The minimum matching energy of tricyclic graphs with given girth and without K_4-subdivision *

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Abstract

The matching energy of a graph was introduced by Gutman and Wagner in 2012 and defined as the sum of the absolute values of zeros of its matching polynomial. In this paper, we completely determine the graph with minimum matching energy in tricyclic graphs with given girth and without K_4-subdivision.

Key words: matching energy; graph energy; tricyclic graph; girth

1 Introduction

All graphs in this paper are finite, connected, simple and undirected. An edge e is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex. A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. For more notations and terminologies that will be used, see [1]. Let $G = (V, E)$ be a graph with order $|V| = n$ and size $|E| = m$. A *matching* in a graph G is a set of pairwise nonadjacent edges. A matching is called *k-matching* if it is of size k. Let $m_k(G)$ denote the number of k-matchings of G, where $m_1(G) = m$ and $m_k(G) = 0$ for $k > \lfloor \frac{n}{2} \rfloor$ or $k < 0$. In addition, define $m_0(G) = 1$. The matching polynomial of graph G is defined as

$$\alpha(G) = \alpha(G, x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k}.$$

*This work is supported by NSFC (11401348) and (11561032) of China, and by China Postdoctoral Science Foundation.

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Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of a graph G. The energy of graph G [8] is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

An important tool of graph energy is the Coulson integral formula [8] (with regard to G be a tree T):

$$E(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k \geq 0} m_k(T) x^{2k} \right] dx. \quad (1)$$

The graph energy has been widely studied by theoretical chemists and mathematicians. For details, see the book on graph energy [20] and surveys [10,12]. There are also some recent results about graph energy, see [22,23].

In 2012, Gutman and Wagner [13] defined the matching energy of a graph G. Let G be a simple graph, and let $\mu_1, \mu_2, \cdots, \mu_n$ be the zeros of its matching polynomial. Then

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Being similar to eq.(1), the matching energy also has a beautiful formula as follows [13]:

$$ME(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k \geq 0} m_k(G) x^{2k} \right] dx. \quad (2)$$

By eq.(2) and the monotonicity of the logarithm function, the matching energy of a graph G is a monotonically increasing function of any $m_k(G)$. This means that if two graphs G and G' satisfy $m_k(G) \leq m_k(G')$ for all $k \geq 1$, then $ME(G) \leq ME(G')$. If, in addition, $m_k(G) < m_k(G')$ for at least one k, then $ME(G) < ME(G')$. It motivates the introduction of a *quasi-order* $\succeq$ as follows: If two graphs G_1 and G_2 have the same order n and size, then

$$G_1 \succeq G_2 \iff m_k(G_1) \geq m_k(G_2) \text{ for } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor.$$

If $G_1 \succeq G_2$ and there exists some k such that $m_k(G_1) > m_k(G_2)$, then we write $G_1 \succ G_2$. If $G_1 \succeq G_2$ we say that G_1 is *m-greater than* G_2, or G_2 is *m-smaller than* G_1. If $G_1 \succeq G_2$ and $G_2 \succeq G_1$, the graphs G_1 and G_2 are said to be *m-equivalent*, denote it by $G_1 \sim G_2$. If $G_1 \succ G_2$ we say that G_1 is *strictly m-greater than* G_2. It is easy to see that $G_1 \succeq G_2 \Longrightarrow ME(G_1) \geq ME(G_2)$ and $G_1 \succ G_2 \Longrightarrow ME(G_1) > ME(G_2)$.

As the research of extremal graph energy is an amusing work, the study on extremal matching energy is also interesting.

A connected simple graph with n vertices and n, $n+1$, $n+2$ edges are called *unicyclic*, *bicyclic*, *tricyclic* graphs, respectively. In [13], the authors gave some elementary results on the matching energy and obtained that $ME(S_n^+) \leq ME(G) \leq ME(C_n)$ for any unicyclic graph G, where S_n^+ is the graph obtained by adding a new edge to the star S_n. In [16], Ji et al. proved that if G is a bicyclic graph with $n \geq 10$ or $n = 8$, $ME(S_n^*) \leq ME(G) \leq ME(P_n^{4,n-4})$. In [15], the authors characterize the connected graphs (and bipartite graphs) of order n having minimum matching energy with m $(n+2 \leq m \leq 2n-4)$ edges. Especially, among all tricyclic graphs of order $n \geq 5$, $ME(G) \geq ME(S_n^{**})$, with equality if and only if $G \cong S_n^{**}$ or $G \cong K_4^{n-4}$. In [3], the tricyclic graph with maximum matching energy is characterized. In [27], the authors characterize the bicyclic graph with given girth having minimum matching energy. For more results about matching energy, see [2, 4–6, 14, 17, 19, 21, 24–26].

Denote by $\mathcal{T}_{n,g}^*$ the set of all connected tricyclic graphs with order n and girth g and do not contain a subdivision of K_4. Clearly, any tricyclic graph in $\mathcal{T}_{n,g}^*$ must contain one of the graphs in Figure 1 as an induced graph, called it a *brace*. The set $\mathcal{T}_{n,g}^*$ can be partitioned into three subsets $\mathcal{T}_{n,g}^1$, $\mathcal{T}_{n,g}^2$ and $\mathcal{T}_{n,g}^3$, where $\mathcal{T}_{n,g}^1$ is the set of all tricyclic graphs which contain a brace of the form (a) or (b) in Figure 1, $\mathcal{T}_{n,g}^2$ is the set of all tricyclic graphs which contain a brace of the form (c) or (d) in Figure 1 and $\mathcal{T}_{n,g}^3$ is the set of all tricyclic graphs which contain a brace of the form (e), (f) or (g) in Figure 1.

Let $P_{r+2}, P_{s+2}, P_{t+2}, P_{q+2}$ be four paths where $P_{r+2} = u_0u_1 \cdots u_{r+1}$, $P_{s+2} = v_0v_1 \cdots v_{s+1}$, $P_{t+2} = w_0w_1 \cdots w_{t+1}$ and $P_{q+2} = x_0x_1 \cdots x_{q+1}$. Let $\beta_1(r,s,t,q)$ be the graph get from $P_{r+2}, P_{s+2}, P_{t+2}, P_{q+2}$ by fusing u_0, v_0, w_0, x_0 to u and fusing $u_{r+1}, v_{s+1}, w_{t+1}, x_{q+1}$ to v. Clearly, $\beta_1(r,s,t,q)$ is the brace of type (e) in Figure 1. Let $T_3 = \beta_1(a,b,b,b)(u)S_{n-a-3b-1}$ where $a = \lfloor \frac{g-2}{2} \rfloor$ and $b = g-2-a$. Clearly, $T_3 \in \mathcal{T}_{n,g}^3$. See Figure 2.

The main result of this paper is the following theorem which gives the graph in $\mathcal{T}_{n,g}^*$ with minimum matching energy.

Theorem 1. *For any graph $G \in \mathcal{T}_{n,g}^*$, $G \succeq T_3$ with equality if and only if $G \cong T_3$.*

2 Preliminaries

We now exhibit some elemental results which will be used in the sequel.

Lemma 2. *[13] Let G be a graph and e one of its edges. Let $G - e$ be the subgraph obtained by deleting from G the edge e, but keeping all the vertices of G. Then*

$$ME(G - e) < ME(G).$$

In [7, 9], two fundamental identities are established as follows.

Lemma 3. *Let G be a graph, then for any edge $e = uv$ and $N(u) = \{v_1(= v), v_2, \cdots, v_t\}$, we have the following two identities:*

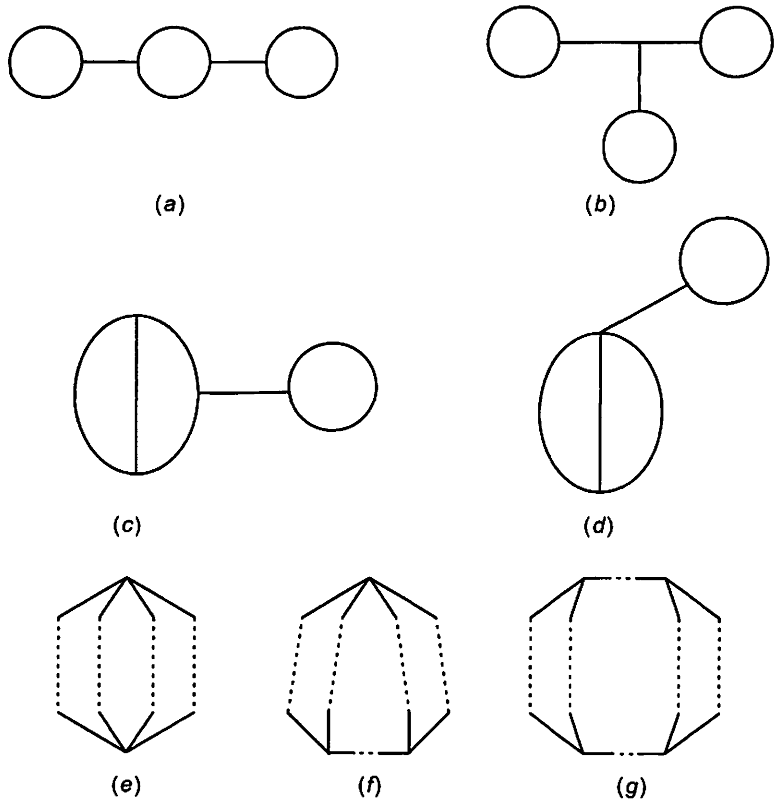

Figure 1: Braces of the graph in $\mathcal{T}^*_{n,g}$.

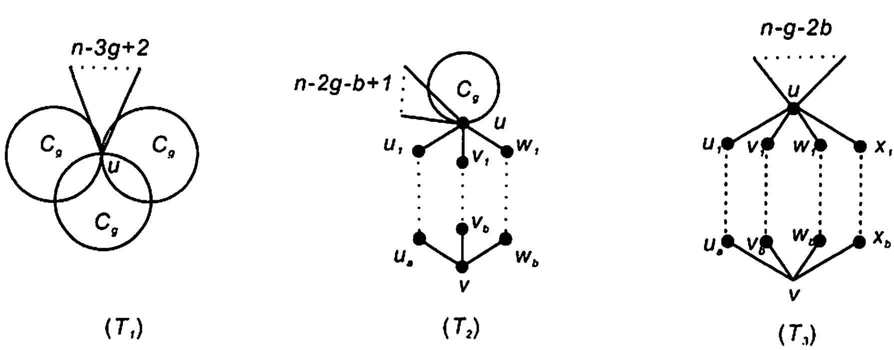

Figure 2: The graphs T_1, T_2 and T_3 where $T_1 \in \mathcal{T}^1_{n,g}$, $T_2 \in \mathcal{T}^2_{n,g}$, $T_3 \in \mathcal{T}^3_{n,g}$, $a = \lfloor \frac{g-2}{2} \rfloor$ and $b = g - 2 - a$.

$$m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v) \tag{3}$$

$$m_k(G) = m_k(G - u) + \sum_{i=1}^{t} m_{k-1}(G - u - v_i). \tag{4}$$

For any G and a star S_{t+1}, let $G(u)S_{t+1}$ denote the graph obtained by identifying the vertex u of G with the center of the star S_{t+1}. For a cycle C, let $G(u)C$ be the graph got from G and C by identifying the vertex u of G with any one vertex of C. It is easy to get the following lemma.

Lemma 4. *Let G be a graph and $u \in V(G)$, then $m_k(G(u)S_{t+1}) = m_k(G) + tm_{k-1}(G - u)$.*

Lemma 5. *[11] If $G_1 \succ G_2$, then $G_1 \cup H \succ G_2 \cup H$, where H is an arbitrary graph.*

Lemma 6. *[13] Suppose that G is a connected graph and T an induced subgraph of G such that T is a tree and T is connected to the rest of G only by a cut vertex v. If T is replaced by a star of the same order, centered at v, then the matching energy decreases (unless T is already such a star). If T is replaced by a path, with one end at v, then the matching energy increases (unless T is already such a path).*

Recall the definition of generalized π-transform in [18]. Say Q is a *branch* of a connected graph G with root u if Q is a connected induced subgraph of G for which u is the only vertex in Q that has a neighbor not in Q. Let P and Q be branches of a component of a graph G with a common root u_0, which is also their only common vertex. Assume that P is a path and u_0 has at least one neighbor in G that does not lie on P or Q. Form a graph from G by relocating the branch Q from u_0 to v where v is the other end vertex of the path P (by deleting edges u_0w and adding new edges vw for every vertex w in Q adjacent to u_0). We refer to the resulting graph as a generalized π-transform of G.

Lemma 7. *[18] If G' is a generalized π-transform of G, then $G' \succ G$ and so $ME(G') > ME(G)$.*

Let G be an arbitrary graph with a specified vertex v. Denote by $\hat{G}_i$ for $i = 1, 2, \cdots, n-1$ the graph obtained from G by adding $n-1$ new vertices to G in the following way: attach at v $i-1$ pendent edges and a path of length $n-i$. It is easy to get the following lemma.

Lemma 8. *[27] Let $\hat{G}_i$ $(i = 1, 2, \cdots, n-1)$ be defined above, then $\hat{G}_1 \succ \hat{G}_2 \succ \cdots \succ \hat{G}_{n-1}$.*

3 The proof of the main results

3.1 The graph in $\mathcal{T}_{n,g}^1$ with minimum matching energy

Let G_1 be a graph shown in Figure 3, where the graph G and the cycle share a common vertex u and there may have some stars rooted on the cycle. The graph G_2, shown in Figure 3, can be got from G_1 by transforming all stars rooted on the cycle to a star rooted on u with the number of vertices not changed.

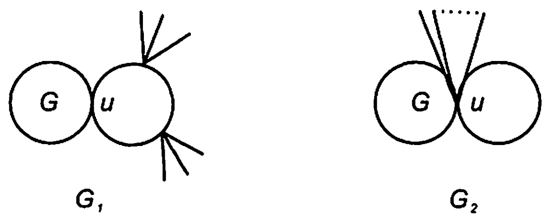

Figure 3: The graph G_1 and G_2.

Lemma 9. *Let G_1 and G_2 be the graph shown in Figure 3, then $G_1 \succeq G_2$ with equality if and only if $G_1 \cong G_2$.*

Proof. Denote the vertices of the cycle in G_1 and G_2 by $u, u_1, u_2, \cdots, u_s$ and $u, u_1', u_2', \cdots, u_s'$ in clockwise respectively. Suppose $G_1 \ncong G_2$, that is, there is at least one star rooted on u_i for some $1 \leq i \leq s$. Suppose there are ℓ stars rooted on the vertices of the cycle in G_1. Without loss of generality, assume that S_{r_i+1} is rooted on u_i where $1 \leq i \leq \ell$. Let $r = \sum_{i=1}^{\ell} r_i$. For convenience, use the notation H_i to denote the graphs defined recursively as follows: let $H_0 = G(u)C_s$; and if H_{i-1} is defined already, then $H_i = H_{i-1}(u_i)S_{r_i+1}$. Clearly, $G_1 = H_\ell$. By Lemma 4,

$$\begin{aligned}
m_k(G_1) &= m_k(H_{\ell-1}) + r_\ell m_{k-1}(H_{\ell-1} - u_\ell) \\
&= m_k(H_{\ell-2}) + r_{\ell-1} m_{k-1}(H_{\ell-2} - u_{\ell-1}) + r_\ell m_{k-1}(H_{\ell-1} - u_\ell) \\
&= \cdots \\
&= m_k(H_0) + \sum_{i=1}^{\ell} r_i m_{k-1}(H_{i-1} - u_i).
\end{aligned}$$

$$m_k(G_2) = m_k(H_0) + r m_{k-1}(H_0 - u).$$

Note that $r = \sum_{i=1}^{\ell} r_i$ and it is not hard to see that $H_{i-1} - u_i \succ H_0 - u_i \succ H_0 - u$ for $i = 1, 2, \cdots, \ell$. The proof completes.

□

Let G_3 be a graph which contains the graph $C_{r_1}(u)C_{r_2}(v)C_{r_3}$, all tree branches of G_3 are stars and all its star branches are rooted on vertices of C_{r_2}. Denote the vertices of C_{r_3} in G_3 by $v, v_1, v_2, \cdots, v_{r_3-1}$ subsequently. G_4 is a graph that can be get from G_3 by deleting the edges vv_1, vv_{r_3-1} and adding the edges uv_1, uv_{r_3-1}. See Figure 4.

Lemma 10. *Let G_3 and G_4 be the graphs defined above, then $G_3 \succ G_4$.*

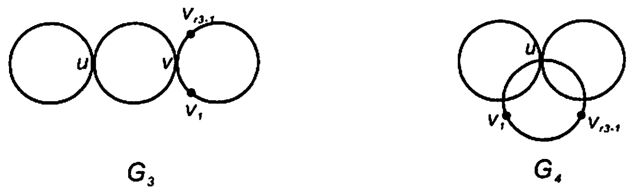

Figure 4: The graph G_3 and G_4.

Proof. By eq.(3),

$$m_k(G_3) = m_k(G_3 - v_1v_2) + m_{k-1}(G_3 - v_1 - v_2)$$

and

$$m_k(G_4) = m_k(G_4 - v_1v_2) + m_{k-1}(G_4 - v_1 - v_2).$$

If $r_3 = 3$, by Lemma 9, $(G_3 - v_1v_2) \succ (G_4 - v_1v_2)$. Now suppose $r_3 \geq 4$. By eq.(3),

$$m_k(G_3 - v_1v_2) = m_k(G_3 - v_1v_2 - v_{r_3-1}v_{r_3-2}) + m_{k-1}(G_3 - v_1v_2 - v_{r_3-1} - v_{r_3-2})$$

and

$$m_k(G_4 - v_1v_2) = m_k(G_4 - v_1v_2 - v_{r_3-1}v_{r_3-2}) + m_{k-1}(G_4 - v_1v_2 - v_{r_3-1} - v_{r_3-2}).$$

By Lemma 9 and Lemma 5, $(G_3 - v_1v_2 - v_{r_3-1}v_{r_3-2}) \succ (G_4 - v_1v_2 - v_{r_3-1}v_{r_3-2})$ and $(G_3 - v_1v_2 - v_{r_3-1} - v_{r_3-2}) \succ (G_4 - v_1v_2 - v_{r_3-1} - v_{r_3-2})$. So $m_k(G_3 - v_1v_2) \geq m_k(G_4 - v_1v_2)$ and the inequality is strict for some k. Similar discussion will prove that $m_{k-1}(G_3 - v_1 - v_2) \geq m_{k-1}(G_4 - v_1 - v_2)$ and the inequality is strict for some k. The proof completes. □

Let G be any graph, $u \in V(G)$ and $r \geq g$. $G_5 = G(u)C_r$ and G_6 can be got from $G(u)C_g$ by attaching $r - g$ pendent edges at u.

Lemma 11. *Let G_5 and G_6 be defined above, then $G_5 \succ G_6$.*

Proof. Denote the vertices of C_r in G_5 by $u, u_1, u_2, \cdots, u_{r-1}$ subsequently and denote the vertices of C_g in G_6 by $u, v_1, v_2, \cdots, v_{g-1}$ subsequently. By eq.(3),

$$m_k(G_5) = m_k(G_5 - u_1u_2) + m_{k-1}(G_5 - u_1 - u_2)$$

and

$$m_k(G_6) = m_k(G_6 - v_1v_2) + m_{k-1}(G_6 - v_1 - v_2).$$

By Lemma 8, $(G_5 - u_1u_2) \succ (G_6 - v_1v_2)$ and $(G_5 - u_1 - u_2) \succ (G_6 - v_1 - v_2)$. The proof completes. □

Let $T_1 \in \mathcal{T}^1_{n,g}$ be the graph in which three cycles of length g share a common vertex u and there are $n - 3g + 2$ pendent edges attached at u. See Figure 2.

Theorem 12. *For any $G \in \mathcal{T}^1_{n,g}$, $G \succeq T_1$ with equality if and only if $G \cong T_1$.*

Proof. If G contains a brace of type (a) in Figure 1. Firstly, by the inverse generalized π-transform and Lemma 6, assume the left and the middle cycles of G share a common vertex, the middle and the right cycles of G share a common vertex, and all tree branches of G are stars. Secondly, by Lemma 9, assume all star branches of G are rooted at vertices of the middle cycle. Lastly, by Lemma 10, Lemma 9 and Lemma 11, $G \succ T_1$.

If G contains a brace of type (b) in Figure 1, assume $G \ncong T_1$. Firstly, by the inverse generalized π-transform and Lemma 6, assume all three cycles of G share a common vertex and all tree branches of G are stars. Then by Lemma 9 and Lemma 11, $G \succ T_1$.

The proof completes.

□

3.2 The graph in $\mathcal{T}^2_{n,g}$ with minimum matching energy

Let $P_{r+2}, P_{s+2}, P_{t+2}$ be three paths where $P_{r+2} = u_0u_1 \cdots u_{r+1}$, $P_{s+2} = v_0v_1 \cdots v_{s+1}$ and $P_{t+2} = w_0w_1 \cdots w_{t+1}$. Let $\theta(r, s, t)$ be the graph got from $P_{r+2}, P_{s+2}, P_{t+2}$ by fusing u_0, v_0, w_0 to u and fusing $u_{r+1}, v_{s+1}, w_{t+1}$ to v.

Lemma 13. *[27] Let G be any bicyclic graph containing $\theta(r, s, t)$ and having n vertices, then $G \succeq \theta(r, s, t)(u)S_{n-r-s-t-1}$ with equality if and only if $G \cong \theta(r, s, t)(u)S_{n-r-s-t-1}$.*

Let G_7 be the graph containing $\theta(r, s, t)(w_i)C_g$ (w_i can be changed to u_i or v_i), having n vertices and all vertices of C_g except w_i having degree 2. Let G_8 be the graph contained $\theta(r, s, t)(u)C_g$ and having $n-r-s-t-g-1$ pendent edges attaching at u. Clearly, $|G_8| = n$.

Lemma 14. *Let G_7 and G_8 be defined above, then $G_7 \succ G_8$.*

Proof. Denote the vertices of C_g in G_7 by $w_i, u'_1, u'_2, \cdots, u'_{g-1}$ subsequently and denote the vertices of C_g in G_8 by $u, u''_1, u''_2, \cdots, u''_{g-1}$ subsequently. By eq.(3),

$$m_k(G_7) = m_k(G_7 - u'_1u'_2) + m_{k-1}(G_7 - u'_1 - u'_2)$$

and

$$m_k(G_8) = m_k(G_8 - u''_1u''_2) + m_{k-1}(G_8 - u''_1 - u''_2).$$

If $g = 3$, by Lemma 13, $(G_7 - u'_1u'_2) \succ (G_8 - u''_1u''_2)$. If $g \geq 4$, by eq.(3),

$$m_k(G_7-u'_1u'_2) = m_k(G_7-u'_1u'_2-u'_{g-1}u'_{g-2})+m_{k-1}(G_7-u'_1u'_2-u'_{g-1}-u'_{g-2})$$

and

$$m_k(G_8-u''_1u''_2) = m_k(G_8-u''_1u''_2-u''_{g-1}u''_{g-2})+m_{k-1}(G_8-u''_1u''_2-u''_{g-1}-u''_{g-2}).$$

By Lemma 13 and Lemma 5, $(G_7 - u_1'u_2' - u_{g-1}'u_{g-2}') \succ (G_8 - u_1''u_2'' - u_{g-1}''u_{g-2}'')$ and $(G_7 - u_1'u_2' - u_{g-1}' - u_{g-2}') \succ (G_8 - u_1''u_2'' - u_{g-1}'' - u_{g-2}'')$. So $m_k(G_7 - u_1'u_2') \geq m_k(G_8 - u_1''u_2'')$ and the inequality is strict for some k.

Similarly, we can prove $m_{k-1}(G_7 - u_1' - u_2') \geq m_{k-1}(G_8 - u_1'' - u_2'')$ and the inequality is strict for some k. This completes the proof.

□

Recall that G_8 is the graph containing $\theta(r,s,t)(u)C_g$ and having $n - r - s - t - g - 1$ pendent edges attaching at u. Suppose now $G_8 \in \mathcal{T}_{n,g}^2$ and $r \leq s \leq t$. Let $T_2 \in \mathcal{T}_{n,g}^2$ be the graph containing $\theta(a,b,b)(u)C_g$ and having $n - a - 2b - g - 1$ pendent edges attaching at u, where $a = \lfloor \frac{g-2}{2} \rfloor$ and $b = g - 2 - a$. See Figure 2.

Lemma 15. *Let $G_8, T_2 \in \mathcal{T}_{n,g}^2$ be defined above, then $G_8 \succeq T_2$ with equality if and only if $G_8 \cong T_2$.*

Proof. Let $H_1 \in \mathcal{T}_{n,g}^2$ be the graph contained $\theta(r,s,s)(u)C_g$ and having $n - r - 2s - g - 1$ pendent edges attaching at u.

Claim 1 If $s < t$, then $G_8 \succ H_1$.

Proof. By eq.(3), we get

$$m_k(G_8) = m_k(G_8 - vw_t) + m_{k-1}(G_8 - v - w_t)$$

and

$$m_k(H_1) = m_k(H_1 - vw_s) + m_{k-1}(H_1 - v - w_s).$$

Because $s < t$, from Lemma 8, $m_k(G_8 - vw_t) \geq m_k(H_1 - vw_s)$ and the inequality is strict for some k, $m_{k-1}(G_8 - v - w_t) \geq m_{k-1}(H_1 - v - w_s)$ and the inequality is strict for some k.

□

Similar discussion like Claim 1, we can assume $r + s + 2 = g$.

If $s - r \geq 2$, let $H_2 \in \mathcal{T}_{n,g}^2$ be the graph containing $\theta(r+1, s-1, s-1)(u)C_g$ and having $n - r - 2s - g$ pendent edges attaching at u.

Claim 2 If $s - r \geq 2$, then $H_1 \succ H_2$.

Proof. By eq.(3),

$$m_k(H_1) = m_k(H_1 - vw_s) + m_{k-1}(H_1 - v - w_s)$$

and

$$m_k(H_2) = m_k(H_2 - vw_{s-1}) + m_{k-1}(H_2 - v - w_{s-1}).$$

From Lemma 8, $m_k(H_1 - vw_s) \geq m_k(H_2 - vw_{s-1})$ and the inequality is strict for some k.

By eq.(3), we get

$$m_{k-1}(H_1 - v - w_s) = m_{k-1}(H_1 - v - w_s - v_s v_{s-1}) + m_{k-2}(H_1 - v - w_s - v_s - v_{s-1})$$

and

$$m_{k-1}(H_2 - v - w_{s-1}) = m_{k-1}(H_2 - v - w_{s-1} - u_{r+1}u_r) + m_{k-2}(H_2 - v - w_{s-1} - u_{r+1} - u_r)$$

From Lemma 8, $m_{k-1}(H_1 - v - w_s - v_s v_{s-1}) \geq m_{k-1}(H_2 - v - w_{s-1} - u_{r+1}u_r)$ and the inequality is strict for some k, $m_{k-2}(H_1 - v - w_s - v_s - v_{s-1}) \geq m_{k-2}(H_2 - v - w_{s-1} - u_{r+1} - u_r)$ and the inequality is strict for some k.

□

From Claim 1 and Claim 2, the proof of the lemma completes.

□

Recall that G_8 is the graph containing $\theta(r,s,t)(u)C_g$ and having $n-r-s-t-g-1$ pendent edges attaching at u. Let G_9 be the graph contained $\theta(r,s,t)(u)C_g$, having n vertices and all vertices of C_g except u having degree 2. Similar to the proof of Lemma 13 in [27], the following lemma can be got.

Lemma 16. *Let G_8 and G_9 be defined above, then $G_9 \succeq G_8$ with equality if and only if $G_9 \cong G_8$.*

Theorem 17. *For any $G \in \mathcal{T}^2_{n,g}$, $G \succeq T_2$ with equality if and only if $G \cong T_2$.*

Proof. The brace of G is of type (c) or (d) in Figure 1. Suppose the brace of G contains a bicyclic graph $\theta(r,s,t)$ for some r,s,t and a cycle C. By Lemma 6, assume all tree branches of G are stars. By inverse generalized π-transform, assume $\theta(r,s,t)$ and the cycle C share a common vertex v_0 in G. By Lemma 9, assume that all the vertices of the cycle C except v_0 are of degree 2. By Lemma 11, assume that the length of the cycle C in G is g.

If the common vertex v_0 of $\theta(r,s,t)$ and the cycle C_g is u_i (or v_i, w_i for some i), by Lemma 14 and Lemma 15, $G \succ T_2$.

If the common vertex v_0 of $\theta(r,s,t)$ and the cycle C_g is u (or v), by Lemma 16 and Lemma 15, $G \succeq T_2$ with equality if and only if $G \cong T_2$.

□

3.3 The graph in $\mathcal{T}^3_{n,g}$ with minimum matching energy

Recall that $\beta_1(r,s,t,q)$ has been introduced in Section 1. Suppose $r \leq s \leq t \leq q$ without loss of generality. Let $G_{10} = \beta_1(r,s,t,q)(u)S_{n-r-s-t-q-1}$. Clearly, $|G_{10}| = n$.

Lemma 18. *Let $G \in \mathcal{T}^3_{n,g}$ be a graph which contains $\beta_1(r,s,t,q)$ as a brace, then $G \succeq G_{10}$ with equality if and only if $G \cong G_{10}$.*

Proof. First prove the following claim.

Claim: $\beta_1(r,s,t,q)$ with a vertex of degree 2 deleted is strictly m-greater than $\beta_1(r,s,t,q)$ with a vertex of degree 4 deleted.

Proof. Choose a vertex of degree two and a vertex of degree four in $\beta_1(r,s,t,q)$, say x_m and u. Let $H = \beta_1(r,s,t,q)$. First suppose $m = 1$.

If $q = 1$, by Lemma 8, $H - x_1 \succ H - x_1 - uu_1 - uv_1 \succ H - u$.

If $q \geq 2$, by eq.(3),

$$m_k(H - x_1) = m_k(H - x_1 - vx_q) + m_{k-1}(H - x_1 - v - x_q)$$

and

$$m_k(H - u) = m_k(H - u - x_q x_{q-1}) + m_{k-1}(H - u - x_q - x_{q-1}).$$

Note that $H - x_1 - v - x_q \cong H - u - x_q - x_{q-1}$, so $m_{k-1}(H - x_1 - v - x_q) = m_{k-1}(H - u - x_q - x_{q-1})$. By the above discussion when $q = 1$, we know $H - x_1 - vx_q \succ H - u - x_q x_{q-1}$.

When $m = q$, the proof is similar to the case $m = 1$. Next suppose $1 < m < q$. By eq.(3),

$$m_k(H - x_m) = m_k(H - x_m - ux_1) + m_{k-1}(H - x_m - u - x_1)$$

and

$$m_k(H - u) = m_k(H - u - x_{m-1}x_m) + m_{k-1}(H - u - x_{m-1} - x_m).$$

Note that $(H - x_m - u - x_1) \cong (H - u - x_{m-1} - x_m)$, so $m_{k-1}(H - x_m - u - x_1) = m_{k-1}(H - u - x_{m-1} - x_m)$.

By eq.(3),

$$m_k(H - x_m - ux_1) = m_k(H - x_m - ux_1 - vx_q) + m_{k-1}(H - x_m - ux_1 - v - x_q)$$

and

$$\begin{aligned} m_k(H - u - x_{m-1}x_m) = & m_k(H - u - x_{m-1}x_m - x_q x_{q-1}) \\ & + m_{k-1}(H - u - x_{m-1}x_m - x_q - x_{q-1}). \end{aligned}$$

Note that $(H - x_m - ux_1 - v - x_q) \cong (H - u - x_{m-1}x_m - x_q - x_{q-1})$, so $m_{k-1}(H - x_m - ux_1 - v - x_q) = m_{k-1}(H - u - x_{m-1}x_m - x_q - x_{q-1})$.

After deleting two edges uu_1, uv_1 of $H - x_m - ux_1 - vx_q$, by Lemma 5 and Lemma 8, $m_k(H - x_m - ux_1 - vx_q - uu_1 - uv_1) \geq m_k(H - u - x_{m-1}x_m - x_q x_{q-1})$ and the inequality is strict for some k. So $m_k(H - x_m - ux_1 - vx_q) \geq m_k(H - u - x_{m-1}x_m - x_q x_{q-1})$ and the inequality is strict for some k. The proof of the Claim completes. □

Next prove the lemma. Suppose $G \ncong G_{10}$.

By Lemma 6, assume that all tree branches at the cycles of G are stars. Without loss of generality, suppose that G is the coalescence of the vertex t_i (new notations for these vertices) in $\beta_1(r,s,t,q)$ and the center of S_{r_i+1} for $i = 1, 2, \cdots, \ell$, and $\sum_{i=1}^{\ell} r_i = n - r - s - t - q - 2$. For convenience, use the notation H_i to denote graphs defined recursively as follows: let $H_0 = \beta_1(r,s,t,q)$; and if H_{i-1} is defined already, then $H_i = H_{i-1}(t_i)S_{r_i+1}$. Note that $H_\ell = G$.

By Lemma 4,

$$
\begin{aligned}
m_k(G) &= m_k(H_{\ell-1}) + r_\ell m_{k-1}(H_{\ell-1} - t_\ell) \\
&= m_k(H_{\ell-2}) + r_{\ell-1} m_{k-1}(H_{\ell-2} - t_{\ell-1}) + r_\ell m_{k-1}(H_{\ell-1} - t_\ell) \\
&= \cdots \\
&= m_k(H_0) + \sum_{i=1}^{\ell} r_i m_{k-1}(H_{i-1} - t_i),
\end{aligned}
$$

and

$$m_k(G_{10}) = m_k(H_0) + (n - r - s - t - q - 2) m_{k-1}(H_0 - u).$$

For $i \geq 2$, $H_0 - t_i$ is a proper subgraph of $H_{i-1} - t_i$, so $H_{i-1} - t_i \succ H_0 - t_i$. By the Claim, $H_0 - t_i \succ H_0 - u$ when t_i is of degree 2 in H_0. Because $G \ncong G_{10}$, $\ell \geq 2$, or $\ell = 1$ and t_1 is of degree 2 in $\beta_1(r, s, t, q)$. The proof completes.

□

Suppose $r \leq s \leq t \leq q$ and $r + s + 2 = g$ without loss of generality. Let $G_{10} = \beta_1(r, s, t, q)(u) S_{n-r-s-t-q-1}$, $G_{11} = \beta_1(r, s, s, s)(u) S_{n-r-3s-1}$. Clearly, $G_{10}, G_{11} \in \mathcal{T}_{n,g}^3$. Similar proof like Claim 1 in Lemma 15, the following lemma can be got.

Lemma 19. *Let G_{10} and G_{11} be defined above, then $G_{10} \succeq G_{11}$ with equality if and only if $G_{10} \cong G_{11}$.*

Recall that $T_3 = \beta_1(a, b, b, b)(u) S_{n-a-3b-1}$ where $a = \lfloor \frac{g-2}{2} \rfloor$ and $b = g - 2 - a$, see Figure 2. Clearly, $T_3 \in \mathcal{T}_{n,g}^3$.

Lemma 20. *Let G_{11} and T_3 be defined above, then $G_{11} \succeq T_3$ with equality if and only if $G_{11} \cong T_3$.*

Proof. Let $H = \beta_1(r+1, s-1, s-1, s-1)(u) S_{n-r-3s+1}$. Just need to prove the following claim.

Claim: If $s - r \geq 2$, then $G_{11} \succ H$.

Proof. By eq.(3),

$$m_k(G_{11}) = m_k(G_{11} - vx_s) + m_{k-1}(G_{11} - v - x_s)$$

and

$$m_k(H) = m_k(H - vx_{s-1}) + m_{k-1}(H - v - x_{s-1}).$$

First compare $m_k(G_{11} - vx_s)$ with $m_k(H - vx_{s-1})$. By eq.(3),

$$m_k(G_{11} - vx_s) = m_k(G_{11} - vx_s - vw_s) + m_{k-1}(G_{11} - vx_s - v - w_s)$$

and

$$m_k(H - vx_{s-1}) = m_k(H - vx_{s-1} - vw_{s-1}) + m_{k-1}(H - vx_{s-1} - v - w_{s-1}).$$

By Lemma 8, $(G_{11}-vx_s-vw_s) \succ (H-vx_{s-1}-vw_{s-1})$, so $m_k(G_{11}-vx_s-vw_s) \geq m_k(H-vx_{s-1}-vw_{s-1})$ and the inequality is strict for some k.

By eq.(3),

$$\begin{aligned} m_{k-1}(G_{11}-vx_s-v-w_s) =& m_{k-1}(G_{11}-vx_s-v-w_s-v_{s-1}v_s) \\ &+ m_{k-2}(G_{11}-vx_s-v-w_s-v_{s-1}-v_s) \end{aligned}$$

and

$$\begin{aligned} m_{k-1}(H-vx_{s-1}-v-w_{s-1}) =& m_{k-1}(H-vx_{s-1}-v-w_{s-1}-u_ru_{r+1}) \\ &+ m_{k-2}(H-vx_{s-1}-v-w_{s-1}-u_r-u_{r+1}). \end{aligned}$$

By Lemma 8, $m_{k-1}(G_{11}-vx_s-v-w_s-v_{s-1}v_s) \geq m_{k-1}(H-vx_{s-1}-v-w_{s-1}-u_ru_{r+1})$ and inequality is strict for some k, and $m_{k-2}(G_{11}-vx_s-v-w_s-v_{s-1}-v_s) \geq m_{k-2}(H-vx_{s-1}-v-w_{s-1}-u_r-u_{r+1})$ and the inequality is strict for some k.

From the above discussion we know that $m_k(G_{11}-vx_s) \geq m_k(H-vx_{s-1})$ and the inequality is strict for some k.

Next compare $m_{k-1}(G_{11}-v-x_s)$ with $m_{k-1}(H-v-x_{s-1})$. By eq.(3),

$$\begin{aligned} m_{k-1}(G_{11}-v-x_s) =& m_{k-1}(G_{11}-v-x_s-v_{s-1}v_s) \\ &+ m_{k-2}(G_{11}-v-x_s-v_{s-1}-v_s) \end{aligned}$$

and

$$\begin{aligned} m_{k-1}(H-v-x_{s-1}) =& m_{k-1}(H-v-x_{s-1}-u_ru_{r+1}) \\ &+ m_{k-2}(H-v-x_{s-1}-u_r-u_{r+1}). \end{aligned}$$

By Lemma 8, $m_{k-1}(G_{11}-v-x_s-v_{s-1}v_s) \geq m_{k-1}(H-v-x_{s-1}-u_ru_{r+1})$ and the inequality is strict for some k, and $m_{k-2}(G_{11}-v-x_s-v_{s-1}-v_s) \geq m_{k-2}(H-v-x_{s-1}-u_r-u_{r+1})$ and the inequality is strict for some k.

From the above discussion, $m_{k-1}(G_{11}-v-x_s) \geq m_{k-1}(H-v-x_{s-1})$ and the inequality is strict for some k.

The proof of the Claim completes.

□

Using the Claim several times as needed, the proof of the lemma completes.

□

From Lemma 18, Lemma 19 and Lemma 20, the following theorem can be got.

Theorem 21. *Let $r \leq s \leq t \leq q$ and $r+s+2=g$. For any graph G in $\mathcal{T}^3_{n,g}$ with $\beta_1(r,s,t,q)$ as its brace, then $G \succeq T_3$ with equality if and only if $G \cong T_3$.*

Let $P_{r+2}, P_{s+2}, P_{t+2}, P_{q+2}, P_{q_1+2}$ be five paths where $P_{r+2} = u_0u_1\cdots u_{r+1}$, $P_{s+2} = v_0v_1\cdots v_{s+1}$, $P_{t+2} = w_0w_1\cdots w_{t+1}$, $P_{q+2} = x_0x_1\cdots x_{q+1}$, $P_{q_1+2} = y_0y_1\cdots y_{q_1+1}$. Let $\beta_2(r,s,t,q,q_1)$ be the graph got from $P_{r+2}, P_{s+2}, P_{t+2}, P_{q+2}, P_{q_1+2}$ by fusing u_0, v_0, w_0, x_0 to u, fusing u_{r+1}, v_{s+1}, y_0 to v_1' and fusing $w_{t+1}, x_{q+1}, y_{q_1+1}$ to v_2'. Clearly, $\beta_2(r,s,t,q,q_1)$ is the brace of type (f) in Figure 1. Let $T_3' = \beta_2(a,b,a,b,0)(u)S_{n-2a-2b-2}$ where $a = \lfloor\frac{g-2}{2}\rfloor$ and $b = g-2-a$. Obviously $T_3' \in \mathcal{T}_{n,g}^3$.

Let $P_{q_2+2} = z_1z_2\cdots z_{q_2+1}$. Let $\beta_3(r,s,t,q,q_1,q_2)$ be the graph got from $P_{r+2}, P_{s+2}, P_{t+2}, P_{q+2}, P_{q_1+2}, P_{q_2+2}$ by fusing u_0, v_0, y_0 to u, fusing w_0, x_0, y_{q_1+1} to v, fusing u_{r+1}, v_{s+1}, z_0 to v_1' and fusing $w_{t+1}, x_{q+1}, z_{q_2+1}$ to v_2'. Clearly, $\beta_3(r,s,t,q,q_1,q_2)$ is the brace of type (g) in Figure 1. Let $T_3'' = \beta_3(a,b,a,b,0,0)(u)S_{n-2a-2b-3}$ where $a = \lfloor\frac{g-2}{2}\rfloor$ and $b = g-2-a$. Obviously $T_3'' \in \mathcal{T}_{n,g}^3$.

Similar discussion like the proof of Theorem 21 can give the following theorem.

Theorem 22. *(1)Let $G \in \mathcal{T}_{n,g}^3$ with $\beta_2(r,s,t,q,q_1)$ as its brace, then $G \succeq T_3'$ with equality if and only if $G \cong T_3'$.*
(2)Let $G \in \mathcal{T}_{n,g}^3$ with $\beta_3(r,s,t,q,q_1,q_2)$ as its brace, then $G \succeq T_3''$ with equality if and only if $G \cong T_3''$.

Theorem 23. *Let $T_3, T_3', T_3'' \in \mathcal{T}_{n,g}^3$ and be defined above, then $T_3 \prec T_3' \prec T_3''$.*

Proof. First prove $T_3 \prec T_3'$. By eq.(3),

$$m_k(T_3) = m_k(T_3 - vx_b) + m_{k-1}(T_3 - v - x_b)$$

and

$$m_k(T_3') = m_k(T_3' - v_2'x_b) + m_{k-1}(T_3' - v_2' - x_b).$$

Note that $T_3 - v - x_b$ is a proper subgraph of $T_3' - v_2' - x_b$, so $(T_3 - v - x_b) \prec (T_3' - v_2' - x_b)$.

It is obvious that $a = b-1$ or $a = b$. When $a = b-1$, $(T_3 - vx_b) \cong (T_3' - v_2'x_b)$; when $a = b$, similar discussion like in Lemma 15, we know $(T_3 - vx_b) \prec (T_3' - v_2'x_b)$.

Next prove $T_3' \prec T_3''$. By eq.(3),

$$m_k(T_3') = m_k(T_3' - v_1'v_2') + m_{k-1}(T_3' - v_1' - v_2')$$

and

$$m_k(T_3'') = m_k(T_3'' - v_1'v_2') + m_{k-1}(T_3'' - v_1' - v_2').$$

From the inverse generalized π-transform, we know that $(T_3' - v_1'v_2') \prec (T_3'' - v_1'v_2')$ and $(T_3' - v_1' - v_2') \prec (T_3'' - v_1' - v_2')$.

□

Combining Theorem 21, Theorem 22, Theorem 23, the following theorem can be got.

Theorem 24. *For any $G \in \mathcal{T}^3_{n,g}$, $G \succeq T_3$ with equality if and only if $G \cong T_3$.*

Theorem 25. *Let $T_1 \in \mathcal{T}^1_{n,g}$, $T_2 \in \mathcal{T}^2_{n,g}$, $T_3 \in \mathcal{T}^3_{n,g}$ be defined previously, then $T_1 \succ T_2 \succ T_3$.*

Proof. First prove $T_1 \succ T_2$. Suppose the vertices of one C_g in T_1 is denoted by $u, x_1, x_2, \cdots, x_{g-1}$. By eq.(3),

$$m_k(T_1) = m_k(T_1 - x_1x_2) + m_{k-1}(T_1 - x_1 - x_2)$$

and

$$m_k(T_2) = m_k(T_2 - vw_b) + m_{k-1}(T_2 - v - w_b).$$

When $g = 3$, $(T_1 - x_1x_2) \cong (T_2 - vw_b)$; when $g \geq 3$, by Lemma 8, $(T_1 - x_1x_2) \succ (T_2 - vw_b)$.

By Lemma 8 and Lemma 2, $(T_1 - x_1 - x_2) \succ (T_2 - v - w_b)$.

Next prove $T_2 \succ T_3$. Suppose the vertices of C_g in T_2 is denoted by $u, y_1, y_2, \cdots, y_{g-1}$. By eq.(3),

$$m_k(T_2) = m_k(T_2 - y_1y_2) + m_{k-1}(T_2 - y_1 - y_2)$$

and

$$m_k(T_3) = m_k(T_3 - vx_b) + m_{k-1}(T_3 - v - x_b).$$

When $g = 3$, $(T_2 - y_1y_2) \cong (T_3 - vx_b)$; when $g \geq 3$, by Lemma 8, $(T_2 - y_1y_2) \succ (T_3 - vx_b)$.

By Lemma 8 and Lemma 2, $(T_2 - y_1 - y_2) \succ (T_3 - v - x_b)$. □

Combing Theorem 12, Theorem 17, Theorem 24 and Theorem 25, the main result Theorem 1 can be got.

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