NBB bases of some lattices of pattern avoiding permutations

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Abstract

In this paper we will determine the NBB bases with respect to a certain standard ordering of atoms of lattices of $321-312-231$-avoiding permutations and of 321-avoiding permutations with the weak Bruhat order. Using our expressions of NBB bases we will calculate the Möbius numbers of these lattices. These values are shown to be related to Fibonacci polynomials.

1 Introduction

In this paper we give expressions of NBB bases of the lattices of 321-avoiding permutations and of $321-312-231$-avoiding permutations with added maximums and calculate the Möbius numbers of these lattices. These values are shown to be related to Fibonacci polynomials which we will introduce in Section 2.

Let $\widehat{S_n}(321-312-231)$ (resp. $\widehat{S_n}(321)$) be the partially ordered set of the $321-312-231$-avoiding (resp. 321-avoiding) permutations with the weak Bruhat order with an added maximum for $n \in \mathbb{N}$. We will determine the NBB bases for $\widehat{S_n}(321-312-231)$ and $\widehat{S_n}(321)$ with respect to a natural total ordering of their atoms. Using some modified Fibonacci polynomials and the expressions of the NBB bases we will determine the Möbius numbers of $\widehat{S_n}(321-312-231)$ and $\widehat{S_n}(321)$.

Let P be a poset and $Int(P)$ be the set of intervals of P. The function $\mu : Int(P) \to \mathbb{Z}$ is called the *Möbius function* if μ satisfies $\sum_{x\leq y\leq z} \mu([x,y]) = \delta_{x,z}$. If P has a maximum element $\widehat{1}$ and a minimum element $\widehat{0}$, we set $\mu(P) := \mu([\widehat{0},\widehat{1}])$. We call $\mu(P)$ the *Möbius number* of P.

Our main result is the following.

Theorem 1.1. *For $n \in \mathbb{N}, n \geq 3$, we have $\mu(\widehat{S_n}(321-312-231)) = \mu(\widehat{S_n}(321)) = F_{n-2}(-1)$ where $F_n(q)$ is a modified Fibonacci polynomial which will be defined in Section 2.3.*

2 Preliminaries

2.1 Bounded Below Sets

In this subsection we introduce a tool to calculate Möbius numbers of lattices which is given in Blass and Sagan's paper [2]. Throughout this subsection L will denote a finite lattice.

We will use $\vee$ for the join (least upper bound) and $\wedge$ for the meet (greatest lower bound) in L. Since L is finite it also has minimum $\widehat{0}$ and maximum $\widehat{1}$. Set $\mu(L) := \mu([\widehat{0}, \widehat{1}])$. We give a combinatorial description of $\mu(L)$. Let $A(L)$ be the set of atoms of L. Endow $A(L)$ with an arbitrary total order which we denote $\trianglelefteq_A$ to distinguish it from $\leq$ in L. In Blass and Sagan's paper the authors give an arbitrary partial order to $A(L)$ but in this paper only the case of a total order will be considered. A nonempty set $D \subset A(L)$ is a *bounded below set* (*BB set* for short) if for every $d \in D$ there is an $a \in A(L)$ such that $a \triangleleft_A d$ and $a < \vee D$. Hence $a \in A(L)$ is simultaneously a strict lower bound for d in the total order $\trianglelefteq_A$ and for $\vee D$ in $\leq$. We will say that $B \subset A(L)$ is an *NBB set* if B does not contain any bounded below set D. In particular an NBB set is not a BB set. We will call B an *NBB base* for $\widehat{1}$ if $\vee B = \widehat{1}$ and B is an NBB set.

The following theorem is a special case of Blass and Sagan's result [2].

Theorem 2.1. *Let $A(L)$ be the set of atoms of a finite lattice L and $\trianglelefteq_A$ be a total order on $A(L)$. Then we have*

$$\mu(L) = \sum_B (-1)^{|B|} \tag{1}$$

where the sum is over all NBB bases of $\widehat{1}$ and $|\cdot|$ denotes cardinality.

Lemma 2.1. *Let $A(L)$ be the set of atoms of a finite lattice L with a total order $\trianglelefteq$ and $X \subset A(L)$ be an NBB base for $\widehat{1}$. Let $a \in A(L)$ be the minimum with respect to $\trianglelefteq$. Then we have $a \in X$.*

Proof. If $a \notin X$ then X is a BB set since for any $x \in X$ we have $a \triangleleft x$ and $x < \vee X = \widehat{1}$. This contradicts the assumption that X is a NBB set. □

2.2 Weak Bruhat Order

In this subsection we will introduce the weak Bruhat order [4]. Let σ be an element of the permutation group S_n, for $n \in \mathbb{N}$. We set $Inv(\sigma) :=$

$\{(i,j)|1 \le i < j \le n, \sigma(i) > \sigma(j)\}$. We write $\sigma \le \tau$ if and only if $Inv(\sigma) \subseteq Inv(\tau)$. This defines the *weak Bruhat order*. In particular the weak Bruhat order is a lattice.

Remark 2.1. For permutations $\sigma(1)\sigma(2)\cdots\sigma(n), \tau(1)\tau(2)\cdots\tau(n) \in S_n$, if $\tau(1)\tau(2)\cdots\tau(n)$ covers $\sigma(1)\sigma(2)\cdots\sigma(n)$ in the weak Bruhat order then we have

1. $\sigma(k) = \tau(k)$ for $k \ne i, j$,
2. $\sigma(i) = \tau(j) = \sigma(j) - 1 = \tau(i) - 1$

for some $1 \le i < j \le n$. This is a straightforward consequence of the definition of the weak Bruhat order. Hence we omit the proof.

2.3 Modified Fibonacci Polynomials

In this subsection we introduce some modified Fibonacci polynomials.

Definition 2.1. *Define the sequences $\{F_n(q)\}_{n\in\mathrm{N}}$ by the following recurrence relation*

1. $F_1(q) = 1$ *and* $F_2(q) = 1$,
2. $F_{k+2}(q) = F_{k+1}(q) + qF_k(q)$ *for* $k \ge 1$.

We call $\{F_n(q)\}_{n\in\mathrm{N}}$ modified Fibonacci polynomials.

The *Fibonacci polynomials* $\{F'_n(q)\}_{n\in\mathrm{N}}$ are defined by the recurrence relation $F'_{k+2}(q) = qF'_{k+1}(q)+F'_k(q)$ for $k \ge 1$ with $F'_1(q) = 1$ and $F'_2(q) = q$ [5]. It is easy to see that $F'_k(q) = q^{k-1}F_k(q^{-2})$.

The right sparse subsets of $[n]$ are the subsets which contain no pair of adjacent elements and which do not contain 1. There are 5 right sparse subsets ϕ, $\{2\}$, $\{3\}, \{4\}, \{2,4\}$ of $[4]$.

A simple calculation yields the following proposition.

Proposition 2.1. *We have*

$$\sum_{X:\text{right sparse set of } [n-1]} q^{|X|} = F_n(q) \tag{2}$$

for $n \in \mathbb{N}$.

3 The case of $321 - 312 - 231$-avoiding permutations

Let $S_n(321, 312, 231)$ be the set of $321 - 312 - 231$-avoiding permutations in S_n. Let $V = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ be any subset of positive integers. The *standardization* of a permutation π on V is the permutation $st(\pi)$ on $[n]$ obtained from π by replacing the letter a_i with the letter i. For example $st(5248) = 3124$.

Lemma 3.1. *For a permutation $\sigma(1)\sigma(2)\cdots\sigma(n) \in S_n(321, 312, 231)$, if $\sigma(i) = \sigma(j) + 1$ for some $1 \leq i < j \leq n$, then we have $j = i + 1$.*

Proof. If $j \geq i + 2$, we have $\sigma(i + 1) \leq \sigma(j) - 1$ or $\sigma(i + 1) \geq \sigma(j) + 2$. In the case of $\sigma(i + 1) \leq \sigma(j) - 1$, we have $st(\sigma(i)\sigma(i + 1)\sigma(j)) = 312$ and this contradicts the assumption that $\sigma(1)\sigma(2)\cdots\sigma(n)$ is an element of $S_n(321, 312, 231)$. In the case of $\sigma(i + 1) \geq \sigma(j) + 2$, we have $st(\sigma(i)\sigma(i + 1)\sigma(j)) = 231$ and this contradicts the assumption that $\sigma(1)\sigma(2)\cdots\sigma(n)$ is an element of $S_n(321, 312, 231)$. Hence we have $j = i + 1$. □

Recall that an order ideal of a poset P is a subset I such that if $x \in I$ and $y \leq x$ then $y \in I$.

Proposition 3.1. *The poset $S_n(321, 312, 231)$ is an order ideal in the weak Bruhat order.*

Proof. For permutations $\tau = \tau(1)\tau(2)\cdots\tau(n) \in S_n(321, 312, 231)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in S_n$, we show that if $\sigma \prec \tau$ then $\sigma \in S_n(321, 312, 231)$.

By Remark 2.1 and Lemma 3.1, we have

1. $\sigma(k) = \tau(k)$ for $k \neq i, i + 1$,
2. $\sigma(i) = m, \sigma(i + 1) = m + 1, \tau(i) = m + 1$ and $\tau(i + 1) = m$

for some $1 \leq i \leq n$ and $1 \leq m \leq n - 1$.

Thus τ and σ coincide everywhere except for the pair of adjacent entries at positions i and $i+1$, which are $m, m+1$ in σ and $m+1, m$ in τ. Clearly the only patterns in σ which possibly do not appear in τ must involve the two elements m and $m + 1$.

Now, it is evident that there cannot be any occurrence of 321 in σ involving m or $m + 1$, since otherwise the same pattern would appear in τ.

Analogous arguments show that an occurrence of 312 in σ involving m or $m + 1$ would imply an occurrence of 321 in τ. Hence σ does not contain a 312 pattern.

Similarly an occurrence of 231 in σ involving m or $m+1$ would imply an occurrence of a 321 pattern in τ. Hence σ does not contain a 231 pattern.

Then we can conclude $\sigma \in S_n(321, 312, 231)$ and therefore $S_n(321, 312, 231)$ is an order ideal in weak Bruhat order.

□

Definition 3.1. *Let $\widehat{S_n}(321-312-231)$ be the partially ordered set of $321-312-231$-avoiding permutations of length n with the weak Bruhat order with an added maximum for $n \in \mathbb{N}$. We denote the added maximum by $\hat{1}$.*

Our poset $\widehat{S_n}(321-312-231)$ is a lattice because the set of $321-312-231$-avoiding permutations is an order ideal of the weak Bruhat order. We denote the join operator of $\widehat{S_n}(321-312-231)$ (resp. the weak Bruhat order) by $\vee_{\{321,312,231\}}$ (resp. $\vee$).

For permutations $\sigma, \tau \in S_n(321, 312, 231)$ $\sigma \vee_{\{321,312,231\}} \tau = \hat{1}$ if $\sigma \vee \tau \notin S_n(321, 312, 231)$ and $\sigma \vee_{\{321,312,231\}} \tau = \sigma \vee \tau$ if $\sigma \vee \tau \in S_n(321, 312, 231)$.

We denote the adjacent transposition that interchanges i and $i+1$ by σ_i. The set of adjacent transpositions $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ is the set of atoms of $\widehat{S_n}(321-312-231)$.

Since the join of σ_i and σ_{i+1} in the weak Bruhat order has a 321-pattern, we can state the following Lemma.

Lemma 3.2. *We have $\sigma_i \vee_{\{321,312,231\}} \sigma_{i+1} = \hat{1}$ for $1 \leq i \leq n-2$.*

We give a total order $\lhd_{\{321,312,231\}}$ as follows,

$$\sigma_1 \lhd_{\{321,312,231\}} \sigma_2 \lhd_{\{321,312,231\}} \cdots \lhd_{\{321,312,231\}} \sigma_{n-1}. \tag{3}$$

Lemma 3.3. *For $1 \leq j_1 < j_2 < \cdots < j_l \leq n-1$ with $j_{p+1} - j_p \geq 2$, we have $\vee_{\{321,312,231\}}\{\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_l}\} = \sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_l}$.*

Proof. The join of $\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_l}$ in the weak Bruhat order is $\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_l}$ since $\sigma_{j_p}\sigma_{j_q} = \sigma_{j_q}\sigma_{j_p}$ for $p, q \in \{j_1, j_2, \ldots j_l\}$. Also we have $\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_l} = 12\cdots(j_1-1)(j_1+1)j_1(j_1+2)\cdots(j_2-1)(j_2+1)j_2(j_1+2)\cdots(j_l-1)(j_l+1)j_l(j_l+2)\cdots n$. Hence the permutation $\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_l}$ has no 321-patterns, 231-patterns and 312-patterns. □

Proposition 3.2. *Let $\{\sigma_{i_1}, \sigma_{i_2}, \ldots \sigma_{i_k}\}$ be a subset of the atoms of $\widehat{S_n}(321-312-231)$ with $i_1 < i_2 < \cdots < i_k$. Then $\{\sigma_{i_1}, \sigma_{i_2}, \ldots \sigma_{i_k}\}$ is an NBB base of $\hat{1}$ if and only if*

1. *$i_1 = 1, i_2 = 2$,*

2. *$i_p + 2 \leq i_{p+1}$ for $2 \leq p \leq k-1$.*

Proof. ($\Rightarrow$)

We have $i_1 = 1$ by Lemma 2.1 since σ_1 is the minimum with respect to $\trianglelefteq$. If $\vee_{\{321,312,231\}}\{\sigma_{i_2}, \sigma_{i_3}, \ldots, \sigma_{i_k}\} = \widehat{1}$ then the set $\{\sigma_{i_2}, \sigma_{i_3}, \ldots, \sigma_{i_k}\}$ is a BB set since $\sigma_1 < \widehat{1}$ and $\sigma_1 \notin \{\sigma_{i_2}, \sigma_{i_3}, \ldots, \sigma_{i_k}\}$. This contradicts the fact that $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$ is an NBB base. Hence we have

$\vee_{\{321,312,231\}}\{\sigma_{i_2}, \sigma_{i_3}, \ldots, \sigma_{i_k}\} \neq \widehat{1}$. From Lemma 3.2 we have $i_p + 2 \leq i_{p+1}$ for $2 \leq p \leq k-1$. If $i_2 \geq 3$, then we would have $\vee_{\{321,312,231\}}\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\} = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ by Lemma 3.3, and this contradicts the assumption that the join of $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$ is $\widehat{1}$. Hence we have $i_2 = 2$.

($\Leftarrow$)

We have $\vee_{\{321,312,231\}}\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\} = \widehat{1}$ since $i_1 = 1, i_2 = 2$ and $\sigma_1 \vee \sigma_2$ has a 321-pattern. For $1 \leq p_1 < p_2 < \cdots < p_l \leq k$, we have to show that $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \ldots, \sigma_{i_{p_l}}\}(\subset \{\sigma_1, \sigma_2, \ldots, \sigma_k\})$ is not a BB set. If $p_1 = 1$, $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \ldots, \sigma_{i_{p_l}}\}$ is not a BB set since σ_1 is the minimum with respect to $\triangleleft$.

If $p_1 \geq 2$, then $\vee_{\{321,312,231\}}\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \ldots, \sigma_{i_{p_l}}\} = \sigma_{i_{p_1}}\sigma_{i_{p_2}}\cdots\sigma_{i_{p_l}}$ by Lemma 3.3. Since $\sigma_p < \vee_{\{321,312,231\}}\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \ldots, \sigma_{i_{p_l}}\}$ if and only if $p \in \{i_{p_1}, i_{p_2}, \ldots, i_{p_l}\}$, the set $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \ldots, \sigma_{i_{p_l}}\}$ is not a BB set. □

Theorem 3.1. $\mu(\widehat{S_n}(321-312-231)) = F_{n-2}(-1)$

Proof. From Theorem 2.1 and Proposition 3.2 we have $\mu(\widehat{S_n}(321-312-231)) = \sum_X (-1)^{|X|}$ where the sum is over $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\} \subset \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ with

1. $i_1 = 1, i_2 = 2$,
2. $i_p + 2 \leq i_{p+1}$ for $2 \leq p \leq k-1$.

Since $i_1 = 1$ and $i_2 = 2$ we have $\mu(\widehat{S_n}(321-312-231)) = (-1)^2 \times \sum_Y (-1)^{|Y|}$ where the sum is over right sparse sets of $[n-2]$. □

4 The case of 321-avoiding permutations

Let $S_n(321)$ be the set of 321-avoiding permutations in S_n.

Proposition 4.1. *The poset $S_n(321)$ is an order ideal in the weak Bruhat order.*

Proof. It is sufficient to show that if σ has a 321-pattern with $\sigma \prec \tau$ in the weak Bruhat order then τ has a 321-pattern. For $\sigma, \tau \in S_n$ with $\sigma \prec \tau$ there exist $1 \leq i < j \leq n$ such that

1. $\sigma(k) = \tau(k)$ for $k \neq i, j$,

2. $\sigma(i) = \tau(j) = \sigma(j) - 1 = \tau(i) - 1$.

from Remark 2.1. Set $\sigma(i) = \tau(j) = m$. If σ contains an occurrence of 321 not involving its elements of index i and j, then obviously the same occurrence appears in τ.

Otherwise, suppose that $\sigma(i) = m$ is involved in an occurrence of 321 in σ. If such an occurrence has elements preceding $\sigma(i)$, then such elements must be greater than $m+1$, analogously, if such an occurrence has elements following $\sigma(i)$, then such elements must be smaller than m. Thus the same occurrence must appear in τ.

A similar argument can be used if $\sigma(j) = m + 1$ is involved in an occurrence of 321. Therefore we can conclude that $S_n(321)$ is an order ideal.

□

Therefore the set of permutations which contain no 321-patterns is an order ideal in the weak Bruhat order.

Definition 4.1. *Let $\widehat{S_n}(321)$ be the partially ordered set of 321-avoiding permutations with the weak Bruhat order with an added maximum for $n \in \mathbb{N}$. We denote the added maximum by $\widehat{1}$.*

Our poset $\widehat{S_n}(321)$ is a lattice because the set of 321-avoiding permutations is an order ideal of the weak Bruhat order. We denote the join operator in $\widehat{S_n}(321)$ (resp. the weak Bruhat order) by $\vee_{\{321\}}$ (resp. $\vee$). For permutations $\sigma, \tau \in S_n(321)$, $\sigma \vee_{\{321\}} \tau = \widehat{1}$ if $\sigma \vee \tau$ has a 321-pattern and $\sigma \vee_{\{321\}} \tau = \sigma \vee \tau$ if $\sigma \vee \tau$ has no 321-patterns.

Lemma 4.1. *We have $\sigma_i \vee_{321} \sigma_{i+1} = \widehat{1}$ for $1 \leq i \leq n-2$.*

Proof. The join of σ_i and σ_{i+1} in the weak Bruhat order has a 321-pattern.

□

We give a total order $\lhd_{321}$ on $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ as follows:

$$\sigma_1 \lhd_{321} \sigma_2 \lhd_{321} \cdots \lhd_{321} \sigma_{n-1}. \quad (4)$$

A proof analogous to that of Lemma 3.3 yields the following result.

Lemma 4.2. *For $1 \leq j_1 < j_2 < \cdots < j_l \leq n-1$ with $j_{p+1} - j_p \geq 2$, we have $\vee_{321}\{\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_l}\} = \sigma_{j_1}\sigma_{j_2} \cdots \sigma_{j_l}$.*

Proposition 4.2. *Let $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$ be a subset of the atoms of $\widehat{S_n}(321)$ with $i_1 < i_2 < \cdots < i_k$. Then $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$ is an NBB base of $\widehat{1}$ if and only if*

1. $i_1 = 1, i_2 = 2,$

2. $i_p + 2 \leq i_{p+1}$ *for* $2 \leq p \leq k-1$.

Proof. ($\Rightarrow$)

We have $i_1 = 1$ by Lemma 2.1 since σ_1 is the minimum with respect to $\trianglelefteq$. If $\vee_{321}\{\sigma_{i_2}, \sigma_{i_3}, \dots \sigma_{i_k}\} = \widehat{1}$ then $\{\sigma_{i_2}, \sigma_{i_3}, \dots, \sigma_{i_k}\}$ is a BB set since $\sigma_1 < \widehat{1}$ and $\sigma_1 \notin \{\sigma_{i_2}, \sigma_{i_3}, \dots, \sigma_{i_k}\}$. This contradicts the fact that $\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}\}$ is an NBB base. Hence we have $\vee_{321}\{\sigma_{i_2}, \sigma_{i_3}, \dots, \sigma_{i_k}\} \neq \widehat{1}$. From Lemma 4.1 we have $i_p + 2 \leq i_{p+1}$ for $2 \leq p \leq k-1$. If $i_2 \geq 3$, then we would have $\vee_{321}\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}\} = \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_k} \neq \widehat{1}$ by Lemma 4.2. This contradicts the assumption that the join of $\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}\}$ is $\widehat{1}$. Hence we have $i_2 = 2$.

($\Leftarrow$)

We have $\vee_{321}\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}\} = \widehat{1}$ since $i_1 = 1, i_2 = 2$ and $\sigma_1 \vee \sigma_2$ has a 321-pattern. For $1 \leq p_1 < p_2 < \cdots < p_l \leq k$, we have to show that $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \dots, \sigma_{i_{p_l}}\}$ is not a BB set. If $p_1 = 1$, $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \dots, \sigma_{i_{p_l}}\}$ is not a BB set because σ_1 is the minimum with respect to $\triangleleft$.

If $p_1 \geq 2$, then $\vee_{321}\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \dots, \sigma_{i_{p_l}}\} = \sigma_{i_{p_1}}\sigma_{i_{p_2}} \cdots \sigma_{i_{p_l}}$ by Lemma 4.2. Hence $\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \dots, \sigma_{i_{p_l}}\}$ is not a BB set,
since $\sigma_p < \vee_{321}\{\sigma_{i_{p_1}}, \sigma_{i_{p_2}}, \dots, \sigma_{i_{p_l}}\}$ if and only if $p \in \{i_{p_1}, i_{p_2}, \dots, i_{p_l}\}$. □

An argument analogous to that of previous section then yields the following result.

Theorem 4.1. $\mu(\widehat{S_n}(321)) = F_{n-2}(-1)$

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