Laplacian and signless Laplacian characteristic polynomial of generalized subdivision corona vertex graph*

Pengli Lu[†] and Yumo Wu
School of Computer and Communication
Lanzhou University of Technology
Lanzhou, 730050, Gansu, P.R. China

Abstract

Let G be a graph with n vertices, $\mathfrak{S}(G)$ the subdivision graph of G. $V(G)$ denotes the set of original vertices of G. The *generalized subdivision corona vertex graph* of G and $H_1, H_2, \ldots, H_n$ is the graph obtained from $\mathfrak{S}(G)$ and $H_1, H_2, \ldots, H_n$ by joining the ith vertex of $V(G)$ to every vertex of H_i. In this paper, we determine the Laplacian (respectively, the signless Laplacian) characteristic polynomial of the generalized subdivision corona vertex graph. As an application, we construct infinitely many pairs of cospectral graphs.

Keywords: Generalized subdivision corona vertex graph, Laplacian characteristic polynomial, signless Laplacian characteristic polynomial, Cospectral graphs

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1 Introduction

All graphs considered in this paper are simple. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The *adjacent matrix* of G is denoted by $A(G)$. The *incident matrix* of G, denoted by $R(G)$, is the $n \times m$ matrix, whose (i, j)−entry is

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[†]Corresponding author. E-mail addresses: lupengli88@163.com (P. Lu), wuyumoxdf@126.com (Y. Wu).

1 if v_i and e_j are adjacent in G and 0 otherwise. The *degree matrix* of G is a diagonal matrix with diagonal entries $d_1, \dots, d_n$, where $d_i = d_G(v_i)$ is the degree of v_i in G.

The *Laplacian matrix* and *signless Laplacian matrix* are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. It's easy to know that $R(G)R(G)^T = A(G) + D(G) = Q(G)$. The characteristic polynomial of an $n \times n$ matrix M is defined as $f_M(x) = \det(xI_n - M)$, where I_n is the identity matrix of size n. In particular, for a graph G, we call $f_{L(G)}(\mu)$ (respectively $f_{Q(G)}(\nu)$)) the *Laplacian* (respectively, *signless Laplacian*) *characteristic polynomial* of G, and its roots the *Laplacian* (respectively, *signless Laplacian*) *eigenvalues* of G. The Laplacian and signless Laplacian eigenvalues of G are denoted as $0 = \mu_1(G) \leqslant \mu_2(G) \leqslant \cdots \leqslant \mu_n(G)$ and $\nu_1(G) \leqslant \nu_2(G) \leqslant \cdots \leqslant \nu_n(G)$ respectively. The Laplacian eigenvalues with their multiplicities are called the $L-$*spectrum* of G. Graphs with the same $L-$spectrum are called $L-$*cospectral graphs*. Similar terminology will be used for $Q(G)$. It is well known [4] that the *subdivision graph* $\mathfrak{S}(G)$ of G is the graph obtained by inserting a new vertex into every edge of G. We denote the set of such new vertices by $I(G)$, and the original ones by $V(G)$. For more review, readers may refer to [1,3,4].

The *corona* of two graphs G and H [5], is the graph obtained by one copy of G and $|V(G)|$ copies of H, all vertices disjoint, and joining the ith vertex of G to every vertex in the ith copy of H. The (usual) corona $G \circ H$ of graphs G and H may be regarded as a specific case of the *rooted product* of graphs G and H^*, where H^* has the root r which is a dominating vertex (a cone point) adjacent to all other points of H^*, and $H^* - r = H$. In other terms, $H^* := \{r\} \circ H \cong K_1 \circ H$, wherein the root vertex r is associated graph K_1. After that many new graph operations based on corona and subdivision graph such as the *edge corona*, the *neighborhood corona*, and the *subdivision-vertex* and *subdivision-edge neighborhood corona* have been introduced and their spectra are computed in [6–9], but none of them has been expended their $|V(G)|$ or $|I(G)|$ copies of H to arbitrary graphs. In this paper we defined a new graph operation. The so-called *subdivision-vertex corona* of G and H is the graph obtained from $\mathfrak{S}(G)$ and $|V(G)|$ copies of H, all vertices disjoint, and joinging the ith vertex of $V(G)$ to every vertex in the ith copy of H. We defined the *generalized subdivision corona vertex graph* by extending the $|V(G)|$ copies of H to arbitrary graphs H_i, for $i = 1, \dots, |V(G)|$, so as to spread research to broader areas on chemistry, physics and computer science.

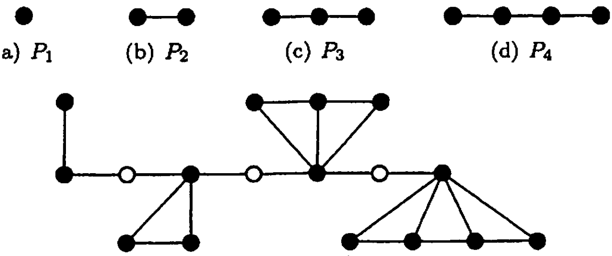


Fig. 1: An example of generalized subdivision corona vertex graph.

The rest of our paper is organized as follows. In Section 2, we give the definition of our new graph operation and several useful tools to obtain our results. The adjacent and degree matrix of generalized subdivision corona vertex graph are showed in Section 3 calculation. Section 4 determines the Laplacian characteristic polynomial of the generalized subdivision corona vertex graph. This result enables us to construct infinitely many pairs of $L-$cospectral graphs. In Section 5 we compute the signless Laplacian characteristic polynomial of the generalized subdivision corona vertex graph and also construct infinitely many pairs of $Q-$cospectral graphs.

2 Preliminaries

Definition 2.1. *Let G be a graph with n vertices and $H_1, \ldots, H_n$ n arbitrary graphs which are not necessarily nonisomorphic with one another. The* generalized subdivision corona vertex graph *of G and $H_1, \ldots, H_n$, denoted by $\mathfrak{S}(G) \odot \bigwedge_{i}^{n} H_i$, is the graph obtained from $\mathfrak{S}(G)$ and $H_1, \ldots, H_n$, all vertices disjoint, and joining the ith vertex of $V(G)$ to every vertex in H_i.*

Remark 1. *All the graphs $H_1, \ldots, H_n$ can be disconnected. The results in this paper is adapted to disconnected graphs, which may be useful for applications.*

Example 2.2. *Let P_n denote a path of order n. Figure 1 depicts the generalized subdivision corona vertex graph of P_4 and $\{H_i | H_i = P_i, i = 1, \ldots, 4\}$.*

With the help of *Schur complement* and *coronal* of a matrix in the lemmas below we obtained our results.

Lemma 2.3. *[10] Let A be an $n \times n$ matrix partitioned as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square matrices. If A_{11} and A_{22} are invertible, then* $\det(A) = \det(A_{22}) \det\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right) = \det(A_{11}) \det\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)$.

Lemma 2.4. *[2, 9] The U-coronal $\Gamma_U(x)$ of an $n \times n$ square matrix U is the sum of the entries of the matrix $(xI_n - U)^{-1}$, that is, $\Gamma_U(x) = \mathbf{1}_n^T(xI_n - U)^{-1}\mathbf{1}_n$, where $\mathbf{1}_n$ is the column vector of size n with all the entries equal one.*

In particular, if U is an $n \times n$ matrix with each row sum equals to a constant t, then $\Gamma_U(x) = \frac{n}{x-t}$. For any graph G with n vertices, the sum of each row of $L(G)$ is equal to 0, then $\Gamma_{L(G)}(\mu) = \frac{n}{\mu}$. For any r-regular graph with n vertices, the sum of each row of $Q(G)$ is equal to $2r$. Thus, we have $\Gamma_{Q(G)}(\nu) = \frac{n}{\nu-2r}$.

The following lemma is a way to build cospectral families. In Section 4 and Section 5 we give Example 4.6 and Example 5.7 for $L-$cospectral and $Q-$cospectal family by using this lemma to make some corollaries clear.

Lemma 2.5. *[4] Let H_1 and H_2 be two A-cospectral (respectively, L-cospectral and Q-cospectral) graphs, and let L be any graph. Define $G_k = L \cup kH_1 \cup (n-k)H_2$. Then the family of graphs $\{G_k | k = 1, 2, \ldots, n\}$ is an A-cospectral (respectively, L-cospectral and Q-cospectral) family.*

3 Adjacent and degree matrix of $\mathfrak{S}(G) \odot \bigwedge_{i}^{n} H_i$

Let G be a graph with n vertices and m edges and H_i an arbitrary graph with t_i vertices, for $i = 1, \ldots, n$. Let $N = m + n$ and $M = t_1 + \cdots + t_n$. Label the vertices of G by $1, 2, \ldots, n$, and the vertices newly inserted in $\mathfrak{S}(G)$ by $n+1, \ldots, n+m$. Label the vertices of H_1 by $n+m+1, n+m+2, \ldots, n+m+t_1$, and the vertices of H_i for $i \geqslant 2$ by $n+m+\sum_{k=1}^{i-1} t_k + 1, n+m+\sum_{k=1}^{i-1} t_k + 2, \ldots, n+m+\sum_{k=1}^{i} t_k$. Let $\mathbf{0}_{m\times n}$ be an $m \times n$ zero matrix with all the entries equal to zero. Usually, we briefly use $\mathbf{0}$ to denote a zero matrix when its size can be read from the context.

The adjacent matrix of $\mathfrak{S}(G) \odot \bigwedge_i^n H_i$ is given by

$$A\left(\mathfrak{S}(G) \odot \bigwedge_i^m H_i\right) = \begin{pmatrix} A(\mathfrak{S}(G)) & C \\ C^T & B \end{pmatrix}, \tag{3.1}$$

where $A(\mathfrak{S}(G)) = \begin{pmatrix} 0 & R(G) \\ R(G)^T & 0 \end{pmatrix}$,

$$C = \begin{pmatrix} \mathbf{1}_{t_1}^T & 0 & 0 & 0 \\ 0 & \mathbf{1}_{t_2}^T & 0 & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{1}_{t_n}^T \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} A(H_1) & 0 & 0 \\ & & \\ 0 & \ddots & 0 \\ 0 & 0 & A(H_n) \end{pmatrix}.$$

The degree matrix of $\mathfrak{S}(G) \odot \bigwedge_i^n H_i$ under this labeling is given by

$$D\left(\mathfrak{S}(G) \odot \bigwedge_i^n H_i\right) = \begin{pmatrix} D(\mathfrak{S}(G)) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & 0 \\ 0 & D(H) + I_M \end{pmatrix}, \tag{3.2}$$

where $D(\mathfrak{S}(G)) = \begin{pmatrix} D(G) & 0 \\ 0 & 2I_m \end{pmatrix}$, $W = \begin{pmatrix} t_1 & 0 & 0 \\ & & \\ 0 & \ddots & 0 \\ 0 & 0 & t_n \end{pmatrix}$, and

$$D(H) = \begin{pmatrix} D(H_1) & 0 & 0 \\ & & \\ 0 & \ddots & 0 \\ 0 & 0 & D(H_n) \end{pmatrix}.$$

4 L–characteristic polynomial of $\mathfrak{S}(G) \odot \bigwedge_i^n H_i$

Theorem 4.1. *Let G be a graph with n vertices and m edges, and H_i an arbitrary graph with t_i vertices for $i = 1, \ldots, n$. The Laplacian characteristic polynomial of $\mathfrak{S}(G) \odot \bigwedge_i^n H_i$ is*

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$

$$= \det\left(\begin{pmatrix} \mu - t_1 - \Gamma_{L(H_1)}(\mu-1) & 0 & 0 & 0 \\ & & & \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \mu - t_n - \Gamma_{L(H_n)}(\mu-1) & 0 \\ 0 & 0 & 0 & \mu I_m \end{pmatrix} - L(\mathfrak{S}(G))\right)$$

$$\cdot \prod_{i=1}^n f_{L(H_i)}(\mu - 1).$$

Proof. By (3.1), (3.2) and the definition of Laplacian matrix, we have

$$L\left(\mathfrak{S}(G)\odot\bigwedge_{i=1}^{n}H_i\right)$$
$$= D\left(\mathfrak{S}(G)\odot\bigwedge_{i=1}^{n}H_i\right) - A\left(\mathfrak{S}(G)\odot\bigwedge_{i=1}^{n}H_i\right)$$
$$= \begin{pmatrix} D(\mathfrak{S}(G)) + \begin{pmatrix} W & 0\\ 0 & 0\end{pmatrix}_{N\times N} & 0 \\ 0 & D(H)+I_M \end{pmatrix} - \begin{pmatrix} A(\mathfrak{S}(G)) & C \\ C^T & B \end{pmatrix}$$
$$= \begin{pmatrix} L(\mathfrak{S}(G)) + \begin{pmatrix} W & 0\\ 0 & 0\end{pmatrix}_{N\times N} & -C \\ -C^T & L(H)+I_M \end{pmatrix},$$

where $L(H) = \begin{pmatrix} L(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L(H_n)\end{pmatrix}$. Then, we have

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_{i}^{n}H_i)}(\mu)$$
$$= \det\begin{pmatrix} \mu I_N - \begin{pmatrix} W & 0\\ 0 & 0\end{pmatrix}_{N\times N} - L(\mathfrak{S}(G)) & C \\ C^T & (\mu-1)I_M - L(H)\end{pmatrix}$$
$$= \det\left(\begin{pmatrix} \mu I_n - W & 0 \\ 0 & \mu I_m\end{pmatrix} - L(\mathfrak{S}(G)) - C\left((\mu-1)I_M - L(H)\right)^{-1}C^T\right)$$
$$\cdot \det\left((\mu-1)I_M - L(H)\right).$$

Notė that

$$\det\left((\mu-1)I_M - L(H)\right) = \det\begin{pmatrix} (\mu-1)I_{t_1} - L(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\mu-1)I_{t_n} - L(H_n)\end{pmatrix}$$
$$= \prod_{i=1}^{n} f_{L(H_i)}(\mu-1),$$

and

$$C\left((\mu-1)I_M - L(H)\right)^{-1}C^T$$
$$= C\begin{pmatrix} (\mu-1)I_{t_1} - L(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\mu-1)I_{t_n} - L(H_n)\end{pmatrix}^{-1} C^T$$
$$= \begin{pmatrix} \mathbf{1}_{t_1}^T\left((\mu-1)I_{t_1} - L(H_1)\right)^{-1}\mathbf{1}_{t_1} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \mathbf{1}_{t_n}^T\left((\mu-1)I_{t_n} - L(H_n)\right)^{-1}\mathbf{1}_{t_n} & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}_{N\times N}$$
$$= \begin{pmatrix} \Gamma_{L(H_1)}(\mu-1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \Gamma_{L(H_n)}(\mu-1) & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}_{N\times N}.$$

Thus,

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$

$$= \det\left(\begin{pmatrix} \mu - t_1 - \Gamma_{L(H_1)}(\mu-1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \mu - t_n - \Gamma_{L(H_n)}(\mu-1) & 0 \\ 0 & 0 & 0 & \mu I_m \end{pmatrix} - L(\mathfrak{S}(G))\right)$$

$$\cdot \prod_{i=1}^{n} f_{L(H_i)}(\mu - 1). \qquad \square$$

Corollary 4.2. *Let G be a graph with n vertices and m edges and H_i an arbitrary graph with t vertices for $i = 1, \ldots, n$, then*

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$

$$= (\mu-1)^{-n} \cdot (\mu-2)^{m-n} \cdot \left(\prod_{i=1}^{n} f_{L(H_i)}(\mu-1)\right)$$

$$\cdot \det\left(\mu^3 I_n - \mu^2\left((3+t)I_n + D(G)\right) + \mu\left(2(t+1)I_n + D(G) + L(G)\right) - L(G)\right).$$

In particular, if $H_i \simeq H$ for $i = 1, \ldots, n$, then

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$

$$= (\mu-1)^{-n} \cdot (\mu-2)^{m-n} \cdot \left(f_{L(H)}(\mu-1)\right)^n$$

$$\cdot \det\left(\mu^3 I_n - \mu^2\left((3+t)I_n + D(G)\right) + \mu\left(2(t+1)I_n + D(G) + L(G)\right) - L(G)\right).$$

Proof. For $i = 1, \ldots, n$, H_i has t vertices, then $\Gamma_{L(H_1)}(\mu-1) = \cdots = \Gamma_{L(H_n)}(\mu-1) = \frac{t}{\mu-1}$. By Theorem 4.1 we have

$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu) = \det\left(\begin{pmatrix} \left(\mu - t - \frac{t}{\mu-1}\right) I_n & 0 \\ 0 & \mu I_m \end{pmatrix} - L(\mathfrak{S}(G))\right) \cdot \prod_{i=1}^{n} f_{L(H_i)}$$

$(\mu-1)$, where $L(\mathfrak{S}(G)) = D(\mathfrak{S}(G)) - A(\mathfrak{S}(G)) = \begin{pmatrix} D(G) & -R(G) \\ -R(G)^T & 2I_m \end{pmatrix}$.

Note that

$$\det\left(\begin{pmatrix} \left(\mu - t - \frac{t}{\mu-1}\right) I_n & 0 \\ 0 & \mu I_m \end{pmatrix} - L(\mathfrak{S}(G))\right)$$

$$= \det\begin{pmatrix} \left(\mu - t - \frac{t}{\mu-1}\right) I_n - D(G) & R(G) \\ R(G)^T & (\mu-2)I_m \end{pmatrix}$$

$$= (\mu-2)^{m-n}(\mu-1)^{-n}\det\left(\mu^3 I_n - \mu^2\left((3+t)I_n + D(G)\right)\right.$$

$$\left. + \mu\left(2(t+1)I_n + D(G) + L(G)\right) - L(G)\right).$$

Thus,

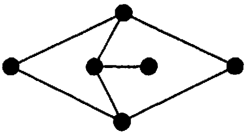
$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$
$$= (\mu-1)^{-n}\cdot(\mu-2)^{m-n}\cdot\left(\prod_{i=1}^{n} f_{L(H_i)}(\mu-1)\right)$$
$$\cdot\det\left(\mu^3 I_n - \mu^2\left((3+t)I_n + D(G)\right) + \mu\left(2(t+1)I_n + D(G) + L(G)\right) - L(G)\right).$$
In particular, if $H_i \simeq H$ for $i = 1,\dots,n$, then
$$f_{L(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\mu)$$
$$= (\mu-1)^{-n}\cdot(\mu-2)^{m-n}\cdot\left(f_{L(H)}(\mu-1)\right)^n$$
$$\cdot\det\left(\mu^3 I_n - \mu^2\left((3+t)I_n + D(G)\right) + \mu\left(2(t+1)I_n + D(G) + L(G)\right) - L(G)\right).$$
□

Corollary 4.3. *Let G be an $r-$regular graph with n vertices and m edges, and H an arbitrary graph with t vertices. Let $\mu_i(G)$ denote the ith Laplacian eigenvalue of G. Then we have*
$$f_{L(G\odot H)}(\mu) = (\mu-2)^{m-n}\cdot\prod_{i=2}^{t}(\mu-1-\mu_i(H))^n$$
$$\cdot\prod_{i=1}^{n}\left(\mu^3-\mu^2(3+t+r)+\mu(2t+2+r+\mu_i(G))-\mu_i(G)\right).$$

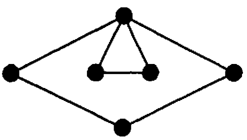
Corollary 4.4. *Let G_1 and G_2 be two L-cospectral r-regular graphs with n vertices and m edges. Let $H_1,\dots,H_n$ be a sequence of graphs such that $\Gamma_{L(H_1)}(\mu) = \cdots = \Gamma_{L(H_n)}(\mu)$. Then $\mathfrak{S}(G_1)\odot\bigwedge_i^n H_i$ and $\mathfrak{S}(G_2)\odot\bigwedge_i^n H_i$ are L-cospectral.*

Corollary 4.5. *Let G be a graph with n vertices and m edges. Let $\{H_i|i = 1,\dots\}$ be a L-cospectral family. Then for any n-subset of sequence $H_{i_1},\dots,H_{i_n}$, the resultant graph $\mathfrak{S}(G)\odot\bigwedge_{k=1}^{n} H_{i_k}$ is L-cospectral to one another.*

Example 4.6. Let H_1 and H_2 be L-cospectral graphs as shown in Figure 2. By a simple computation, we know that $\Gamma(\mu) := \Gamma_{L(H_1)}(\mu) = \Gamma_{L(H_2)}(\mu) = \frac{6}{x}$. Let $L = K_4$ be a complete graph with 4 vertices and $n = 4$. By Lemma 2.5, $\{G_1, G_2, G_3, G_4\}$ is a L-cospectral family. Note that the L-coronal of disjoint union of some graphs equals the sum of the L-coronals of all such graphs. Thus, $\Gamma_{L(G_k)}(\mu) = \Gamma_{L(K_4)}(\mu) + 4\Gamma(\mu)$, for $k = 1,2,3,4$. Now, let $G = C_5$ be a circle graph with 5 vertices. By Corollary 4.5, we can choose any five graphs, denoted by $G_{j_1}, G_{j_2}, G_{j_3}, G_{j_4}, G_{j_5}$,



(a) H_1



(b) H_2

Fig. 2: Two L-cospectral graphs

from $\{G_1, G_1, G_2, G_2, G_3, G_3, G_4, G_4\}$ to obtain a generalized subdivision corona vertex graph $\mathfrak{S}(K_4) \odot \bigwedge_{k=1}^{5} G_{j_k}$. Clearly, all resultant graphs are L-cospectral, whose Laplacian characteristic polynomial can be computed by Corollary 4.2.

5 Q−characteristic polynomial of $\mathfrak{S}(G) \odot \bigwedge_{i}^{n} H_i$

Theorem 5.1. *Let G be a graph with n vertices and m edges, and H_i an arbitrary graph with t_i vertices for $i = 1, \ldots, n$. The signless Laplacian characteristic polynomial of $\mathfrak{S}(G) \odot \bigwedge_{i}^{n} H_i$ is*

$$f_{Q(\mathfrak{S}(G)\odot\bigwedge_{i}^{n} H_i)}(\nu)$$

$$= \det\left(\begin{pmatrix} \nu - t_1 - \Gamma_{Q(H_1)}(\nu-1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \nu - t_n - \Gamma_{Q(H_n)}(\nu-1) & 0 \\ 0 & 0 & 0 & \nu I_m \end{pmatrix} - Q(\mathfrak{S}(G))\right)$$

$$\cdot \prod_{i=1}^{n} f_{Q(H_i)}(\nu - 1).$$

Proof. By (3.1), (3.2) and the definition of signless Laplacian matrix, we have

$$Q\left(\mathfrak{S}(G)\odot\bigwedge_{i}^{n} H_i\right)$$

$$= D\left(\mathfrak{S}(G)\odot\bigwedge_{i}^{n} H_i\right) + A\left(\mathfrak{S}(G)\odot\bigwedge_{i}^{n} H_i\right)$$

$$= \begin{pmatrix} Q(\mathfrak{S}(G)) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & C \\ C^T & Q(H) + I_M \end{pmatrix},$$

where $Q(H) = \begin{pmatrix} Q(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q(H_n) \end{pmatrix}$. Then we have

$$f_{Q(\mathfrak{S}(G)\odot \bigwedge_i^n H_i)}(\nu)$$

$$= \det \begin{pmatrix} \nu I_N - Q(\mathfrak{S}(G)) - \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & -C \\ -C^T & (\nu-1)I_M - Q(H) \end{pmatrix}$$

$$= \det \left(\begin{pmatrix} \nu I_n - W & 0 \\ 0 & \nu I_m \end{pmatrix} - Q(\mathfrak{S}(G)) - C\left((\nu-1)I_M - Q(H)\right)^{-1} C^T \right)$$

$$\cdot \det\left((\nu-1)I_M - Q(H)\right).$$

Note that

$$\det\left((\nu-1)I_M - Q(H)\right) = \det \begin{pmatrix} (\nu-1)I_{t_1} - Q(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\nu-1)I_{t_n} - Q(H_n) \end{pmatrix}$$

$$= \prod_{i=1}^{n} f_{Q(H_i)}(\nu-1),$$

and

$$C\left((\nu-1)I_M - Q(H)\right)^{-1} C^T = \begin{pmatrix} \Gamma_{Q(H_1)}(\nu-1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \Gamma_{Q(H_n)}(\nu-1) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{N\times N}.$$

Thus,

$$f_{Q(\mathfrak{S}(G)\odot \bigwedge_i^n H_i)}(\nu)$$

$$= \det \left(\begin{pmatrix} \nu - t_1 - \Gamma_{Q(H_1)}(\nu-1) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \nu - t_n - \Gamma_{Q(H_n)}(\nu-1) & 0 \\ 0 & 0 & 0 & \nu I_m \end{pmatrix} - Q(\mathfrak{S}(G)) \right)$$

$$\cdot \prod_{i=1}^{n} f_{Q(H_i)}(\nu-1).$$

□

Corollary 5.2. *Let G be a graph with n vertices and m edges, and H_i an arbitrary graph with t vertices and $\Gamma_{Q(H_i)}(\nu) = \Gamma_{Q(H)}(\nu)$, for $i = 1, \ldots, n$. Then we have*

$$f_{Q(\mathfrak{S}(G)\odot \bigwedge_i^n H_i)}(\nu)$$

$$= (\nu-2)^{m-n} \cdot \left(\prod_{i=1}^{n} f_{Q(H_i)}(\nu-1) \right) \cdot \det\Big(\nu^2 I_n - \nu\big((t+2+\Gamma_{Q(H)}(\nu-1))I_n$$

$$+ D(G)\big) + 2\big((t+\Gamma_{Q(H)}(\nu-1))I_n + D(G)\big) - Q(G)\Big).$$

In particular, if $H_i \simeq H$ for $i = 1, \ldots, n$, then

$$f_{Q(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\nu)$$

$$= (\nu-2)^{m-n}\left(f_{Q(H)}(\nu-1)\right)^n \cdot \det\Big(\nu^2 I_n - \nu\Big((t+2+\Gamma_{Q(H)}(\nu-1))I_n$$

$$+ D(G)\Big) + 2\Big((t+\Gamma_{Q(H)}(\nu-1))I_n + D(G)\Big) - Q(G)\Big).$$

Proof. By Theorem 5.1 we have

$$f_{Q(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\nu) = \det\left(\begin{pmatrix}(\nu - t - \Gamma_{Q(H)}(\nu-1))\, I_n & \mathbf{0} \\ \mathbf{0} & \nu I_m\end{pmatrix} - Q(\mathfrak{S}(G))\right)$$

$$\cdot \prod_{i=1}^{n} f_{Q(H_i)}(\nu-1),$$

where $Q(\mathfrak{S}(G)) = D(\mathfrak{S}(G)) + A(\mathfrak{S}(G)) = \begin{pmatrix} D(G) & R(G) \\ R(G)^T & 2I_m\end{pmatrix}$.

Note that

$$\det\left(\begin{pmatrix}(\nu - t - \Gamma_{Q(H)}(\nu-1))\, I_n & \mathbf{0} \\ \mathbf{0} & \nu I_m\end{pmatrix} - Q(\mathfrak{S}(G))\right)$$

$$= \det\begin{pmatrix}(\nu - t - \Gamma_{Q(H)}(\nu-1))\, I_n - D(G) & -R(G) \\ -R(G)^T & (\nu-2)I_m\end{pmatrix}$$

$$= (\nu-2)^{m-n} \cdot \det\Big(\nu^2 I_n - \nu\left((t+2+\Gamma_{Q(H)}(\nu-1))\, I_n + D(G)\right)$$

$$+ 2\left((t+\Gamma_{Q(H)}(\nu-1))\, I_n + D(G)\right) - Q(G)\Big).$$

Thus,

$$f_{Q(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\nu)$$

$$= (\nu-2)^{m-n} \cdot \left(\prod_{i=1}^{n} f_{Q(H_i)}(\nu-1)\right) \cdot \det\Big(\nu^2 I_n - \nu\Big((t+2+\Gamma_{Q(H)}(\nu-1))I_n$$

$$+ D(G)\Big) + 2\Big((t+\Gamma_{Q(H)}(\nu-1))I_n + D(G)\Big) - Q(G)\Big).$$

In particular, if $H_i \simeq H$ for $i = 1, \ldots, n$, then

$$f_{Q(\mathfrak{S}(G)\odot\bigwedge_i^n H_i)}(\nu)$$

$$= (\nu-2)^{m-n}\left(f_{Q(H)}(\nu-1)\right)^n \cdot \det\Big(\nu^2 I_n - \nu\Big((t+2+\Gamma_{Q(H)}(\nu-1))I_n$$

$$+ D(G)\Big) + 2\Big((t+\Gamma_{Q(H)}(\nu-1))I_n + D(G)\Big) - Q(G)\Big).$$

□

Corollary 5.3. *Let G be a graph with n vertices and m edges, and H_i an r−regular graph with t vertices, for $i = 1, \ldots, n$. We have*

$$f_{Q(\mathfrak{S}(G)\odot \bigwedge_{i=1}^{n} H_i)}(\nu)$$

$$= (\nu-2)^{m-n} \cdot \left(\prod_{i=1}^{n} f_{Q(H_i)}(\nu-1)\right) \cdot \det\left(\nu^2 I_n - \nu\left(\left(t+2+\frac{t}{\nu-2r-1}\right) I_n + D(G)\right) + 2\left(\left(t+\frac{t}{\nu-2r-1}\right) I_n + D(G)\right) - Q(G)\right).$$

Corollary 5.4. *Let G be an $r-$regular graph with n vertices and m edges, and H an arbitrary graph with t vertices. Let $\nu_i(G)$ denote the ith signless Laplacian eigenvalue of G. We have*

$$f_{Q(\mathfrak{S}(G)\odot \bigwedge_{i=1}^{n} H_i)}(\nu)$$

$$= (\nu-2)^{m-n} \cdot \prod_{i=1}^{t} (\nu - 1 - \nu_i(H))^n$$

$$\cdot \prod_{i=1}^{n} \left(\nu^2 - \nu\left(t+2+r+\Gamma_{Q(H)}(\nu-1)\right) + 2\left(t+r+\Gamma_{Q(H)}(\nu-1)\right) - \nu_i(G)\right).$$

Corollary 5.5. *Let G_1 and G_2 be two Q-cospectral r-regular graphs with n vertices and m edges. Let $H_1, \ldots, H_n$ be a sequence of graphs such that $\Gamma_{Q(H_1)}(\nu) = \cdots = \Gamma_{Q(H_n)}(\nu)$. Then $\mathfrak{S}(G_1) \odot \bigwedge_{i}^{n} H_i$ and $\mathfrak{S}(G_2) \odot \bigwedge_{i}^{n} H_i$ are Q-cospectral.*

Corollary 5.6. *Let G be a graph with n vertices and m edges. Let $\{H_i | i = 1, \ldots\}$ be a Q-cospectral family and $\Gamma_{Q(H_i)}(\nu) = \Gamma_{Q(H)}(\nu)$. Then for any n-subset of sequence $H_{i_1}, \ldots, H_{i_n}$, the resultant graph $\mathfrak{S}(G) \odot \bigwedge_{k=1}^{n} H_{i_k}$ is Q-cospectral to one another.*

Example 5.7. Let H_3 and H_4 be Q-cospectral graphs as shown in Figure 3. By a simple computation, we know that $\Gamma(\nu) := \Gamma_{Q(H_3)}(\nu) = \Gamma_{Q(H_4)}(\nu) = \frac{8x^4-60x^3+148^2-132x+32}{x^5-11x^4+43x^3-72x^2+48x-8}$. Let $L = K_4$ and $n = 4$. By Lemma 2.5, $\{G_1, G_2, G_3, G_4\}$ is a Q-cospectral family. Note that the Q-coronal of disjoint union of some graphs equals the sum of the Q-coronals of all such graphs. Thus, $\Gamma_{Q(G_k)}(\nu) = \Gamma_{Q(K_4)}(\nu) + 4\Gamma(\nu)$, for $k = 1, 2, 3, 4$. Now, let $G = C_5$. By Corollary 5.6, we can choose any five graphs, denoted by $G_{j_1}, G_{j_2}, G_{j_3}, G_{j_4}, G_{j_5}$, from $\{G_1, G_1, G_2, G_2, G_3, G_3, G_4, G_4\}$ to obtain a generalized subdivision corona vertex graph $\mathfrak{S}(C_5) \odot \bigwedge_{k=1}^{5} G_{j_k}$. Clearly, all

(a) H_3 (b) H_4

Fig. 3: Two Q-cospectral graphs

resultant graphs are Q-cospectral, whose signless Laplacian characteristic polynomial can be computed by Corollary 5.2.

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