The majorization theorem and signless Dirichlet spectral radius of connected graphs *

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Abstract

*Supported by the National Natural Science Foundation of China (No.11526140; No.11571235), Shanghai Natural Science Foundation of China (No.16ZR1422400), Excellent Teacher's Program of Shanghai Municipal Education Commission (No. ZZshjr15027) and partially supported by the Program of Shanghai Finance University (No. SHFUKT15-13).

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ARS COMBINATORIA 132(2017), pp. 49-58

Let $D(G)$ denote the signless Dirichlet spectral radius of the graph G with at least a pendant vertex, and π_1 and π_2 be two nonincreasing unicyclic graphic degree sequences with the same frequency of number 1.

In this paper, the signless Dirichlet spectral radius of connected graphs with a given degree sequence is studied. The results are used to prove a majorization theorem of unicyclic graphs. We prove that if $\pi_1 \trianglelefteq \pi_2$, then $D(G_1) \leq D(G_2)$ with equality if and only if $\pi_1 = \pi_2$, where G_1 and G_2 are the graphs with the largest signless Dirichlet spectral radius among all unicyclic graphs with degree sequences π_1 and π_2, respectively. Moreover, the graphs with the largest signless Dirichlet spectral radius among all unicyclic graphs with k pendant vertices are characterized.

Key words: Signless Dirichlet spectral radius; Degree sequence; Unicyclic graph; Majorization.

AMS Classifications: 05C50.

1 Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$, $N(u)$ and $d(u)$ denote the neighborhood and degree of u, respectively. A non-increasing positive integers sequence $\pi = (d_0, d_1, \cdots, d_{n-1})$ is called the *graphic degree sequence* if there exists a simple graph with degree sequence π. The signless Laplacian matrix of G is defined as $Q(G) = D(G)+A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. Let $\pi_1 = (d_0, d_1, \cdots, d_{n-1})$ and $\pi_2 = (d'_0, d'_1, \cdots, d'_{n-1})$ be two nonincreasing positive sequences. If $\sum_{i=0}^{t} d_i \leq \sum_{i=0}^{t} d'_i$ for $t = 0, 1, \cdots, n-2$ and $\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d'_i$, then π_2 is said to *majorize* π_1, and is denoted by $\pi_1 \trianglelefteq \pi_2$ (see [8]). Let

$\lambda(G)$ and $\mu(G)$ denote adjacent spectral radius and the signless Laplacian spectral radius of G, respectively. Bıyıkoğlu et al. in [1] and Zhang in [8] considered the relation between the maximal spectral radius and signless Laplacian spectral radius of the graphs in two classes of trees with given degree sequences, respectively. They proved the following result:

Theorem 1.1 *([1],[8]) Let π_1 and π_2 be degree sequences of trees on n vertices. Assume that G_1 and G_2 are the graphs with the maximal signless Laplacian spectral radius (resp. spectral radius) among all trees with given degree sequences π_1 and π_2, respectively. If $\pi_1 \trianglelefteq \pi_2$, then $\mu(G_1) \leq \mu(G_2)$ (resp. $\lambda(G_1) \leq \lambda(G_2)$) with equality holds if and only if $\pi_1 = \pi_2$.*

Such a theorem is called *the majorization theorem.* Let Ω_π be the set of unicyclic graphs with given degree sequence π. In the sequel, Liu et al. in [4] and Zhang in [9] proved the majorization theorems for unicyclic graph.

Theorem 1.2 *([4],[9]) Let π_1 and π_2 be unicyclic graphic degree sequences on n vertices. Assume that G_1 and G_2 are the graphs with the maximal signless Laplacian spectral radius (resp. spectral radius) in Ω_{π_1} and Ω_{π_2}, respectively. If $\pi_1 \trianglelefteq \pi_2$, then $\mu(G_1) \leq \mu(G_2)$ (resp. $\lambda(G_1) \leq \lambda(G_2)$) with equality holds if and only if $\pi_1 = \pi_2$.*

Subsequently, Liu et al. in [5] proved that the majorization theorem hlods for pseudographs. Let G be a simple connected graph with pendant vertex set ∂V and nonpendant vertex set V_0. In this paper, we assume that ∂V is not empty. The signless Dirichlet eigenvalue is a real number λ such that there exists a function $f \neq 0$ on $V(G)$ such that

$$\begin{cases} Q(G)f(u) = \lambda f(u) & u \in V_0, \\ f(u) = 0 & u \in \partial V. \end{cases}$$

The signless Dirichlet spectral radius $D(G)$ is the largest signless Dirichlet eigenvalue (see [7]). Zhang et al. in [7] proved a majorization theorem for trees concerning signless Dirichlet spectral radius.

Theorem 1.3 *([7]) Let π_1 and π_2 be two tree degree sequences such that they have same frequency of the number 1. If $\pi_1 \trianglelefteq \pi_2$, then $D(T_1) \leq D(T_2)$ with equality holds if and only if $\pi_1 = \pi_2$, where T_1 and T_2 are the graphs with the maximal signless Dirichlet spectral radius among all trees with given degree sequences π_1 and π_2, respectively.*

Let $\mathcal{G}$ be the set of graphs with at least a pendant vertex. G is called an *extremal* graph in $\mathcal{G}$, if G has the maximal signless Dirichlet spectral radius in $\mathcal{G}$. Motivated by the above results, we will show that the majorization theorem holds for unicyclic graphs, and our main result can be stated as follows:

Theorem 1.4 *Let π_1 and π_2 be two unicyclic degree sequences such that they have same frequency of the number 1. If $\pi_1 \trianglelefteq \pi_2$, then $D(G_1) \leq D(G_2)$ with equality holds if and only if $\pi_1 = \pi_2$, where G_1 and G_2 are the extremal graphs in Ω_{π_1} and Ω_{π_2}, respectively.*

2 The signless Dirichlet spectral radius of connected graphs

In this section, we will study signless Dirichlet spectral radius of connected graphs with same degree sequence. Denote by

$$\Delta_G(f) = \frac{< Qf, f >}{< f, f >} = \frac{\sum\limits_{uv\in E}(f(u) + f(v))^2}{\sum\limits_{v\in V} f^2(v)}$$

the Rayleigh quotient of signless Laplacian matrix. Then we have

Lemma 2.1 *([7]) Let G be a graph such that ∂V is not empty. Denote by S the set of all real-valued functions f on $V(G)$ with $f(u) = 0$ for any $u \in \partial V$. Then*

(1)$D(G) = \max_{f \in S} \Delta_G(f)$. Moreover, if $\Delta_G(f) = \lambda(G)$ for a function $f \in S$, then f is an eigenfunction of $\lambda(G)$.

(2) $D(G) > 0$. Moreover, if f is an eigenfunction of $D(G)$, then $f(v) > 0$ for all $v \in V_0(G)$ or $f(v) < 0$ for all $v \in V_0(G)$.

In [7], the unit eigenvector f of $D(G)$ is called a *Dirichlet Perron vector* of G if $f(v) > 0$ for all $v \in V_0(G)$. Denote by $G - uv$ the graph obtained from G by deleting an edge uv in G and by $G + uv$ the graph obtained from G by adding an edge uv.

Lemma 2.2 *([7]) Let G be a graph such that ∂V is not empty. Assume $u, v, x \in V_0$ and $y \in V(G)$ such that $uv, xy \in E(G)$ and $ux, yv \notin E(G)$. Let f be the Dirichlet Perron vector of G and $G' = G - uv - xy + ux + yv$. Then $D(G') \geq D(G)$ if $f(u) \geq f(y)$ and $f(x) \geq f(v)$. Moreover, $D(G') > D(G)$ if one of the two inequalities is strict.*

Lemma 2.3 *([7]) Let G be a graph such that ∂V is not empty, and P be a path from a non-pendant vertex v_1 to another non-pendant vertex v_2. Suppose that $v_1u_i \in E(G)$, $v_2u_i \notin E(G)$ and u_i is not on the path P for $i = 1, 2, \cdots, t$ with $t \leq d(v_1) - 2$. By deleting the t edges $v_1u_1, v_1u_2, \cdots, v_1u_t$ and adding the t edges $v_2u_1, v_2u_2, \cdots, v_2u_t$ we get a new graph G'. Let f be the Dirichlet Perron vector of G. Then if $f(v_1) \leq f(v_2)$, we have*

$$D(G') > D(G).$$

Let $\mathcal{F}_\pi$ be a set of connected graphs with same degree sequence π, where the frequency of the number 1 in π is at least 1. It is easy to see that the following corollary holds.

Corollary 2.4 *Let G be an extremal graph in $\mathcal{F}_\pi$ and f be the Dirichlet Perron vector of G. If $x, y \in V(G)$ and $f(x) \geq f(y)$, then $d(x) \geq d(y)$.*

Lemma 2.5 *Let $G \in \mathcal{F}_\pi$ and f be the Dirichlet Perron vector of G. If there are three vertices $x, y, z \in V(G)$ such that $xy \in E(G)$, $xz \notin E(G)$, $f(x) \geq f(v)$ for all $v \in N(z)$ and $f(x) \geq f(z) > f(y)$, G is not the extremal graph in $\mathcal{F}_\pi$.*

Proof. Assume that G is the extremal graph in $\mathcal{F}_\pi$. Since $f(z) > f(y) \geq 0$, z is not a pendant vertex and has at least two neighbors. Furthermore, $d(z) \geq d(y)$ by corollary 2.4. There must exist a vertex $w_1 \in N(z)$ such that $w_1 y \notin E(G)$ and $w_1 \neq y$. Otherwise, every vertex $v \in N(z)$ with $v \neq y$ is adjacent to y. We have $N(z) \setminus \{y\} \subseteq N(y) \setminus \{x\}$, which implies $d(y) > d(z)$,a contradiction.

Let P be a path from x to z. Then there exists $w_2 \in V(P) \cap N(z)$. We consider two cases.

Case 1: $y \notin V(P)$. Let $G_1 = G - xy - zw_2 + xz + yw_2$ if $w_2 y \notin E(G)$ or $G_1 = G - xy - zw_1 + xz + yw_1$ if $yw_2 \in E(G)$. Since $f(x) \geq f(z) > f(y)$, $f(x) \geq f(w_1)$ and $f(x) \geq f(w_2)$, we have $G_1 \in \mathcal{F}_\pi$ and $D(G_1) > D(G)$ by Lemma 2.2. It is a contradiction to our assume that G is the extremal graph in $\mathcal{F}_\pi$.

Case 2: $y \in V(P)$. If $N(z) \setminus \{w_2\} \subseteq N(y)$, then $y \neq w_2$ and $yw_2 \notin E(G)$ by the above discussion. Let $G_2 = G - xy - zw_2 + xz + yw_2$. Since $f(z) > f(y)$ and $f(x) \geq f(w_2)$, we have $G_2 \in \mathcal{F}_\pi$ and $D(G_2) > D(G)$ by Lemma 2.2. It is a contradiction to our assume that G is the extremal graph in $\mathcal{F}_\pi$. Now assume that there exists a vertex w_3 such that $w_3 \in N(z) \setminus \{w_2\}$ and $w_3 \notin N(y)$. Let $G_3 = G - xy - zw_3 + yw_3 + xz$. Since $f(z) > f(y)$ and $f(x) \geq f(w_3)$, we have $G_3 \in \mathcal{F}_\pi$ and $D(G_3) > D(G)$ by Lemma 2.2, a contradiction. So G is not the extremal graph in $\mathcal{F}_\pi$. ■

Theorem 2.6 *Let G be an extremal graph in $\mathcal{F}_\pi$ and f be the Dirichlet Perron vector of G. Then the vertices of G can be relabeled $\{v_0, v_1, \cdots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq f(v_2) \geq \cdots \geq f(v_{n-1})$, $d(v_0) \geq d(v_1) \geq d(v_2) \geq \cdots \geq d(v_{n-1})$, and $dist(v_0) \leq dist(v_1) \leq dist(v_2) \leq \cdots \leq dist(v_{n-1})$, where $dist(v_i)$ is the distance between v_i and v_0.*

Proof. Assume that $V(G) = \{v_0, v_1, \cdots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq f(v_2) \geq \cdots \geq f(v_{n-1})$. Then by Corollary 2.4, we have $d(v_0) \geq d(v_1) \geq d(v_2) \geq \cdots \geq d(v_{n-1})$. Furthermore $d(v_i) = d(v_j)$ if $f(v_i) = f(v_j)$. In the following, we prove $dist(v_0) \leq dist(v_1) \leq dist(v_2) \leq \cdots \leq dist(v_{n-1})$ by induction. Clearly, $dist(v_0) \leq dist(v_1)$. Assume that $dist(v_0) \leq dist(v_1) \leq dist(v_2) \leq \cdots \leq dist(v_i)$. We will prove that $dist(v_i) \leq dist(v_{i+1})$. Let $V_1 = \{v_0, v_1, \cdots, v_i\}$ and $V_2 = \{v_{i+1}, v_{i+2}, \cdots, v_{n-1}\}$. Note that G is connected. There exists the smallest integer $s \in \{0, 1, 2, \cdots, i\}$ such that $N(v_s) \cap V_2$ is not an empty set. Without loss of generality, assume $f(v_i) > f(v_{i+1})$. In the following we prove $dist(v_i) \leq dist(v_{i+1})$.

First, we prove $dist(v_i) \leq dist(v_s) + 1$. Otherwise, we have $dist(v_i) > dist(v_s) + 1$. Clearly, $v_i \neq v_s$ and $v_i \notin N(v_s)$. Let $v_p \in V_2 \cap N(v_s)$. Then there are three vertices v_p, v_i, v_s such that $v_s v_p \in E(G)$ and $f(v_s) \geq f(v_i) > f(v_{i+1}) \geq f(v_p)$. Let v_q be any vertex in $N(v_i)$. If $v_q \in V_2$, we have $f(v_s) > f(v_{i+1}) \geq f(v_q)$. If $v_q \in V_1$, we have $dist(v_s) < dist(v_i) - 1 \leq dist(v_q) + 1 - 1 = dist(v_q)$. Thus $s < q$ such that $f(v_s) \geq f(v_q)$. So we have $f(v_s) \geq f(v_q)$ for any $v_q \in N(v_i)$. Then G is not an extremal graph in $\mathcal{F}_\pi$ by Lemma 2.5, a contradiction. So $dist(v_i) \leq dist(v_s) + 1$. Let P be the shortest path from v_0 to v_{i+1} and v_t be the last vertex which belongs to V_1 on the path P. Note that $N(v_t) \cap V_2$ is not an empty set. So $dist(v_s) \leq dist(v_t)$. We have $dist(v_{i+1}) \geq dist(v_t) + 1 \geq dist(v_s) + 1 \geq dist(v_i)$. ■

3 The majorization theorem for unicyclic graph

In this section, we will give the majorization theorem for unicyclic graph involving signless Dirichlet spectral radius.

Lemma 3.1 *([8]) Let π_1 and π_2 be two nonincreasing graphic degree sequences, where $\pi_1 = (d_0, d_1, \cdots, d_{n-1}), \pi_2 = (d'_0, d'_1, \cdots, d'_{n-1})$. If $\pi_1 \trianglelefteq \pi_2$, then there exist a series of graphic degree* *sequences $\pi'_1, \pi'_2, \cdots, \pi'_s$ such that $\pi_1 \trianglelefteq \pi'_1 \trianglelefteq \pi'_2 \trianglelefteq \cdots \trianglelefteq \pi'_s \trianglelefteq \pi_2$, and only two components of π_i and π_{i+1} are different by 1.*

Proof of Theorem 1.4: Let G_1 and G_2 be the extremal graphs in Ω_{π_1} and Ω_{π_2}, respectively. Assume that f is the Dirichlet Perron vector of G_1. Then by Theorem 2.6, the vertices of G_1 can be relabeled $\{v_0, v_1, \cdots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq f(v_2) \geq \cdots \geq f(v_{n-1})$, $d(v_0) \geq d(v_1) \geq d(v_2) \geq \cdots \geq d(v_{n-1})$, and $dist(v_0) \leq dist(v_1) \leq dist(v_2) \leq \cdots \leq dist(v_{n-1})$. By Lemma 3.1, without loss of generality, assume $\pi_1 = (d_0, d_1, \cdots, d_{n-1})$ and $\pi_2 = (d'_0, d'_1, \cdots, d'_{n-1})$ such that $d_r = d'_r - 1$, $d_s = d'_s + 1$ with $0 \leq r < s \leq n-1$, and $d_m = d'_m$ for $m \neq r, s$. Note that π_1 and π_2 have the same frequency of the number 1. We have $d_s = d'_s + 1 \geq 2 + 1 = 3$. Since G_1 is a unicyclic graph, there exists a vertex v_t which is not in any path from v_r to v_s such that $v_t \in N(v_s)$. Let $G = G_1 - v_s v_t + v_r v_t$. Since $f(v_r) \geq f(v_s)$, we have $G \in \Omega_{\pi_2}$ and $D(G_1) < D(G) \leq D(G_2)$ by Lemma 2.3. ■

Let U_k be the unicyclic graph obtained from C_3 by attaching k paths of almost equal lengths at one vertex, where C_3 is a cyclic of length 3. Then we have the following result

Lemma 3.2 *Let $\nu = (k, 2, \cdots, 2, 1, \cdots, 1)$ such that the frequency of the number 1 is k. Then U_k is the only extremal graph in Ω_ν.*

Proof. Let G be an extremal graph in Ω_ν and f be the Dirichlet Perron vector of G. Assume that C is the cyclic of G. Then by Theorem 2.6, the vertices of G can be relabeled $\{v_0, v_1, \cdots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq f(v_2) \geq \cdots \geq f(v_{n-1})$, $d(v_0) \geq d(v_1) \geq d(v_2) \geq \cdots \geq d(v_{n-1})$, and $dist(v_0) \leq dist(v_1) \leq dist(v_2) \leq \cdots \leq dist(v_{n-1})$. So we have $d(v_0) = k$, $d(v_1) = d(v_2) = \cdots = d(v_{n-k-1}) = 2$ and $d(v_{n-k}) = \cdots = d(v_{n-1}) = 1$. Clearly, $v_0 \in V(C)$.

Claim: $f(x) > f(u)$ for any $x \in V(C)$ and $u \in V(G) \setminus V(C)$.

Let $xy \in E(C)$. Without loss of generality, assume $x \neq v_0, y \neq v_0$ and $u \notin \partial V$. Let $u z_1 z_2 \cdots z_m$ be the path such that

$d(u) = d(z_1) = d(z_2) = \cdots = d(z_{m-1}) = 2$ and $d(z_m) = 1$. Since $f(x) > f(z_m)$ and $f(y) > f(z_m)$, we have $min\{f(x), f(y)\} > f(z_{m-1})$. Otherwise, let $G_1 = G - z_{m-1}z_m - xy + z_{m-1}x + z_m y$ if $f(y) \le f(z_{m-1})$ or $G_1 = G - z_{m-1}z_m - xy + z_{m-1}y + z_m x$ if $f(x) \le f(z_{m-1})$. Then $G_1 \in \Omega_\nu$ and $D(G_1) > D(G)$ by Lemma 2.2. It is a contradiction to our assume that G is the extremal graph in Ω_ν. If $min\{f(x), f(y)\} \le f(z_{m-2})$, let $G_2 = G - z_{m-2}z_{m-1} - xy + z_{m-2}x + z_{m-1}y$ if $f(y) \le f(z_{m-2})$ or $G_2 = G - z_{m-2}z_{m-1} - xy + z_{m-2}y + z_{m-1}x$ if $f(x) \le f(z_{m-2})$. Then $G_2 \in \Omega_\nu$ and $D(G_2) > D(G)$ by Lemma 2.2, a contradiction. So $min\{f(x), f(y)\} > f(z_{m-2})$. By repeating the similar discussion as above, we have $f(x) \ge min\{f(x), f(y)\} > f(u)$.

By the claim, we have $v_0, v_1, v_2 \in V(C)$. If $v_1v_2 \notin E(C)$, there exists $v_s \in V(C)$ and $v_t \notin V(C)$ such that $v_1v_s \in E(C)$ and $v_t \in N(v_0)$. Clearly, $v_s \notin N(v_0)$ and $f(v_s) > f(v_t)$ by the claim. Let $G' = G - v_0v_t - v_1v_s + v_0v_s + v_1v_t$. Then $G' \in \Omega_\nu$ and $D(G') > D(G)$ by Lemma 2.2, a contradiction. So $v_1v_2 \in E(C)$. Since $dist(v_0) \le dist(v_1) \le dist(v_2) \le \cdots \le dist(v_{n-1})$, we have $|dist(w) - dist(z)| \le 1$ for any $w, z \in \partial V$. So $G = U_k$ and U_k is the only extremal graph in Ω_ν. ■

Theorem 3.3 *Let G be a unicyclic graph with k pendant vertices. Then $D(G) \ge D(U_k)$ with equality holds if and only if $G = U_k$.*

Proof. Let π be the degree sequence of G. Clearly, $\pi \trianglelefteq \nu$. Then we have $D(G) \ge D(U_k)$ with equality holds if and only if $G = U_k$ by Theorem 1.4 and Lemma 3.2. ■

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