Toughness and $[a, b]$-factors with prescribed properties*

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Abstract

In this paper, we consider the relationship between toughness and the existence of $[a, b]$-factors with inclusion/exclusion properties. We obtain that if $t(G) \geq a - 1 + \frac{a-1}{b}$ with $b > a > 2$ where a, b are two integers, then for any two given edges e_1 and e_2, there exist an $[a, b]$-factor including e_1, e_2; and an $[a, b]$-factor including e_1 and excluding e_2; as well as an $[a, b]$-factor excluding e_1, e_2. Furthermore, it is shown that the results are best possible in some sense.

Keywords: $[a, b]$-factor; toughness; inclusion/exclusion properties

1 Introduction

All graphs considered are simple and finite. We refer the reader to [2] for terminologies and notations not defined here.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, let $N_G(S)$ denote the union of $N_G(x)$ for every $x \in S$. We use $G[S]$ and $G - S$ to denote the subgraph induced by S and $V(G) - S$.

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A subset $S \subseteq V(G)$ is called an independent set(a covering set) if every edge of G is incident with at most(at least) one vertex of S. For any disjoint subsets S, $T \subseteq V(G)$, $E_G(S,T)$ denotes the set of edges with one end in S and the other in T and $e_G(S,T) = |E_G(S,T)|$.

Let $f : V(G) \to N$ be an integer function. For any subset $X \subseteq V(G)$, we denote $f(X) = \sum_{x \in X} f(x)$ and $f(\emptyset) = 0$. A spanning subgraph F of G is called an f-factor of G satisfying $d_F(x) = f(x)$ for any $x \in V(G)$. When $f(x) = k$ for all $x \in V(G)$, F is called a k-factor. Let g and f be two integer-valued functions defined on $V(G)$ with $g(x) \le f(x)$ for any $x \in V(G)$. A *(g, f)-factor* of G is a spanning subgraph F satisfying $g(x) \le d_F(x) \le f(x)$ for any $x \in V(G)$. F is called an *$[a, b]$-factor* if $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$.

Chvátal [7] first introduced the concept of *toughness*, $t(G)$, denoted by

$$t(G) = min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \ge 2\},$$

where $\omega(G - S)$ denotes the number of components of $G - S$ and G is not a complete graph. If G is complete, then $t(G) = \infty$. A graph G is *k-tough* if $t(G) \ge k$.

Chvátal mainly studied the relationship between toughness and the existence of Hamilton cycles and k-factors. He conjectured that every k-tough graph G has a k-factor if $k|V(G)|$ is even(k is a positive integer).

Enomoto et al.[8] confirmed Chvátal's conjecture and showed that the result is sharp.

Theorem 1.1. ([*8*]) *Let G be a graph. If G is k-tough, $|V(G)| \ge k+1$ and $k|V(G)|$ is even, then G has a k-factor.*

Theorem 1.2. ([*8*]) *Let G be a graph with $|V(G)| \ge k+1$ and $k|V(G)|$ is even. For any positive number ε, there exists a $(k-\varepsilon)$-tough graph G which has no k-factors.*

Chen [4] improved Theorem 1.1 by considering k-factors which contain a specified edge or exclude a specified edge under the similar conditions.

Theorem 1.3. ([*4*]) *Let G be a graph and $k \ge 2$. If $t(G) \ge k$ and $k|V(G)|$ is even, then for every edge e of G, there exists a k-factor which contains the given edge e, and there also exists a k-factor which does not contain e.*

Katerinis and Wang [11] further extended Theorem 1.1 by considering the existence of 2-factors in terms of toughness with inclusion/exclusion properties involved two edges.

Theorem 1.4. ([*11*]) *Let G be a 2-tough graph with at least 5 vertices and let e_1, e_2 be a pair of arbitrarily given edges of G. Then*

(a) there exists a 2-factor in G containing e_1, e_2;

(b) there exists a 2-factor in G avoiding e_1, e_2;

(c) there exists a 2-factor in G containing e_1 and avoiding e_2.

As a generalization of Chvátal's conjecture, Katerinis [10] studied the relationship between toughness and the existence of f-factors, as well as $[a,b]$-factor.

Theorem 1.5. ([10]) *Let G be a graph of order n and a, b be two positive integers with $b \geq a$. If $t(G) \geq a - 1 + \frac{a}{b}$ and $a|V(G)| \equiv 0 \pmod 2$ when $a = b$, then G has an $[a,b]$-factor.*

When $a = 2$, Chen [5] obtained a stronger result.

Theorem 1.6. ([5]) *Let G be a graph of order at least 3 and $b > 2$. If $t(G) \geq 1 + \frac{1}{b}$, then G has a $[2,b]$-factor.*

Since the toughness condition about k-factors is sharp, we [3] considered the relationship between toughness condition and the existence of $[a,b]$-factors for $b > a \geq 2$. We observed the bound of toughness condition in Theorem 1.7 is sharp. The result improved the toughness conditions in Theorem 1.5 and Theorem 1.6.

Theorem 1.7. ([3]) *Let G be a graph of order n and a, b be two positive integers with $b > a \geq 2$. If $t(G) \geq a - 1 + \frac{a-1}{b}$, then G has an $[a,b]$-factor.*

Much work has been contributed to the existence of factors with given properties ([1], [14],[15]). In this paper, we consider the existence of $[a,b]$-factors with inclusion/exclusion properties under the condition of toughness when $b > a > 2$.

Theorem 1.8. *Let a, b be two positive integers with $b > a > 2$ and e_1, e_2 be two distinct edges of a graph G. If $t(G) \geq a - 1 + \frac{a-1}{b}$, then G contains an $[a,b]$-factor containing e_1 and e_2; and an $[a,b]$-factor containing e_1 and excluding e_2; as well as an $[a,b]$-factor excluding e_1 and e_2.*

2 Preliminary lemmas

In order to prove the main theorem, we first give the characterization of (g,f)-factors due to Heinrich [9].

Theorem 2.1. ([9]) *Let G be a graph and g, f be integer-valued functions defined on $V(G)$. If $g(x) < f(x)$ for every $x \in V(G)$, then G has a (g,f)-factor if and only if for any subset S of $V(G)$,*

$$g(T) - d_{G-S}(T) \leq f(S),$$

where $T = \{x | x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.

The following lemma can be deduced from Theorem 2.1.

Lemma 2.2. ([12]) *Let G be a graph and g, f be integer-valued functions defined on V(G) such that $g(x) < f(x) \le d_G(x)$ for every $x \in V(G)$. Let E_1 and E_2 be two disjoint subsets of E(G), then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if for any disjoint subsets S and T of E(G)*

$$g(T) - d_{G-S}(T) \le f(S) - \alpha(S,T;E_1,E_2) - \beta(S,T;E_1,E_2),$$

where $U = V(G) - S - T$, $\alpha(S,T;E_1,E_2) = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,U)|$ and $\beta(S,T;E_1,E_2) = 2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T,U)|$.

In addition, the lemmas below are essential to the proof of our main theorem.

Lemma 2.3. ([13]) *Let G be a graph and $H = G[T]$ such that $\delta(H) \ge 1$ and $1 \le d_G(x) \le k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \ge 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \le \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots, k-1$.

Lemma 2.4. ([3]) *Let G be a graph and $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \ge 2$. Then H has a maximal independent set I and a covering set $C = V(G) - I$ satisfying*

$$|V(H)| \le (k - \frac{1}{k+1})i' + \sum_{j=0}^{k-2}(j+1)i''_j, \quad |C| \le (k-1-\frac{1}{k+1})i' + \sum_{j=0}^{k-2} j i''_j,$$

where $i' = |I'| = |\{x | x \in I,\ d_H(x) = k-1 = d_G(x)\}|$, $i''_j = |\{x | x \in I'' = I - I',\ d_H(x) = j < d_G(x)\}|$.

3 Proof of the main result

We also need the following lemmas to prove our main theorem.

Lemma 3.1. ([7]) *If a graph G is not complete, then $t(G) \le \frac{1}{2}\delta(G)$.*

Lemma 3.2. *Let G be a graph with toughness $t(G) \ge a - 1 + \frac{a-1}{b}$, where a, b are integers satisfying $b > a > 2$. Let S, T be a pair of disjoint subsets of V(G). If $S \ne \emptyset$ and $T \ne \emptyset$, then*

$$a|T| - d_{G-S}(T) \le b|S| - 4.$$

Proof of Lemma 3.2. By the contrary, suppose that there exists a pair of disjoint subsets S, T of $V(G)$ with $|S| > 0$, $|T| > 0$ satisfying $a|T| - d_{G-S}(T) > b|S| - 4$. That is,

$$a|T| - d_{G-S}(T) \geq b|S| - 3. \tag{1}$$

Moreover, suppose that S, T is a pair of minimal sets respect to (1). Then by the minimality of S and T we obtain the following claim.
Claim 1.([15])

(1) *Given S, if T is a minimal set with respect to* (1), *then* $d_{G-S}(x) < a$ *for all* $x \in T$.

(2) *Given T, if S is a minimal set with respect to* (1), *then* $d_T(x) > b$ *for all* $x \in S$.

Let $H' = G[T]$. If there exist components of H' which are isomorphic to K_a, let m be the number of these components. Set $H = H' - mK_a - T_0$, where $T_0 = \{x \in T | d_{G-S}(x) = 0\}$. Denote $t_0 = |T_0|$.

If $|V(H)| = 0$, we get $\omega(G - S) = t_0 + m$. By (1), we have $at_0 + ma \geq b|S| - 3$. That is, $1 \leq |S| \leq \frac{a}{b}(t_0 + m) + \frac{3}{b}$. If $\omega(G - S) = 1$, then either $m = 1$ or $t_0 = 1$. It follows that $b - 1 \geq a \geq b|S| - 3$. And it implies that $|S| = 1$ since $S \neq \emptyset$ and $b > a > 2$. Then there exists one vertex x in T such that $d_G(x) \leq a - 1 + |S| = a$. Since $\delta(G) \geq 2t(G) \geq 2(a-1)+2\frac{a-1}{b} > a$, a contradiction. If $\omega(G-S) = t_0+m > 1$, we have

$$a - 1 + \frac{a-1}{b} \leq t(G) \leq \frac{|S|}{\omega(G-S)} = \frac{|S|}{t_0 + m} \leq \frac{a}{b} + \frac{3}{b(t_0 + m)} \leq \frac{a}{b} + \frac{3}{2b}.$$

That is, $b(a - 1) \leq \frac{5}{2}$, a contradiction.

Now we consider that $|V(H)| > 0$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = a - 1$ for any vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 2.4, H_1 has a maximal independent set I_1 and a covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq (a - \frac{1}{a+1})i'_1 + \sum_{j=0}^{a-2}(j + 1)i''_j, \quad |C_1| \leq (a - 1 - \frac{1}{a+1})i'_1 + \sum_{j=0}^{a-2} ji''_j,$$

where $i'_1 = |I'_1| = |\{x | x \in I_1,\ d_{H_1}(x) = a - 1 = d_{G-S}(x)\}|$,
$i''_j = |\{x | x \in I''_1 = I_1 - I'_1,\ d_{H_1}(x) = j < d_{G-S}(x)\}|$, $0 \leq j \leq a - 2$.

On the other hand, let $T_j = \{x \in V(H_2) | d_{G-S}(x) = j\}$ for $1 \leq j \leq a - 1$. By the definition of H_2, we know that there exists one vertex with degree at most $a - 2$ in $G - S$ from each component of H_2. According to Lemma 2.3, H_2 has a maximal independent set I_2 and a covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{a-1}(a - j)c_j \leq \sum_{j=1}^{a-1}(a - 2)(a - j)i_j,$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for $j = 1, \ldots, a-1$.

Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup (N_G(I_1'') \cap W) \cup C_2 \cup (N_G(I_2) \cap W)$. Then

$$|U| \le |S| + |C_1| + \sum_{j=0}^{a-2}(a-1-j)i_j'' + \sum_{j=0}^{a-1} j i_j,$$

$$\omega(G-U) \ge m + t_0 + i_1' + \sum_{j=0}^{a-2} i_j'' + \sum_{j=0}^{a-1} i_j.$$

Now we show that $|U| \ge t(G)\omega(G-U)$.

It holds obviously when $\omega(G-U) > 1$. When $\omega(G-U) = 1$, by the previous discussion we obtain that $t_0 = m = 0$, then $|I_1|+|I_2| = 1$, hence for any independent vertex $x \in T$, $d_{G-S}(x) + |S| \ge \delta(G) \ge 2t(G) > t(G)$, and $|U| \ge d_{G-S}(x) + |S| > t(G)$. □

Therefore

$$|S| + |C_1| + \sum_{j=0}^{a-2}(a-1-j)i_j'' + \sum_{j=0}^{a-1} j i_j \ge t(G)(m + t_0 + i_1' + \sum_{j=0}^{a-2} i_j'' + \sum_{j=0}^{a-1} i_j). \quad (2)$$

From (1) we have

$$a(t_0 + m) + |V(H_1)| + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j \ge b|S| - 3.$$

It follows that

$$a(t_0 + m) + |V(H_1)| + b|C_1| + b\sum_{j=0}^{a-2}(a-1-j)i_j'' + \sum_{j=1}^{a-1}(a-j)c_j$$

$$\ge bt(G)(m + t_0 + i_1' + \sum_{j=0}^{a-2} i_j'') + \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j - 3.$$

That is

$$|V(H_1)| + b|C_1| + b\sum_{j=0}^{a-2}(a-1-j)i_j'' + \sum_{j=1}^{a-1}(a-j)c_j$$

$$\ge bt(G)(i_1' + \sum_{j=0}^{a-2} i_j'') + \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j$$

$$+(bt(G) - a)(t_0 + m) - 3$$

$$\ge bt(G)(i_1' + \sum_{j=0}^{a-2} i_j'') + \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j$$

$$+(ba - b - 1)(t_0 + m) - 3.$$

By Lemma 2.4, we have

$$|V(H_1)| + b|C_1| + b\sum_{j=0}^{a-2}(a-1-j)i''_j$$

$$\leq (a - \frac{1}{a+1} + b(a-1-\frac{1}{a+1}))i'_1 + \sum_{j=0}^{a-2}(j+1+bj)i''_j$$

$$+b\sum_{j=0}^{a-2}(a-1-j)i''_j$$

$$= ((a-1)(b+1)+1-\frac{b+1}{a+1})i'_1 + \sum_{j=0}^{a-2}(ba-b+j+1)i''_j.$$

Therefore

$$\sum_{j=1}^{a-1}(a-2)(a-j)i_j + ((a-1)(b+1)+1-\frac{b+1}{a+1})i'_1$$

$$+\sum_{j=0}^{a-2}(ba-b+j+1)i''_j$$

$$\geq bt(G)i'_1 + bt(G)\sum_{j=0}^{a-2}i''_j + \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j$$

$$+(ba-b-1)(t_0+m)-3$$

$$\geq (b+1)(a-1)i'_1 + bt(G)\sum_{j=0}^{a-2}i''_j + \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j$$

$$+(ba-b-1)(t_0+m)-3.$$

Finally,

$$\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{j=0}^{a-2}(ba-b+j+1)i''_j$$

$$\geq \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j + bt(G)\sum_{j=0}^{a-2}i''_j$$

$$+(ba-b-1)(t_0+m)-3.$$

Now we consider the following cases.

Case 1. $t_0 + m > 0$.

In this case, we have

$$\sum_{j=1}^{a-1}(a-2)(a-j)i_j+\sum_{j=0}^{a-2}(ba-b+j+1)i''_j$$

$$>\sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j+bt(G)\sum_{j=0}^{a-2}i''_j.$$

Thus at least one of the following cases must hold.

Subcase 1.1 There exists at least one j satisfying $(a-2)(a-j)>bt(G)-bj-a+j$. Then $t(G)<\frac{a^2-a+(b-a+1)j}{b}\le a-1+\frac{a-1}{b}(j\le a-1)$, a contradiction.

Subcase 1.2 $ba-b+j+1>bt(G)$ for some $j\in\{0,1,2,\dots,a-2\}$. It follows that $t(G)<a-1+\frac{a-1}{b}$, a contradiction.

Case 2. $t_0+m=0$.

In this case, we first show the following claim.

Claim 2. $C_1\cup C_2\neq\emptyset$.

Proof. If $C_1\cup C_2=\emptyset$, then $|T|=i''_0+\sum_{j=1}^{a-1}i_j$. Combined with (1) and (2), we have

$$\sum_{j=1}^{a-1}(a-2)(bt(G)-bj-a+j)i_j+(bt(G)-b(a-1)-1)i''_0\le 3.$$

Since $t(G)\ge a-1+\frac{a-1}{b}$ and $j\le a-1$, we get

$$(a-2)|T|\le\sum_{j=1}^{a-1}(b(a-1)+(1-b)j-1)i_j+(a-2)i''_0\le 3.$$

By Claim 1, $|T|\ge b+1>4(b>a>2)$, a contradiction. □

Next we show that for any vertex $x\in C_i$, $d_{I_i}(x)=1(i=1,2)$. Without loss of generality, we may assume that for any vertex $x\in C_2$, $d_{I_2}(x)=1$. If there exists one vertex in C_2 with at least two neighbors in I_2, then

$$|U|\le|S|+|C_1|+\sum_{j=0}^{a-2}(a-1-j)i''_j+\sum_{j=0}^{a-1}ji_j-1.$$

And

$$|S|\ge t(G)(i'_1+\sum_{j=0}^{a-2}i''_j+\sum_{j=0}^{a-1}i_j)-(|C_1|+\sum_{j=0}^{a-2}(a-1-j)i''_j+\sum_{j=0}^{a-1}ji_j)+1.$$

According to (1), it follows that

$$\begin{aligned}&|V(H_1)| + b|C_1| + b\sum_{j=0}^{a-2}(a-1-j)i''_j + \sum_{j=1}^{a-1}(a-j)c_j\\ &\geq bt(G)(i'_1 + \sum_{j=0}^{a-2} i''_j) + \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j + b - 3\\ &> bt(G)(i'_1 + \sum_{j=0}^{a-2} i''_j) + \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j.\end{aligned}$$

By the previous discussion, we obtain that

$$\begin{aligned}&\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{j=0}^{a-2}(ba - b + j + 1)i''_j\\ &> \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j + bt(G)\sum_{j=0}^{a-2} i''_j.\end{aligned}$$

Similarly to Case 1, we also obtain a contradiction. □

Now, let $x \in C_2$ and $U' = U - \{x\}$. Then

$$\omega(G - U') = \omega(G - U) \geq i'_1 + \sum_{j=0}^{a-2} i''_j + \sum_{j=0}^{a-1} i_j$$

as $d_{I_2}(x) = 1$. And

$$|U'| = |U| - 1 \leq |S| + |C_1| + \sum_{j=0}^{a-2}(a-1-j)i''_j + \sum_{j=0}^{a-1} ji_j - 1.$$

Similarly, we have $|U'| \geq t(G)(i'_1 + \sum_{j=0}^{a-2} i''_j + \sum_{j=0}^{a-1} i_j)$ and we also obtain that

$$\begin{aligned}&\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{j=0}^{a-2}(ba - b + j + 1)i''_j\\ &> \sum_{j=1}^{a-1}(bt(G) - bj - a + j)i_j + bt(G)\sum_{j=0}^{a-2} i_j,\end{aligned}$$

a contradiction.

The proof is complete. □

Now we begin to prove our main results.

Proof of Theorem 1.8. Let E_1, E_2 be two edge sets with $E_1 \cup E_2 = \{e_1, e_2\}$. The theorem holds if and only if G contains an $[a, b]$-factor F such that $E_1 \subseteq E(F)$,

$E_2 \cap E(F) = \emptyset$ where E_1 or E_2 may be empty. By the contrary, suppose that G does not contain such an $[a, b]$-factor F. Then, by Lemma 2.2, there exists a pair of disjoint subsets S, T of $V(G)$ such that

$$a|T| - d_{G-S}(T) > b|S| - \alpha(S, T; E_1, E_2) - \beta(S, T; E_1, E_2), \tag{3}$$

where $W = V(G) - S - T$, $\alpha(S, T; E_1, E_2) = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S, W)|$ and $\beta(S, T; E_1, E_2) = 2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T, W)|$.

Meanwhile, as $t(G) \geq a - 1 + \frac{a-1}{b}$, by Theorem 1.7, G contains an $[a, b]$-factor. Therefore,

$$a|T| - d_{G-S}(T) \leq b|S|. \tag{4}$$

Now we show the following claim.

Claim. $S \neq \emptyset$ and $T \neq \emptyset$.

Proof. If $S \cup T = \emptyset$, then $\alpha(S, T; E_1, E_2) = \beta(S, T; E_1, E_2) = 0$, and $a|T| - d_{G-S}(T) > b|S|$, a contradiction to (4).

Then we consider the following cases.

Case 1. $S = \emptyset$ and $T \neq \emptyset$. Then $\alpha(S, T; E_1, E_2) = 0$. And we obtain that $\beta(S, T; E_1, E_2) \neq 0$ from (3) and (4). It follows that $E_2 \neq \emptyset$. Hence either $E_2 = \{e_2\}$ or $E_2 = \{e_1, e_2\}$.

If $E_2 = \{e_2\}$, then $E_1 = \{e_1\}$, which is the case of containing e_1 and excluding e_2. According to (3) again,

$$a|T| - d_G(T) > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T, W)|.$$

And $a|T| - d_G(T) \leq (a - \delta(G))|T| \leq (a - 2t(G))|T| \leq (2 - a - \frac{2(a-1)}{b})|T| < (2 - a)|T|$. It yields that $(2 - a)|T| > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T, W)| > -2$.

If $|T| \geq 2$, then $2(2 - a) > (2 - a)|T| > -2$, a contradiction as $a > 2$.

If $|T| = 1$, $2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T, W)| \leq 1$, then $2 - a = (2 - a)|T| > -1$, a contradiction, too.

If $E_2 = \{e_1, e_2\}$, then $E_1 = \emptyset$, which is the case of excluding e_1 and e_2. Then

$$a|T| - d_G(T) > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T, W)|.$$

And since $\delta(G) \geq 2t(G) > a + 1$, that is, $\delta(G) \geq a + 2$,

$$a|T| - d_G(T) \leq (a - \delta(G))|T| \leq -2|T|.$$

If $|T| \geq 2$, then

$$-2|T| > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T, W)| \geq -4,$$

a contradiction.

If $|T| = 1$, $2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T, W)| \leq 1$, then

$$-2 > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T, W)| > -1,$$

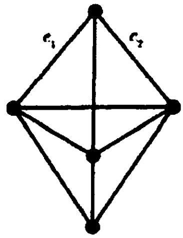

Figure 1: A graph contains no $[2,b]$-factor excluding e_1, e_2 with toughness $\frac{3}{2}$

a contradiction, too. □

Case 2. $S \neq \emptyset$ and $T = \emptyset$. Then $\beta(S,T;E_1,E_2) = 0$. Meanwhile, we obtain that $\alpha(S,T;E_1,E_2) \neq 0$. It follows that $E_1 \neq \emptyset$. Hence either $E_1 = \{e_1\}$ or $E_1 = \{e_1, e_2\}$.

If $E_1 = \{e_1\}$, then $E_2 = \{e_2\}$, which is the case of including e_1 and excluding e_2. From (3), we have $b|S| < \alpha(S,T;E_1,E_2) = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,W)| \leq 2$, which is impossible since $b > a > 2$.

$E_1 = \{e_1, e_2\}$, then $E_2 = \emptyset$, which is the case of containing e_1 and e_2. And

$$b|S| - 2|E_1 \cap E_G(S)| - |E_1 \cap E_G(S,W)| < 0.$$

Then $4|S| < 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,W)|$ as $b > a > 2$. We get a contradiction since $2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,W)| \leq 4$.

This complete the proof of the claim.

Now since $S \neq \emptyset$ and $T \neq \emptyset$, by Lemma 3.2, we have

$$a|T| - d_{G-S}(T) \leq b|S| - 4.$$

But $\alpha(S,T;E_1,E_2) + \beta(S,T;E_1,E_2) \leq 4$, it follows from (3) that

$$a|T| - d_{G-S}(T) > b|S| - 4,$$

a contradiction.

The proof is complete. □

Remark 1. The bound of toughness in Theorem 1.8 is sharp. To see this, consider the graph $G:\ V(G) = V(A) \cup V(B) \cup V(C)$ where A, B and C are disjoint with $A = K_{(nb+1)(a-1)}$, $B = (nb+1)K_{a-1}$ and $C = K_{n(a-1)}$. Set other edges in G are a perfect matching between A and B and all the pairs between B and C. This follows that $t(G) = \frac{(nb+1)(a-1)+n(a-1)}{nb+1} < a - 1 + \frac{a-1}{b}$, $t(G) \to a - 1 + \frac{a-1}{b}$ when $n \to \infty$. By Theorem 1.7, we get that G has no $[a,b]$-factor. And it follows immediately that the bound of toughness in Theorem 1.8 is also sharp.

Remark 2. When $a = 2$, see Figure 1. The graph G in Figure 1 contains no $[2,b]$-factor excluding e_1, e_2 with $t(G) = \frac{3}{2} > a - 1 + \frac{a-1}{b}$ for $b > a = 2$.

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