GAUSSIAN FIBONACCI AND GAUSSIAN LUCAS p–NUMBERS

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ABSTRACT. In this paper we define and study the Gaussian Fibonacci and Gaussian Lucas p–numbers. We give generating functions, Binet formulas, explicit formulas, matrix representations and sums of Gaussian Fibonacci p–numbers by matrix methods . For $p = 1$ these Gaussian Fibonacci and Gaussian Lucas p–numbers reduce to the Gaussian Fibonacci and the Gaussian Lucas numbers.

1. INTRODUCTION

A. F. Horadam [17] introduced the concept of the complex Fibonacci numbers and established some quite general identities concerning them. J. R. Jordan [19] extended some relationships which are known for the usual Fibonacci sequences to the Gaussian Fibonacci and Gaussian Lucas sequences.

The Gaussian Fibonacci sequence in [19] is defined as $GF_0 = i$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. One can see that

$$GF_n = F_n + iF_{n-1} \tag{1.1}$$

where F_n is the usual nth Fibonacci number.

The Gaussian Lucas sequence in [19] is defined in a way similar to Gaussian Fibonacci sequence as $GL_0 = 2 - i$, $GL_1 = 1 + 2i$, and $GL_n = GL_{n-1} + GL_{n-2}$ for $n > 1$. Also it can be seen that

$$GL_n = L_n + iL_{n-1} \tag{1.2}$$

where L_n is the usual nth Lucas number.

There are many important generalizations of Fibonacci and Lucas numbers. The generalized Fibonacci and Lucas p–numbers [41] are examples of them and are defined by:

$$F_{p,n} = F_{p,n-1} + F_{p,n-p-1} \tag{1.3}$$

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with initial conditions $F_{p,0} = 0$ and $F_{p,n} = 1$, $n = 1, 2, ..., p$, and

$$L_{p,n} = L_{p,n-1} + L_{p,n-p-1} \tag{1.4}$$

with initial conditions $L_{p,0} = p + 1$ and $L_{p,n} = 1$, $n = 1, 2, ..., p$ where $p = 0, 1, 2, ...$ and $n = 0, \pm 1, \pm 2, \pm 3, ...$

The complex Fibonacci numbers and Gaussian Fibonacci numbers are studied by some other authors [3, 14, 18]. The complex Fibonacci polynomials were defined and studied in [16] by Horadam. Harman [14] give a new approach toward the extension of Fibonacci numbers into the complex plane. Before this study there were two different methods for defining such numbers studied by Horadam [17] and Berzsenyi [3]. Harman [14] generalized both of the methods. In [5, 6, 9, 11, 26] theories of the generalized Fibonacci and Lucas polynomials are developed. Yu and Liang [45] derive some identities involving the partial derivative sequences of the bivariate Fibonacci polynomials $F_n(x, y)$ and the bivariate Lucas polynomials $L_n(x, y)$. Djordjevic [7, 8] considered the generating functions, explicit formulas and partial derivative sequences of the generalized Fibonacci and Lucas polynomials. Good [12] points out that the square root of the Golden Ratio is the real part of a simple periodic continued fraction but using (complex) Gaussian integers $a + ib$ instead of the natural integers. For more information one can see [4, 10, 11, 13, 15, 18, 22, 27, 28, 42, 44, 46].

Asci and Gurel in [1] defined bivariate Gaussian Fibonacci and Gaussian Lucas polynomials and extended some relationships which are known for the usual Fibonacci sequence, polynomials and bivariate polynomials to the bivariate Gaussian Fibonacci polynomials. Also the authors in [2] define the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. They give generating functions, Binet formulas, explicit formulas and Q matrix of these numbers. They also present explicit combinatorial and determinantal expressions, study negatively subscripted numbers and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers they give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers

Stakhov and Rozin in [31] established theory of Binet formulas for Fibonacci and Lucas p−numbers. Stakhov [30, 35] generalized Fibonacci Q−matrix, Fibonacci matrices, and the "Cassini formula". Also Stakhov in [29, 32, 33, 34, 36, 37, 38, 39, 40] introduced algorithmic measurement theory and generalized principle of the golden section. Tuglu et al. in [21, 43] study the bivariate Fibonacci and Lucas p−polynomials ($p \geq 0$ is integer) from which, specifying x, y and p, bivariate Fibonacci and Lucas polynomials, bivariate Pell and Pell-Lucas polynomials, Jacobsthal and Jacobsthal–Lucas polynomials, Fibonacci and Lucas p−polynomials, Fibonacci and Lucas p–numbers, Pell and Pell–Lucas p–numbers and Chebyshev polynomials of the first and second kind, are obtained. Tuglu et al.

[20] introduced $m-$extension of Fibonacci and Lucas $p-$numbers. Lee et al. in [23, 24, 25] derived a generalized Binet formula for $k-$generalized Fibonacci sequence by using determinants and gave several representation of $k-$generalized Fibonacci numbers.

In this article we define and study the Gaussian Fibonacci $p-$Numbers and Gaussian Lucas $p-$Numbers. We give generating functions, Binet formulas, explicit formulas, matrix representations and sums of Gaussian Fibonacci $p-$numbers by matrix methods. By defining these Gaussian $p-$Fibonacci and Gaussian $p-$Lucas Numbers for special cases for $p = 1$, GF_n is the Gaussian Fibonacci and GL_n is the Gaussian Lucas numbers defined in [19].

2. Gaussian Fibonacci and Gaussian Lucas $p-$Numbers

Definition 1. *Let p be an integer the Gaussian Fibonacci p−numbers* $\{GF_{p,n}\}_{n=0}^{\infty}$ *are defined by the following recurrence relation*

$$GF_{p,n} = GF_{p,n-1} + GF_{p,n-p-1}, \quad n > p \tag{2.1}$$

with initial conditions $GF_{p,0} = i$, $GF_{p,n} = 1$, $n = 1, 2, ..., p$

It can be easily seen that

$$GF_{p,n} = F_{p,n} + iF_{p,n-p}$$

where $F_{p,n}$ is the nth Fibonacci $p-$number.

Definition 2. *The Gaussian Lucas p−numbers* $\{GL_{p,n}\}_{n=0}^{\infty}$ *are defined by the following recurrence relation*

$$GL_{p,n} = GL_{p,n-1} + GL_{p,n-p-1}, \quad n > p \tag{2.2}$$

with initial conditions $GL_{p,0} = p + 1 - pi$, $GL_{p,n} = 1$, $n = 1, 2, 3, ..., p - 1$ *and* $GL_{p,p} = 1 + (p + 1)\,i$

Also

$$GL_{p,n} = L_{p,n} + iL_{p,n-p}$$

where $L_{p,n}$is the nth Lucas $p-$number.

2.1. Some Properties of Gaussian Fibonacci and Gaussian Lucas $p-$Numbers.

Theorem 1. *The generating function for the Gaussian Fibonacci p-numbers is*

$$g(t) = \sum_{n=0}^{\infty} GF_{p,n}t^n = \frac{t + i(1 - t)}{1 - t - t^{p+1}}$$

and for the Gaussian Lucas p−numbers is

$$h(t) = \sum_{n=0}^{\infty} GL_{p,n}t^n = \frac{p + 1 - pt - i\left[p - pt - (p + 1)\,t^p\right]}{1 - t - t^{p+1}}.$$

Proof. Let $g(t)$ be the generating function of the Gaussian Fibonacci p, $GF_{p,n}$. Then

$$\begin{aligned} g(t) - tg(t) - t^{p+1}g(t) &= GF_{p,0} + tGF_{p,1} - tGF_{p,0} \\ &\quad + \sum_{n=2}^{\infty} t^n GF_{p,n} - GF_{p,n-1} - GF_{p,n-2} \\ &= t + i(1-t) \end{aligned}$$

By taking $g(t)$ in parenthesis we get

$$g(t) = \frac{t + i(1-t)}{1 - t - t^{p+1}}.$$

The proof is completed. □

Binet's formulas are well known and studied in the theory of Fibonacci numbers. Let $x_1, x_2, x_3, ..., x_p, x_{p+1}$ be the different roots of the characteristic equation of the recurrence relation (1.3). Then the Binet formula of the Fibonacci p-numbers are given in [31].

Now we can give the Binet formula for the Gaussian Fibonacci p-numbers and the Gaussian Lucas $p-$numbers

Theorem 2. *For $n \geq 0$*

$$GF_{p,n} = k_1 x_1^n + k_2 x_2^n + ... + k_{p+1} x_{p+1}^n + i\left(k_1 x_1^{n-p} + k_2 x_2^{n-p} + ... + k_{p+1} x_{p+1}^{n-p}\right)$$

and

$$GL_{p,n} = t_1 x_1^n + t_2 x_2^n + ... + t_{p+1} x_{p+1}^n + i\left(t_1 x_1^{n-p} + t_2 x_2^{n-p} + ... + t_{p+1} x_{p+1}^{n-p}\right).$$

Proof. Theorem can be proved by mathematical induction on n. □

Theorem 3. *The explicit formulas of Gaussian Fibonacci $p-$numbers and Gaussian Lucas $p-$numbers for $n \geq 1$ and $p \geq 1$ are*

$$GF_{p,n} = \sum_{k=0}^{\lfloor \frac{n-1}{p+1} \rfloor} \binom{n-pk-1}{k} + i \sum_{k=0}^{\lfloor \frac{n-p-1}{p+1} \rfloor} \binom{n-p-pk-1}{k}$$

and

$$GL_n = \sum_{k=0}^{\lfloor \frac{n}{p+1} \rfloor} \frac{n}{n-pk}\binom{n-pk}{k} + i \sum_{k=0}^{\lfloor \frac{n-p}{p+1} \rfloor} \frac{n-p}{n-p-pk}\binom{n-p-pk}{k}.$$

Proof. Theorem can be proved by mathematical induction on n. □

Taking $p = 1$ gives

Corollary 1. [1] *Usual explicit formulas of Gaussian Fibonacci numbers and Gaussian Lucas numbers are*

$$GF_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} + i \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-k-2}{k}$$

and

$$GL_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-k-1} \binom{n-k-1}{k}$$

Theorem 4. *For $n > p$*

$$GL_{p,n} = GF_{p,n+1} + pGF_{p,n-p}$$

Proof. It can be seen from the generating functions. □

Corollary 2. [19] *For $n \geq 1$*

$$GL_n = GF_{n+1} + GF_{n-1}.$$

Theorem 5. *The sums of the Gaussian Fibonacci and Gaussian Lucas $p-$numbers are given as:*

(i) $\sum_{k=0}^{n} GF_{p,k} = GF_{p,n+p+1} - 1$

(ii) $\sum_{k=0}^{n} GL_{p,k} = GL_{p,n+p+1} - [1 + (p+1)\, i]$

Proof. By the definition of Gaussian Fibonacci $p-$numbers;

$$GF_{p,n} = GF_{p,n-1} + GF_{p,n-p-1} \tag{2.3}$$

Replace n with $n + p + 1$ in identity (2.3) to obtain

$$GF_{p,n+p} = GF_{p,n+p+1} - GF_{p,n}$$

and further

$$\begin{aligned} \sum_{k=0}^{n} GF_{p,k+p} &= \sum_{k=0}^{n} GF_{p,k+p+1} - \sum_{k=0}^{n} GF_{p,k} \\ &= \sum_{k=0}^{n} GF_{p,k} + \sum_{k=n+1}^{n+p+1} GF_{p,k} - \sum_{k=0}^{p} GF_{p,k} - \sum_{k=0}^{n} GF_{p,k} \\ &= \sum_{k=0}^{p} GF_{p,k+n+1} - \sum_{k=0}^{p} GF_{p,k} \\ &= \sum_{k=0}^{p} (GF_{p,k+n+1} - GF_{p,k}) \end{aligned}$$

and by definition of Gaussian Fibonacci $p-$numbers

$$\begin{aligned}\sum_{k=0}^{n} GF_{p,k+p+1} &= GF_{p,n+p+1} - GF_{p,p} + \sum_{k=p}^{n+p} GF_{p,k} \\ &= GF_{p,n+p+1} - GF_{p,p} + \sum_{k=0}^{n} GF_{p,k+p}\end{aligned}$$

and thus

$$\sum_{k=0}^{n} GF_{p,k+p+1} - \sum_{k=0}^{n} GF_{p,k+p} = GF_{p,n+p+1} - GF_{p,p}$$

So we get the solution

$$\begin{aligned}\sum_{k=0}^{n} GF_{p,k} &= GF_{p,n+p+1} - GF_{p,p} \\ &= GF_{p,n+p+1} - 1.\end{aligned}$$

□

Corollary 3. [19] *For* $n \geq 0$

$$\sum_{k=0}^{n} GF_k = GF_{n+2} - 1$$

Corollary 4. [19] *For* $n \geq 0$

$$\sum_{k=0}^{n} GL_k = GL_{n+2} - (1+2i)$$

Lemma 1. [43] *For* $n \geq 0$

$$\begin{aligned}\sum_{k=0}^{n} F_{p,k}L_{p,n-k} &= (n+p)F_{p,n} \\ \sum_{k=0}^{n} F_{p,k+1}L_{p,n-k} &= (n+p+1)\,F_{n+1} \\ \sum_{k=0}^{n} F_{p,k+1}L_{p,n-k+1} &= (n+1)\,F_{p,n+2} \\ \sum_{k=0}^{n} F_{p,k}L_{p,n-k+1} &= nF_{p,n+1}\end{aligned}$$

Theorem 6. *For* $n \geq 0$

$$\sum_{k=0}^{n} GF_{p,k}GL_{p,n-k} = -2i(n+p)F_{p,n} + (i+1)(2n+p+1)F_{p,n+1} - (n+1)F_{p,n+2}$$

Proof. By the definition and lemma 1

$$\begin{aligned}\sum_{k=0}^{n} GF_{p,k}GL_{p,n-k} &= \sum_{k=0}^{n} (F_{p,k} + iF_{p,k-p})(L_{p,n-k} + iL_{p,n-k-p}) \\ &= \sum_{k=0}^{n} [F_{p,k}L_{p,n-k} + iF_{p,k}L_{p,n-k-p} + iF_{p,k-p}L_{p,n-k} \\ &\quad -F_{p,k-p}L_{p,n-k-p}]\end{aligned}$$

It can be easily seen that

$$F_{p,k-p} = F_{p,k+1} - F_{p,k}$$

$$L_{p,n-k-p} = L_{p,n-k+1} - L_{p,n-k}$$

and

$$\begin{aligned}\sum_{k=0}^{n} GF_{p,k}GL_{p,n-k} &= \sum_{k=0}^{n} [F_{p,k}L_{p,n-k} + iF_{p,k}(L_{p,n-k+1} - L_{p,n-k}) \\ &\quad +i(F_{p,k+1} - F_{p,k})L_{p,n-k} \\ &\quad -(F_{p,k+1} - F_{p,k})(L_{p,n-k+1} - L_{p,n-k})] \\ &= \sum_{k=0}^{n} [F_{p,k}L_{p,n-k} + iF_{p,k}L_{p,n-k+1} \\ &\quad -iF_{p,k}L_{p,n-k} + iF_{p,k+1}L_{p,n-k} - iF_{p,k}L_{p,n-k} \\ &\quad -(F_{p,k+1}L_{p,n-k+1} - F_{p,k+1}L_{p,n-k} \\ &\quad -F_{p,k}L_{p,n-k+1} + F_{p,k}L_{p,n-k}))]\end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{n} [iF_{p,k}L_{p,n-k+1} - iF_{p,k}L_{p,n-k} + iF_{p,k+1}L_{p,n-k} - iF_{p,k}L_{p,n-k} \\ &\quad -F_{p,k+1}L_{p,n-k+1} + F_{p,k+1}L_{p,n-k} + F_{p,k}L_{p,n-k+1}] \\ &= (i+1)\sum_{k=0}^{n} F_{p,k}L_{p,n-k+1} - 2i\sum_{k=0}^{n} F_{p,k}L_{p,n-k} \\ &\quad +(i+1)\sum_{k=0}^{n} F_{p,k+1}L_{p,n-k} - \sum_{k=0}^{n} F_{p,k+1}L_{p,n-k+1}\end{aligned}$$

by lemma 1

$$\sum_{k=0}^{n} GF_{p,k}GL_{p,n-k} = (i+1)\,nF_{p,n+1} - 2i(n+p)F_{p,n}$$
$$+ (i+1)\,(n+p+1)\,F_{p,n+1} - (n+1)\,F_{p,n+2}$$
$$= -2i(n+p)F_{p,n} + (i+1)\,(2n+p+1)\,F_{p,n+1}$$
$$- (n+1)\,F_{p,n+2}.$$

□

Theorem 7. *For $m, n \geq 1$, then*

$$GF_{p,m+n} = F_{p,m}GF_{p,n+1} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,n-p+k}.$$

Proof. The formula is trivially true for $n = p$ and $n = p+1$.

$$GF_{p,m+p} = F_{p,m}GF_{p,p+1} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,k}$$

and

$$GF_{p,m+p+1} = F_{p,m}GF_{p,p+2} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,k+1}$$

Assume it is true for $n = t - p$ and $n = t$ where $t \geq p+1$, that is,

$$GF_{p,m+t-p} = F_{p,m}GF_{p,t-p+1} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,t-2p+k} \tag{2.4}$$

$$GF_{p,m+t} = F_{p,m}GF_{p,t+1} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,t-p+k}. \tag{2.5}$$

Adding identities (2.4) and (2.5)

$$GF_{p,m+t-p} + GF_{p,m+t} = F_{p,m}\left(GF_{p,t-p+1} + GF_{p,t+1}\right)$$
$$+ \sum_{k=1}^{p} F_{p,m-k}\left(GF_{p,t-p+k} + GF_{p,t-2p+k}\right)$$

and further

$$GF_{p,m+t+1} = F_{p,m}GF_{p,t+2} + F_{p,m-1}\left(GF_{p,t-p+1} + GF_{p,t-2p+1}\right)$$
$$+ F_{p,m-2}\left(GF_{p,t-p+2} + GF_{p,t-2p+2}\right)$$
$$+ F_{p,m-3}\left(GF_{p,t-p+3} + GF_{p,t-2p+3}\right)$$
$$+ \ldots + F_{p,m-p}\left(GF_{p,t} + GF_{p,t-p}\right).$$

By the definition of recurrence relation (2.1)

$$\begin{aligned} GF_{p,m+t+1} &= F_{p,m}GF_{p,t+2} + F_{p,m-1}GF_{p,t-p+2} + F_{p,m-2}GF_{p,t-p+3} \\ &\quad +\ldots + F_{p,m-p}GF_{p,t+1} \\ &= F_{p,m}GF_{p,t+2} + \sum_{k=1}^{p} F_{p,m-k}GF_{p,t-p+k+1}. \end{aligned}$$

□

Corollary 5. [1] *For* $m, n \geq 1$

$$GF_{m+n} = F_m GF_{n+1} + F_{m-1}GF_n.$$

Now we introduce the matrices Q_p and R_p that play the role of the Q-matrix. Let Q_p and R_p denote the $(p+1) \times (p+1)$ matrices defined as

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(p+1)\times(p+1)}$$

and

$$R_p = \begin{bmatrix} 1+i & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & i \\ 1 & 1 & 1 & \cdots & 1 & i & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & i & \cdots & 0 & 0 & 0 \\ 1 & i & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(p+1)\times(p+1)}.$$

Then we can give the following theorem:

Theorem 8. *Let* $n \geq 1$. *Then*

$$Q_p^n R_p = \begin{bmatrix} GF_{p,n+p+1} & GF_{p,n+p} & \cdots & GF_{p,n+2} & GF_{p,n+1} \\ GF_{p,n+p} & GF_{p,n+p-1} & \cdots & GF_{p,n+1} & GF_{p,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ GF_{p,n+2} & GF_{p,n+1} & \cdots & GF_{p,n-p+3} & GF_{p,n-p+2} \\ GF_{p,n+1} & GF_{p,n} & \cdots & GF_{p,n-p+2} & GF_{p,n-p+1} \end{bmatrix}$$

Proof. Theorem can be proved by mathematical induction on n. □

Now we introduce the matrices B and C for finding the sums of Gaussian Fibonacci $p-$numbers. For $p \geq 1$ let $B = (b_{ij})$ denote the $(p+2) \times (p+2)$

matrix such that

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{2.6}$$

or

$$B = \begin{bmatrix} 1 & 0 \quad 0 \quad 0 \quad \ldots \quad 0 \\ 1 & \\ 0 & \\ 0 & Q_p \\ \ldots & \\ 0 & \end{bmatrix},$$

let $C = (c_{ij})$ denote the $(p+2) \times (p+2)$ matrix such that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1+i & 1+i & 1 & 1 & \cdots & 1 & 1 & 1 \\ i & 1 & 1 & 1 & \cdots & 1 & 1 & i \\ 0 & 1 & 1 & 1 & \cdots & 1 & i & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & i & \cdots & 0 & 0 & 0 \\ 0 & 1 & i & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

or

$$C = \begin{bmatrix} 1 & 0 \quad 0 \quad \cdots \quad 0 \quad 0 \\ 1+i & \\ i & \\ 0 & R_p \\ \vdots & \\ 0 & \end{bmatrix}$$

and let E_n also be the $(p+2) \times (p+2)$ square matrix such that

$$E_n = \begin{bmatrix} 1 & 0 \quad 0 \quad \ldots \quad 0 \quad 0 \\ S_n & \\ S_{n-1} & \\ S_{n-2} & Q_p^n \\ \ldots & \\ S_{n-p} & \end{bmatrix} \tag{2.8}$$

where S_n denote sums of Gaussian Fibonacci $p-$numbers defined as

$$S_n = \sum_{k=0}^{n} GF_{p,k}.$$

Then we have the following theorem:

Theorem 9. *Let B, C and E_n be the matrices in (2.6), (2.7) and (2.8) respectively. Then for $n \geq 1$*

$$B^n C = E_n.$$

Proof. By induction method. If $n = 1$, then from the definition of the matrix E_n and Gaussian Fibonacci $p-$numbers,

$$BC = E_1$$

Assume that the theorem holds for $n \geq 1$

$$B^n C = E_n.$$

Then for $n+1$ we have

$$\begin{aligned} B^{n+1}C &= BB^nC = BE_n \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ 0 & & & Q_p & & \\ \ldots & & & & & \\ 0 & & & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ S_n & & & & & \\ S_{n-1} & & & & & \\ S_{n-2} & & & Q_p^n & & \\ \ldots & & & & & \\ S_{n-p} & & & & & \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ S_{n+1} & & & & & \\ S_n & & & & & \\ S_{n-1} & & & Q_p^{n+1} & & \\ \ldots & & & & & \\ S_{n-p+1} & & & & & \end{bmatrix} \\ &= E_{n+1}. \end{aligned}$$

□

Theorem 10. *(Cassini Identity) For $n \geq 0$*

$$
\begin{vmatrix}
GF_{p,n+p+1} & GF_{p,n+p} & \cdots & GF_{p,n+2} & GF_{p,n+1} \\
GF_{p,n+p} & GF_{p,n+p-1} & \cdots & GF_{p,n+1} & GF_{p,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
GF_{p,n+2} & GF_{p,n+1} & \cdots & GF_{p,n-p+3} & GF_{p,n-p+2} \\
GF_{p,n+1} & GF_{p,n} & \cdots & GF_{p,n-p+2} & GF_{p,n-p+1}
\end{vmatrix}
$$

$$
= (-1)^{pn} \begin{vmatrix}
GF_{p,p+1} & GF_{p,p} & \cdots & GF_{p,1} \\
GF_{p,p} & GF_{p,p-1} & \cdots & GF_{p,0} \\
\vdots & \vdots & \vdots & \vdots \\
GF_{p,2} & GF_{p,1} & \cdots & GF_{p,2-p} \\
GF_{p,1} & GF_{p,0} & \cdots & GF_{p,1-p}
\end{vmatrix}.
$$

Corollary 6. [19] *For* $n \geq 0$

$$GF_{n+2}GF_n - GF_{n+1}^2 = (-1)^{n+1}(2-i).$$

3. Conclusion

In this paper we define and study the Gaussian Fibonacci $p-$Numbers and Gaussian Lucas $p-$Numbers. We give generating functions, Binet formulas, explicit formulas, matrix representations and sums of Gaussian Fibonacci $p-$numbers by matrix methods. By defining these Gaussian Fibonacci and Gaussian Lucas $p-$Numbers for special cases for $p = 1$, GF_n is the Gaussian Fibonacci and GL_n is the Gaussian Lucas numbers.

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