0, P_1, \ldots, P_{k-1}$ are represented by k disjoint segments in the plane from left to right, respectively. Next, $C_0, C_1, \ldots, C_{m-1}$ are respectively represented by m pairwise disjoint circles which satisfy the following conditions:

(1) Each of $C_0, C_1, \ldots, C_{m-1}$ is drawn between $P_{\lceil \frac{k}{2} \rceil - 1}$ and $P_{\lceil \frac{k}{2} \rceil}$,

(2) The orientation of each of $C_2, C_4, \ldots, C_{l_1}$ and C_{m-1} is defined clockwise by indices of vertices from small to large, where l_1 is the largest even number which is less than $m-1$,

(3) The orientation of each of $C_1, C_3, \cdots, C_{l_2}$ and C_0 is defined anticlockwise by indices of vertices from small to large, where l_2 is the largest odd number which is less than $m-1$.

For $i = 1, 2, \ldots, k-1$, $v_{im-1}v_{im}$ is drawn between P_{i-1} and P_i. For $j = 0, 1, \ldots, km-1$, v_j joins to u_j such that any two of $v_1u_1, v_2u_2, \cdots, v_{km-1}v_{km-1}$ do not cross each other. At last, v_0v_{km-1} is drawn such that it does not intersect any other edge.

Thus a drawing of $P(km, m)$ in the plane is completed, which is denoted by $Dr(P(km, m))$. For example, $Dr(P(25, 5))$ is shown in Figure 1.

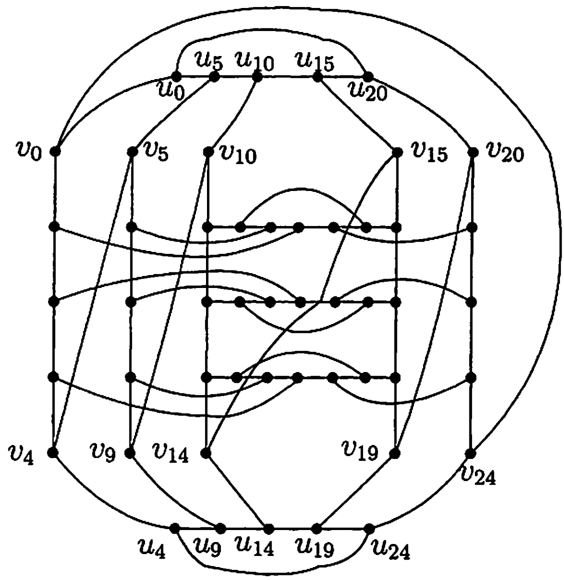

Figure 1 A drawing of $P(25, 5)$ in the plane

We will construct an embedding of $P(km, m)$ in a surface from $Dr(P(km, m))$ by adding tubes to the sphere. By *adding a tube to a surface* S, we mean that we cut two holes D_1 and D_2 in S, respectively, and orient the boundary of each hole, then we attach a tube T to S in such a way that the rim of one of the ends of T coincides with the boundary of D_1 and the rim the other end of T coincides with the boundary of D_2.

Lemma 2.1 $\gamma(P(3m, m)) \leq \lfloor \frac{m-1}{2} \rfloor$.

Proof We now construct an embedding of $P(3m, m)$ in the surface of genus $\lfloor \frac{m-1}{2} \rfloor$ from $Dr(P(3m, m))$ by adding tubes to the sphere. If $m \equiv 0 \pmod 2$, then the tube T_1 is added to the sphere such that it strides over the edge $v_{m+1}v_{m+2}$ satisfying the condition that its one end nears v_{m+1} and another nears

v_{m+2}. Next, $v_{m+1}v_{m+2}$ is drawn in T_1. For $j = 2, \ldots, \frac{m-2}{2}$, we add the tube T_j to the present surface such that its two ends are situated between P_0 and P_1 satisfying the condition that its one end nears v_{m+2j-1} and another nears v_{m+2j}. Then $v_{m+2j-1}v_{m+2j}$ is drawn in T_j. At last, both edges $v_{m-1}v_m$ and $v_{2m-1}v_{2m}$ are drawn through $T_1, T_2, \ldots, T_{\frac{m-2}{2}}$. Thus we get an embedding of $P(3m, m)$ in the surface of genus $\frac{m-2}{2}$(which is equal to $\lfloor \frac{m-1}{2} \rfloor$). For example, an embedding of $P(12, 4)$ in the torus is constructed as in Figure 2.

We observe that if all vertices in the cycle C_{m-2} are deleted from $P(3(m+1), m+1)$ and $v_{m-1}v_{m-2}$, $v_{2m-1}v_{2m-2}$ and $v_{3m-1}v_{3m-2}$ are contracted into a vertex, respectively, then the obtained graph is isomorphic to $P(3m, m)$. So $\gamma(P(3m, m)) \leq \gamma(P(3(m+1), m+1)) \leq \frac{m-1}{2} = \lfloor \frac{m-1}{2} \rfloor$ if $m \equiv 1 \pmod 2$. Thus we complete the proof. □

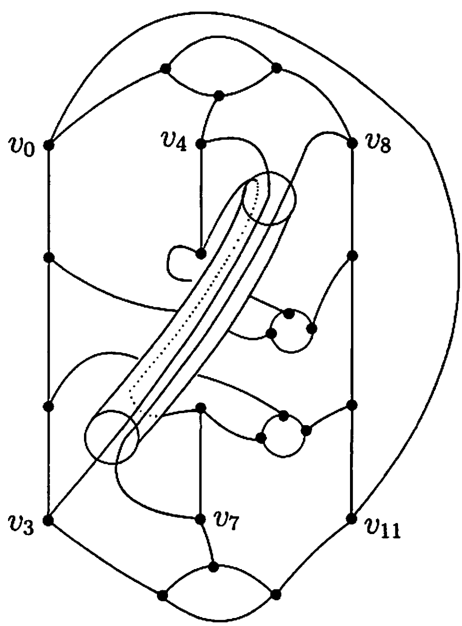

Figure 2 A construction of an embedding of $P(12, 4)$ in the torus

Lemma 2.2 *If $k \geq 4$ and $m > k$, then*

$$\gamma(P(km,m)) \leq \lceil \frac{m-2}{2} \rceil \lceil \frac{k-2}{2} \rceil + \begin{cases} \frac{k-2}{2}, & \text{if } k \equiv 0 \pmod 2, \\ \frac{k-3}{2}, & \text{if } k \equiv 1 \pmod 2. \end{cases}$$

Proof We now construct an embedding of $P(km,m)$ in the desired surface from $Dr(P(km,m))$. If $k \geq 5$, for $i = 1,2,\ldots,\lceil\frac{m-2}{2}\rceil - 1$ and $j = 1,2,\ldots,\lceil\frac{k-2}{2}\rceil - 1$, we add the tube $T_{i,j}$ to the sphere such that its two ends are situated between P_{2j-1} and P_{2j} satisfying the condition that its one end nears $v_{(2j-1)m+(2i-1)}$ and another nears $v_{(2j-1)m+2i}$. Next, both edges $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ and $v_{2jm+(2i-1)}v_{2jm+2i}$ are drawn in $T_{i,j}$. If $k = 4$, there is nothing to do. We now consider two cases.

Case 1 $k \equiv 0 \pmod 2$. For $i = 1,2,\ldots,\lceil\frac{m-2}{2}\rceil - 1$, we add the tube $T_{i,\frac{k-2}{2}}$ to the present surface such that its two ends are situated between P_{k-3} and P_{k-2} satisfying the condition that one end nears $v_{(k-3)m+(2i-1)}$ and another nears $v_{(k-3)m+2i}$. Then both edges $v_{(k-3)m+(2i-1)}v_{(k-3)m+2i}$ and $v_{(k-2)m+(2i-1)}v_{(k-2)m+2i}$ are drawn in it. Next, for $j = 1,2,\ldots,\frac{k-2}{2}$, we add the tube $T_{\lceil\frac{m-2}{2}\rceil,j}$ to the present surface such that it strides over the edge $v_{2jm-1}v_{2jm-2}$ if $m \equiv 1 \pmod 2$ or the edge $v_{2jm-2}v_{2jm-3}$ if $m \equiv 0 \pmod 2$. Then $v_{2jm-1}v_{2jm-2}$ is drawn in $T_{\lceil\frac{m-2}{2}\rceil,j}$ if $m \equiv 1 \pmod 2$, or $v_{2jm-2}v_{2jm-3}$ is drawn in $T_{\lceil\frac{m-2}{2}\rceil,j}$ if $m \equiv 0 \pmod 2$. For $j = 1,2,\cdots,\frac{k-2}{2}$, both edges $v_{(2j-1)m-1}v_{(2j-1)m}$ and $v_{2jm-1}v_{2jm}$ are drawn through $T_{1,j}, T_{2,j},\ldots,T_{\lceil\frac{m-2}{2}\rceil,j}$.

We now add the tube $T'_{\lceil\frac{m-2}{2}\rceil,\frac{k-2}{2}}$ to the present surface such that it strides over the edge $v_{(k-1)m-1}v_{(k-1)m-2}$ if $m \equiv 1 \pmod 2$, or the edge $v_{(k-1)m-2}v_{(k-1)m-3}$ if $m \equiv 0 \pmod 2$. Then $v_{(k-1)m-1}v_{(k-1)m-2}$ is drawn in $T'_{\lceil\frac{m-2}{2}\rceil,\frac{k-2}{2}}$ if $m \equiv 1 \pmod 2$, or $v_{(k-1)m-2}v_{(k-1)m-3}$ drawn in $T'_{\lceil\frac{m-2}{2}\rceil,\frac{k-2}{2}}$ if $m \equiv 0 \pmod 2$. Next, $v_{(k-1)m-1}v_{(k-1)m}$ is newly drawn such that it parallels P_{k-1}, then it travels under $T'_{\lceil\frac{m-2}{2}\rceil,\frac{k-2}{2}}$. If $k = 4$, then we complete the desired embedding. Otherwise, for $j = 1,2,\ldots,\frac{k-4}{2}$, we add the tube $T'_{\lceil\frac{m-2}{2}\rceil,j}$ to the present surface, such that it strides over the edge $v_{(2j+1)m-1}v_{(2j+1)m-2}$ if $m \equiv 1 \pmod 2$, or

$v_{(2j+1)m-2}v_{(2j+1)m-3}$ if $m \equiv 0 \pmod 2$. Then $v_{(2j+1)m-1}v_{(2j+1)m-2}$ is drawn in $T'_{\lceil \frac{m-2}{2} \rceil, j}$ if $m \equiv 1 \pmod 2$, or $v_{(2j+1)m-2}v_{(2j+1)m-3}$ is drawn in $T'_{\lceil \frac{m-2}{2} \rceil, j}$ if $m \equiv 0 \pmod 2$. Thus we obtain an embedding of $P(km, m)$ in the surface of genus $(\lceil \frac{m-2}{2} \rceil - 1)(\lceil \frac{k-2}{2} \rceil - 1) + (\lceil \frac{m-2}{2} \rceil - 1) + \lceil \frac{k-2}{2} \rceil + \frac{k-2}{2} (= \lceil \frac{m-2}{2} \rceil \lceil \frac{k-2}{2} \rceil + \frac{k-2}{2})$.

Case 2 $k \equiv 1 \pmod 2$. For $i = 1, 2, \ldots, \lceil \frac{m-2}{2} \rceil - 1$, we add the tube $T_{i, \frac{k-1}{2}}$ to the present surface such that its two ends are situated between P_{k-2} and P_{k-1} satisfying the condition that one end nears $v_{(k-2)m+(2i-1)}$ and another nears $v_{(k-2)m+2i}$. Then the edge $v_{(k-2)m+(2i-1)}v_{(k-2)m+2i}$ is drawn in it. Next, the tube $T_{\lceil \frac{m-2}{2} \rceil, \frac{k-1}{2}}$ is added to the present surface such that it strides over the edge $v_{(k-1)m-3}v_{(k-1)m-2}$ if $m \equiv 0 \pmod 2$, or the edge $v_{(k-1)m-2}v_{(k-1)m-1}$ if $m \equiv 1 \pmod 2$. Then $v_{(k-1)m-3}v_{(k-1)m-2}$ is drawn in $T_{\lceil \frac{m-2}{2} \rceil, \frac{k-1}{2}}$ if $m \equiv 0 \pmod 2$, or $v_{(k-1)m-2}v_{(k-1)m-1}$ drawn in $T_{\lceil \frac{m-2}{2} \rceil, \frac{k-1}{2}}$ if $m \equiv 1 \pmod 2$. For $j = 1, 2, \ldots, \frac{k-3}{2}$, we add the tube $T_{\lceil \frac{m-2}{2} \rceil, j}$ to the present surface such that its two ends are situated between P_{2j-1} and P_{2j} satisfying the condition that one end nears v_{2jm-3} and another nears v_{2jm-2} if $m \equiv 0 \pmod 2$, or one end nears v_{2jm-2} and another nears v_{2jm-1} if $m \equiv 1 \pmod 2$. For $j = 1, 2, \ldots, \frac{k-3}{2}$, $v_{2jm-3}v_{2jm-2}$ and $v_{(2j+1)m-3}v_{(2j+1)m-2}$ are drawn in $T_{\lceil \frac{m-2}{2} \rceil, j}$ if $m \equiv 0 \pmod 2$, or $v_{2jm-2}v_{2jm-1}$ and $v_{(2j+1)m-2}v_{(2j+1)m-1}$ are drawn in $T_{\lceil \frac{m-2}{2} \rceil, j}$ if $m \equiv 1 \pmod 2$.

Next, for $j = 1, 2, \ldots, \lceil \frac{k-2}{2} \rceil - 1$, the edge $v_{2jm-1}v_{2jm}$ is drawn through $T_{1,j}, T_{2,j}, \ldots, T_{\lceil \frac{m-2}{2} \rceil, j}$. Both edges $v_{(k-2)m-1}v_{(k-2)m}$ and $v_{(k-1)m-1}v_{(k-1)m}$ are drawn through $T_{1, \frac{k-1}{2}}, \ldots, T_{\lceil \frac{m-2}{2} \rceil, \frac{k-1}{2}}$. At last, for $t = 1, 2, \ldots, \frac{k-3}{2}$, we add the tube T'_t to the present surface such that its two ends situate between P_{2t-1} and P_{2t} such that one end nears $v_{(2t-1)m-1}$ and another nears $v_{(2t-1)m}$. Then the edge $v_{(2t-1)m-1}v_{(2t-1)m}$ is drawn in T'_t. Thus we get an embedding of $P(km, m)$ in the surface of genus $(\lceil \frac{m-2}{2} \rceil - 1)(\lceil \frac{k-2}{2} \rceil - 1) + (\lceil \frac{m-2}{2} \rceil - 1) + \frac{k-1}{2} + \frac{k-3}{2} (= \lceil \frac{m-2}{2} \rceil \lceil \frac{k-2}{2} \rceil + \frac{k-3}{2})$.□

Lemma 2.3 *If* $5 \le m \le k$, *then* $\gamma(P(km, m)) \le \lceil \frac{m-2}{2} \rceil \lceil \frac{k-1}{2} \rceil + \lceil \frac{k-m}{2} \rceil$.

Proof We will also construct an embedding of $P(km, m)$ in the desired surface from $Dr(P(km, m))$. We consider two cases.

Case 1 $k \equiv 1 \pmod 2$. There are two cases to be considered.

Subcase 1.1 $m \equiv 0 \pmod 2$. For $i = 1, 2, \ldots, \frac{m-2}{2}$ and $j = 1, 2, \ldots, \frac{m-2}{2}$, we add the tube $T_{i,j}$ to the sphere by the following rules.

(i) If $i + j < \frac{m}{2}$, then two ends of $T_{i,j}$ are situated between P_{2j-1} and P_{2j} such that one end nears $v_{(2j-1)m+(2i-1)}$ and another nears $v_{(2j-1)m+2i}$. Next, $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ and $v_{2jm+(2i-1)}v_{2jm+2i}$ are drawn in $T_{i,j}$.

(ii) If $i+j = \frac{m}{2}$, then $T_{i,j}$ strides over $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ such that one end nears $v_{(2j-1)m+(2i-1)}$ and another nears $v_{(2j-1)m+2i}$. Next, $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ is drawn in it.

(iii) If $i + j > \frac{m}{2}$, then two ends of $T_{i,j}$ are situated between P_{2j-2} and P_{2j-1} such that one end nears $v_{(2j-2)m+(2i-1)}$ and another nears $v_{(2j-2)m+2i}$. Next, $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ and $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ are drawn in $T_{i,j}$.

For $j = 1, 2, \ldots, \frac{m-2}{2}$, both edges $v_{(2j-1)m-1}v_{(2j-1)m}$ and $v_{2jm-1}v_{2jm}$ are drawn through tubes $T_{1,j}, T_{2,j}, \ldots, T_{\frac{m-2}{2},j}$.

For $i = 1, 2, \ldots, \frac{m-2}{2}$ and $j = \frac{m}{2}, \frac{m}{2} + 1, \ldots, \frac{k-1}{2}$, we add the tube $T_{i,j}$ to the present surface such that its ends are situated between P_{2j-2} and P_{2j-1} satisfying the condition that its one end nears $v_{(2j-2)m+(2i-1)}$ and another nears $v_{(2j-2)m+2i}$. Next, $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ and $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ are drawn on $T_{i,j}$. For $i = 1, 2, \ldots, \frac{m-2}{2}$, $v_{(2j-1)m-1}v_{(2j-1)m}$ is drawn through tubes $T_{1,j}, \ldots, T_{\frac{m-2}{2},j}$.

For $j = \frac{m+2}{2}, \frac{m+2}{2} + 1, \ldots, \frac{k+1}{2}$, we add the tube T'_j between P_{2j-3} and P_{2j-2} such that its one end nears $v_{(2j-2)m}$ and another nears $v_{(2j-2)m-1}$. Next, $v_{(2j-2)m}v_{(2j-2)m-1}$ is drawn on T'_j.

Thus we eventually obtain an embedding of $P(km, m)$ in the surface of genus $\frac{(m-2)(m-2)}{4} + \frac{(m-2)(k-m+1)}{4} + \frac{k-m+1}{2}$ $(= \lceil \frac{m-2}{2} \rceil \lceil \frac{k-1}{2} \rceil + \lceil \frac{k-m}{2} \rceil)$.

Subcase 1.2 $m \equiv 1 \pmod 2$. If $m < k$, then $m + 1 \leq k$ and $m + 1 \equiv 0 \pmod 2$. Since $P(km, m)$ is isomorphic to a minor of $P(k(m+1), m+1)$, $\gamma(P(km,m)) \le \gamma(P(k(m+1), m+1)) \le \frac{(m-1)(k-1)}{4} + \frac{k-m}{2}$. If $m = k$, we can construct an embedding of $P(km, m)$ in the surface of genus $\frac{(m-1)(k-1)}{4}$ using the similar method to that in the former four paragraphs in subcase 1.1. The differences are that m is replaced by $m+1$ and that if $i = \frac{m-1}{2}$ then there is only one edge which is drawn on the tube. For example, the way of adding tubes to construct an embedding of $P(25, 5)$ in the surface S_4 is shown in Figure 3.

Therefore, $\gamma(P(km,m)) \le \lceil \frac{m-2}{2} \rceil \lceil \frac{k-1}{2} \rceil + \lceil \frac{k-m}{2} \rceil$ if $k \equiv 1 \pmod 2$.

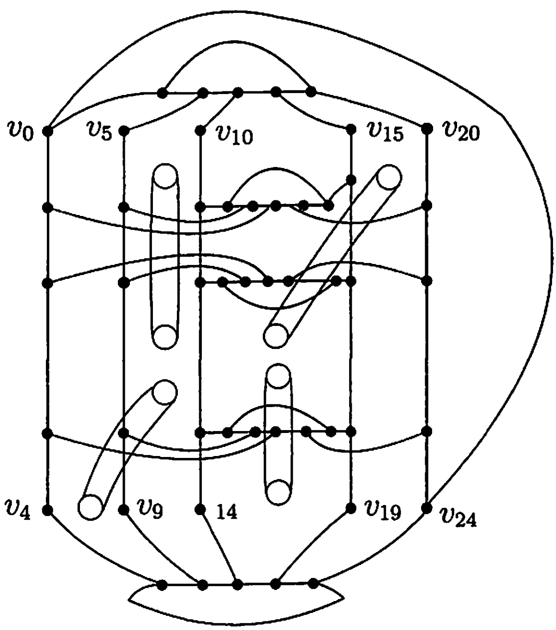

Figure 3 The way of adding tubes to form an embedding of $P(25, 5)$

Case 2 $k \equiv 0 \pmod 2$. We consider two cases.

Subcase 2.1 $m \equiv 0 \pmod 2$.

For $i = 1, 2, \ldots, \frac{m-2}{2}$, and $j = 1, 2, \ldots, \frac{m}{2}$, we add the tube $T_{i,j}$ to the sphere by the following rules.

(i) If $i + j < \frac{m+2}{2}$, then two ends of $T_{i,j}$ are situated between P_{2j-2} and P_{2j-1} such that one end nears $v_{(2j-2)m+(2i-1)}$ and another nears $v_{(2j-2)m+2i}$. Next, $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ and $v_{(2j-1)m+(2i-1)}v_{(2j-1)m+2i}$ are drawn in $T_{i,j}$.

(ii) If $i+j=\frac{m+2}{2}$, then the tube $T_{i,j}$ strides over the edge $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ such that its one end nears $v_{(2j-2)m+(2i-1)}$ and another nears $v_{(2j-2)m+2i}$. Next, $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ is drawn in it.

(iii) If $i+j>\frac{m+2}{2}$, then two ends of $T_{i,j}$ are situated between P_{2j-3} and P_{2j-2} such that one end nears $v_{(2j-3)m+(2i-1)}$ and another nears $v_{(2j-3)m+2i}$. Next, $v_{(2j-3)m+(2i-1)}v_{(2j-3)m+2i}$ and $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ are drawn in $T_{i,j}$.

For $j=2,3,\ldots,\frac{m}{2}$, $v_{(2j-2)m-1}v_{(2j-2)m}$ and $v_{(2j-1)m-1}v_{(2j-1)m}$ are drawn through tubes $T_{1,j},T_{2,j},\ldots,T_{\frac{m-2}{2},j}$. Also, the edge $v_{m-1}v_m$ is drawn through tubes $T_{1,1},T_{2,1},\ldots,T_{\frac{m-2}{2},1}$.

For $i=1,2,\ldots,\frac{m-2}{2}$ and $j=\frac{m+2}{2},\frac{m+2}{2}+1,\ldots,\frac{k}{2}$, we add the tube $T_{i,j}$ to the present surface such that its ends are situated between P_{2j-3} and P_{2j-2} satisfying the condition that one end nears $v_{(2j-3)m+(2i-1)}$ and another nears $v_{(2j-3)m+2i}$. Next, both edges $v_{(2j-3)m+(2i-1)}v_{(2j-3)m+2i}$ and $v_{(2j-2)m+(2i-1)}v_{(2j-2)m+2i}$ are drawn in $T_{i,j}$. For $j=\frac{m+2}{2},\frac{m+2}{2}+1,\ldots,\frac{k}{2}$, $v_{(2j-2)m-1}v_{(2j-2)m}$ is drawn through tubes $T_{1,j},\ldots,T_{\frac{m-2}{2},j}$.

For $j=\frac{m+2}{2},\frac{m+2}{2}+1,\ldots,\frac{k}{2}$, we add the tube T'_j between P_{2j-2} and P_{2j-1} such that its one end nears $v_{(2j-1)m}$ and another nears $v_{(2j-1)m-1}$. Next, $v_{(2j-1)m}v_{(2j-1)m-1}$ is drawn in T'_j.

Thus we eventually obtain an embedding of $P(km,m)$ in the surface of genus $\frac{m(m-2)}{4}+\frac{(m-2)(k-m)}{4}+\frac{k-m}{2}(=\lceil\frac{m-2}{2}\rceil\lceil\frac{k-1}{2}\rceil+\lceil\frac{k-m}{2}\rceil)$.

Subcase 2.2 $m\equiv 1 \pmod 2$. If $m<k$, then $m+1\leq k$ and $m+1\equiv 0 \pmod 2$. Since $P(km,m)$ is isomorphic to a minor of $P(k(m+1),m+1)$, $\gamma(P(km,m))\leq\gamma(P(k(m+1),m+1))\leq\frac{(m-1)(k)}{4}+\frac{k-m-1}{2}(=\lceil\frac{m-2}{2}\rceil\lceil\frac{k-1}{2}\rceil+\lceil\frac{k-m}{2}\rceil)$.

Therefore, $\gamma(P(km,m))\leq\lceil\frac{m-2}{2}\rceil\lceil\frac{k-1}{2}\rceil+\lceil\frac{k-m}{2}\rceil$ if $k\equiv 0 \pmod 2$. □

Lemma 2.4 *If $k \geq 4$, then*

$$\gamma(P(4k,4)) \leq \begin{cases} \frac{2k-3}{3}, & \text{if } k \equiv 0 \pmod 3, \\ \frac{2k-2}{3}, & \text{if } k \equiv 1 \pmod 3, \\ \frac{2k-1}{3}, & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Proof We will also construct an embedding of $P(4k, 4)$ in the desired surface from $Dr(P(4k, k))$ by adding tubes to the sphere. Suppose that $k = 3t + s$, where $t \geq 1$ and $s = 0, 1$, or 2.

For $i = 1, 2, \ldots, t$, we add the tube T_i to the sphere such that it strides over the edge $v_{4(3i-2)+1}v_{4(3i-2)+2}$. Next, $v_{4(3i-2)-1}v_{4(3i-2)}$, $v_{4(3i-2)+1}v_{4(3i-2)+2}$ and $v_{4(3i-2)+3}v_{4(3i-2)+4}$ are drawn in it.

For $j = 1, 2, \ldots, t-1$, we add the tube T'_j to the present surface such that its one end nears $v_{4(3i-1)+1}$ and another nears $v_{4(3i-1)+2}$. Next, $v_{4(3i-1)+1}v_{4(3i-1)+2}$, $v_{4(3i-1)+3}v_{4(3i-1)+4}$ and $v_{4(3i-1)+5}v_{4(3i-1)+6}$ are drawn in it.

If $s = 0$, then we have obtained an embedding of $P(4k, 4)$ in the surface of genus $2t - 1$ (i.e., $\frac{2k-3}{3}$).

If $s = 1$, then we add a tube T" to the present surface such that its one end nears v_{12t-3} and another nears v_{12t-2}. Next, $v_{12t-3}v_{12t-2}$ and $v_{12t-1}v_{12t}$ are drawn in it. Then we obtain an embedding of $P(4k, 4)$ in the surface of genus $2t$ (i.e., $\frac{2k-2}{3}$).

If $s = 2$, then we add two tubes $T"_1$ and $T"_2$ to the present surface such that one end of $T"_1$ nears v_{12t-3} and another nears v_{12t-2} and one end of $T"_2$ nears v_{12t+1} and another nears v_{12t+2}. Next, $v_{12t-3}v_{12t-2}$ and $v_{12t-1}v_{12t}$ are drawn in $T"_1$, and $v_{12t+1}v_{12t+2}$ and $v_{12t+3}v_{12t+4}$ are drawn in $T"_2$. Then we obtain an embedding of $P(4k, 4)$ in the surface of genus $2t + 1$ (i.e., $\frac{2k-1}{3}$). □

Since $P(3k, 3)$ is a minor of $P(4k, 4)$, we have the following result.

Lemma 2.5 *If $k \geq 4$, then*

$$\gamma(P(3k,3)) \leq \begin{cases} \frac{2k-3}{3}, & \text{if } k \equiv 0 \pmod 3, \\ \frac{2k-2}{3}, & \text{if } k \equiv 1 \pmod 3, \\ \frac{2k-1}{3}, & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

3 A lower bound of the orientable genus of $P(km, m)$

Let $H(km, m)$ be the graph obtained from $P(km, m)$ by the cycle C_i being contracted into a vertex z_i for $i = 0, 1, \cdots, k-1$. We call z_i a *singular vertex*. Any edge incident with a singular vertex is called a *spoke*. It is obvious that the principal cycle of $P(km, m)$ is reserved in $H(km, m)$, which is still called the *principal cycle*. Since $H(km, m)$ is a minor of $P(km, m)$, $\gamma(P(km, m)) \geq \gamma(H(km, m))$. We now consider a lower bound of the orientable genus of $H(km, m)$.

Lemma 3.1 *Suppose Π is a 2-cell embedding of $H(km, m)$ in some surface. If a facial walk in Π contains a spoke, then it contains even spokes. Moreover, if a facial walk in Π contains $2t$ spokes, then it contains at least t edges in the principal cycle of $H(km, m)$.*

Proof Suppose that W is a facial walk which contains a spoke. We observe that any two singular vertices in W are not adjacent to each other, and that each appearance of any singular vertex in W must correspond to two spokes. So W has even spokes, say $2t$ spokes. Since any edge in the principal cycle is incident with at most two spokes, W has at least t edges in the principal cycle. □

Lemma 3.2 *Suppose Π is a 2-cell embedding of $H(km, m)$ in some surface. Let a_2 be the number of all facial walks in which each has exactly two spokes. Then $a_2 \leq \frac{km}{m-1}$.*

Proof Suppose W is a facial walk containing exactly two spokes. Then the two spokes must be incident with each other. So the induced subgraph by all edges in W which are in the principal cycle is a path. Let P be the path. Obviously, the length of P is less than km. Since two ends of P are adjacent the same singular vertex, the number of all edges in P is a multiple of m, say jm by the definition of $H(km, m)$. Clearly, $1 \leq j \leq k-1$. Hence, W has $jm + 2$ edges.

We claim that there are at least $jm - 2$ edges of W such that each can not appear in any other facial walk which has exactly two spokes. In fact, suppose that $P = v_{i_1}v_{i_2}\dots v_{i_{jm+1}}$. Then each edge in $v_{i_2}\cdots v_{i_{jm}}$ can not be in any other facial walk which contains exactly two spokes. Otherwise, there is a spoke which is incident with one of $v_{i_2}, v_{i_{jm}}$ such that it crosses some edge in Π, a contradiction.

For $j = 1, 2, \dots, k-1$, let each facial walk with length $jm+2$ that contains exactly two spokes correspond to $jm + (jm-2)$ edges in the principal cycle. For $j = 1, 2, \dots, k-1$, let b_j be the number of facial walks of Π in which each has exactly two spokes and has length $jm + 2$. Since any edge of the principal cycle may be in two facial walks or appears twice in the same facial walk, we have that $\sum_{j=1}^{k-1}[jm + (jm-2)]b_j \leq 2km$, i.e., $\sum_{j=1}^{k-1}(jm-1)b_j \leq km$. Since $\sum_{j=1}^{k-1}(jm-1)b_j \geq (m-1)\sum_{j=1}^{k-1}b_j$ and $\sum_{j=1}^{k-1}b_j = a_2$, we have that $(m-1)a_2 \leq km$, i.e.,$a_2 \leq \frac{km}{m-1}$.

□

Lemma 3.3 *Suppose that $m \geq 4$. Suppose Π is a 2-cell embedding of $H(km, m)$ in some surface. Let r_i be the number of facial walks with length i in Π. Let*

$$\Phi = [(km-6)r_6 + (km-7)r_7 + \dots + (km-m-1)r_{m+1}]$$
$$+(km-m-2)r_{m+2} + \dots + (m-2)r_{(k-1)m+2}$$
$$+[(m-3)r_{(k-1)m+3} + \dots + r_{km-1}],$$

Then

$$\Phi \leq \tfrac{1}{2}(km)^2 - 3km + \tfrac{km}{2m-2}(km-2m+2).$$

Proof Since $m \geq 4$, we observe that the length of any facial walk in Π is at least six. Also, we observe that if a facial walk in Π does not contain any spoke, then it must be the principal cycle. So a facial walk with length at most $km-1$ must contains even spokes by Lemma 3.1. Moreover, a facial walk containing $2t$ spokes has length at least $3t$ by Lemma 3.1. Suppose $km - 1 = 3q + s$, where $s = 0, 1$ or 2. So r_i can be written the sum of $r'_{j,2}, \dots, r'_{j,2q}$, where $r'_{j,2k}$ is the number of facial walks in which each has length j and contains exactly $2k$ spokes. By the proof of Lemma 3.2, we have known that a facial walk containing exactly two spokes has length $m+2$. So we have that

$$\Phi \leq (km-m-2)(r'_{m+2,2}+\cdots)+(km-6)(r'_{6,4}+\cdots)+$$
$$(km-9)(r'_{9,6}+\cdots)+\cdots+(km-3q)r'_{3q,2q}. \qquad (1)$$

For $i=1,2,\ldots,k-1$, let a_{2i} be the number of facial walks in which each contains exactly $2i$ spokes. We have that

$$\Phi \leq (km-m-2)a_2+(km-6)a_4+(km-9)a_6+\cdots+(km-3q)a_{2q}. \qquad (2)$$

Since each spoke may appears at most two times in a facial walk in Π, we have the following formula. For $i=1,2,\ldots,q$,

$$a_{2i} \leq \frac{1}{2i}[2km-2a_2-4a_4-\cdots-(2i-2)a_{2i-2}]$$
$$=\frac{1}{i}[km-a_2-2a_4-\cdots-(i-1)a_{2i-2}]. \qquad (3)$$

Now a_{2q} is substituted by formula (3). Then we obtain that

$$\Phi \leq \frac{1}{q}km(km-3q)+(km-\frac{km}{q}-m+1)a_2+(km-\frac{2km}{q})a_4+\cdots$$
$$+[km-\frac{(q-1)km}{q}]a_{2q-2}$$
$$=\frac{k^2m^2}{q}-3km-(\frac{q-1}{q}km-m+1)a_2+\frac{q-2}{q}kma_4+\cdots+\frac{1}{q}kma_{2q-2}. \qquad (4)$$

Next, a_{2q-2} is substituted by formula (3). Then we obtain that

$$\Phi \leq [\frac{1}{q}+\frac{1}{q(q-1)}]k^2m^2-3km-[(\frac{q-1}{q}-\frac{1}{q(q-1)})km-m+1]a_2+$$
$$[\frac{q-2}{q}-\frac{2}{q(q-1)}]kma_4+\cdots+[\frac{2}{q}-\frac{q-2}{q(q-1)}]kma_{2q-4}$$
$$=[\frac{1}{q}+\frac{1}{q(q-1)}]k^2m^2-3km-[(1-\frac{1}{q-1})km-m+1]a_2+$$

$$(1-\frac{2}{q-1})kma_4+\cdots+(1-\frac{q-2}{q-1})kma_{2q-4}. \quad (5)$$

Next, a_{2q-4} is substituted by formula (3), then $a_{2q-6},\ldots,a_4$ one by one. By similar argument to the above, we obtain that

$$\Phi \leq [\frac{1}{q}+\frac{1}{q(q-1)}+\cdots+\frac{1}{3\cdot 2}]k^2m^2-3km+[(1-\frac{1}{2})km-m+1]a_2$$

$$=(\frac{1}{q}+\frac{1}{q-1}-\frac{1}{q}+\cdots+\frac{1}{2}-\frac{1}{3})k^2m^2-3km-(\frac{1}{2}km-m+1)a_2$$

$$=\frac{1}{2}k^2m^2-3km+(\frac{1}{2}km-m+1)a_2. \quad (6)$$

By Lemma 3.2, $a_2 \leq \frac{km}{m-1}$. We have that

$$\Phi \leq \frac{1}{2}k^2m^2-3km+\frac{km}{2m-2}(km-2m+2). \quad (7)$$

□

Theorem 3.4 *Suppose that* $m \geq 4$. *Then* $\gamma(P(km,m)) \geq \gamma(H(km,m)) \geq \frac{km}{4}-\frac{m}{2}-\frac{km}{4m-4}+1.$

Proof It is obvious that $\gamma(P(km,m)) \geq \gamma(H(km,m))$. Suppose that $\gamma(H(km,m)) = g$. Let Π be an embedding of $H(km,m)$ in the surface S_g. Then it is a 2-cell embedding. Let r_i be the number of facial walks with length i in Π. Since $m \geq 4$, we observe that the length of any facial walk is at leat six. Let r be the number of faces of Π. Let $|V(H(km,m))| = |V|$ and $|E(H(km,m))| = |E|$. Then $r = r_6+r_7+\ldots$, and $2|E| = 6r_6+7r_7+\ldots$.

Since $kmr = km(r_6+r_7+\ldots) = [(km-6)r_6+(km-7)r_7+\ldots+r_{km-1}]+[6r_6+7r_7+\ldots]$, $r \leq \frac{1}{km}[(km-6)r_6+(km-7)r_7+\ldots+r_{km-1}]+\frac{2|E|}{km}$. By Lemma 3.3 and the fact that $|E| = 2km$, we have that

$$r \leq \frac{1}{2}km+\frac{km-2m+2}{2m-2}+1. \quad (8)$$

By Euler's formula, $|V|-|E|+r = 2-2g$. We have that $g = 1+\frac{1}{2}(|E|-|V|-r)$. By formula (8), we obtain the following formula,

$$g \geq 1 + \frac{|E|}{2} - \frac{|V|}{2} - \frac{km}{4} - \frac{km - 2m + 2}{4m - 2} - \frac{1}{2}. \tag{9}$$

Considering that $|V| = km + m$ and $|E| = 2km$, we have that

$$\begin{aligned} g &\geq 1 + \frac{km}{2} - \frac{m}{2} - \frac{km}{4} - \frac{km - 2m + 2}{4m - 4} - \frac{1}{2} \\ &= \frac{km}{4} - \frac{m}{2} - \frac{km}{4m - 4} + 1. \end{aligned} \tag{10}$$

Theorem 3.5 *If $m \geq 4$, then $\gamma(P(4m, m)) = \lceil \frac{m}{2} \rceil$.*

Proof By Lemma 2.2, $\gamma(P(4m, m)) \leq \frac{m}{2}$ if $m \equiv 0 \pmod 2$ and $m \geq 6$, and $\gamma(P(4m, m)) \leq \frac{m+1}{2}$ if $m \equiv 1 \pmod 2$. By Lemma 2.4, $\gamma(P(16, 4)) \leq 2$.

By Theorem 3.4, $\gamma(P(4m, m)) \geq \lceil \frac{m}{2} + 1 - \frac{m}{m-1} \rceil = \lceil \frac{m}{2} - \frac{1}{m-1} \rceil$. If $m \equiv 0 \pmod 2$, then $\gamma(P(4m, m)) \geq \frac{m}{2}$. If $m \equiv 1 \pmod 2$, let $m = 2t + 1$. Since $m \geq 5$, $\gamma(P(4m, m)) \geq t + 1$, i.e., $\gamma(P(4m, m)) \geq \frac{m+1}{2}$. Thus we complete the proof. □

Theorem 3.6 *If $m \equiv 0 \pmod 2$ and $m \geq 6$, then $\gamma(P(6m, m)) = m$.*

Proof By Lemma 2.2 and 2.3, $\gamma(P(6m, m)) \leq m$. By Theorem 3.4, $\gamma(P(6m, m)) \geq \lceil m + 1 - \frac{3m}{2m-2} \rceil = m + \lceil -\frac{m+2}{2m-2} \rceil$. Since $m \geq 6$, $m + 2 < 2m - 2$. Then $\gamma(P(6m, m)) \geq m$. Hence, $\gamma(P(6m, m)) = m$. □

Theorem 3.7 *If $k \geq 4$, then*

$$\gamma(P(4k, 4)) = \begin{cases} \frac{2k-3}{3}, & \text{if } k \equiv 0 \pmod 3, \\ \frac{2k-2}{3}, & \text{if } k \equiv 1 \pmod 3, \\ \frac{2k-1}{3}, & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Proof By Lemma 3.4, $\gamma(P(4k, 4)) \geq \lceil \frac{2k}{3} - 1 \rceil$. Let $k = 3t + s$, where $s = 0, 1$, or 2. If $s = 0$, then $\gamma(P(4k, 4)) \geq 2t - 1 = \frac{2k-3}{3}$. If $s = 1$, then $\gamma(P(4k, 4)) \geq 2t = \frac{2k-2}{3}$. If $s = 2$, then

$\gamma(P(4k,4)) \geq 2t+1 = \frac{2k-1}{3}$. By lemma 2.4, we obtain the desired result. □

In the end of the section, we now determine orientable genera of several other graphs. By Lemma 2.1 and Lemma 3.4, we can show that $\gamma(P(9,3)) = 1$. Since $P(12,4)$ has a minor isomorphic to $P(9,3)$, we have that $\gamma(P(12,3)) \geq 1$. By Lemma 2.1, $\gamma(P(12,3)) = 1$. By Lemma 2.2 and Lemma 3.4, we can show that $\gamma(P(25,5)) = 4$.

4 The orientable genus of $P(3m, m)$

In Section 2 we have given an upper bound of $\gamma(P(3m,m))$. We need a proper lower bound of $\gamma(P(3m,m))$. Let us begin with a lemma.

Lemma 4.1 *[3] If the blocks of the graph G are $G_1, G_2, \cdots, G_n$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_n)$.*

Suppose that $t \geq 2$, and suppose that $Q_1, Q_2, \cdots, Q_t$ is a sequence of graphs such that Q_i is isomorphic to $K_{3,3}$ with vertex partition $\{x_{i,1}, x_{i,3}, x_{i,5}\} \cup \{x_{i,2}, x_{i,4}, x_{i,6}\}$ for $i = 1, 2, \ldots, t$. For $i = 1, 2, \ldots, t-1$, $x_{i,6}$ is identified with $x_{i+1,1}$. Then the above obtained graph is called a t-chain of $(K_{3,3})'s$.

Considering the orientable genus of $K_{3,3}$ is one, the below result follows from Lemma 4.1.

Theorem 4.2 *The orientable genus of an n-chain of $(K_{3,3})$'s is n.*

We now consider the relation of $P(3m,m)$ and t-chain of $(K_{3,3})'s$.

Lemma 4.3 *If $m \geq 5$ and $m \equiv 1 \pmod 2$, then $P(3m,m)$ has a minor isomorphic to $\frac{m-1}{2}$-chain of $(K_{3,3})$'s.*

Proof Let D_i be the induced subgraph of $P(3m,m)$ by the vertex set $\{v_i, v_{m+i}, v_{2m+i}\} \cup \{u_i, u_{m+i}, u_{2m+i}\}$ for $i = 2, 4, \ldots, m-3$. Then D_i is contracted into a vertex. Next, three edges $v_0v_{3m-1}, v_{m-1}v_m$ and $v_{2m-1}v_{2m}$ are deleted. Thus, there are six vertices each has degree two. For each vertex of degree two, an edge incident with it is contracted. Then we obtain a graph isomorphic to $\frac{m-1}{2}$-chain of $(K_{3,3})$'s. □

Theorem 4.4 $\gamma(P(3m, m)) = \lfloor\frac{m-1}{2}\rfloor$.

Proof Recall that we have shown $\gamma(P(9,3)) = \gamma(P(12,4)) = 1$ in Section 3. Now we consider the case that $m \geq 5$. Since $\gamma(P(3m, m)) \leq \lfloor\frac{m-1}{2}\rfloor$ by Lemma 2.1, it is sufficient to show $\gamma(P(3m, m)) \geq \lfloor\frac{m-1}{2}\rfloor$.

If $m \equiv 1 \pmod 2$, $P(3m, m)$ has a minor isomorphic to $\frac{m-1}{2}$-chain of $(K_{3,3})$'s by Lemma 4.3. So $\gamma(P(3m, m)) \geq \frac{m-1}{2} \geq \lfloor\frac{m-1}{2}\rfloor$ by Theorem 4.2.

If $m \equiv 0 \pmod 2$, then $\lfloor\frac{m-1}{2}\rfloor = \lfloor\frac{m-2}{2}\rfloor$. Since $P(3m, m)$ has a minor isomorphic to $P(3(m-1), m-1)$, $\gamma(P(3m, m)) \geq \gamma(P(3(m-1), m-1)) \geq \lfloor\frac{m-2}{2}\rfloor$ by the above paragraph. Thus, we complete the proof. □

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