Riordan arrays and hyperharmonic numbers

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Abstract

In this paper, we establish some general identities involving the weighted row sums of a Riordan array and hyperharmonic numbers. From these general identities, we deduce some particular identities involving other special combinatorial sequences, such as the Stirling numbers, the ordered Bell numbers, the Fibonacci numbers, the Lucas numbers and the binomial coefficients.

Keywords Hyperharmonic numbers ; Stirling numbers; Ordered Bell number; Lucas numbers; Fibonacci numbers

1. Introduction and preliminaries

The hyperharmonic numbers are denoted by $H_n^{(k)}$ and are defined by (see [3])

$$H_n^{(k+1)} = \sum_{i=1}^{n} H_i^{(k)},$$

for $k, n \geq 1$. In particular, $H_n^{(0)} = 1/n$ for $n \geq 1$ and $H_n^{(1)} = H_n = \sum_{k=1}^{n} \frac{1}{k}$. Moreover, define $H_n^{(k)} = 0$ for $k < 0$ or $n \leq 0$. The generating function of the hyperharmonic numbers is

$$H^{(k)}(x) = \sum_{n=0}^{\infty} H_n^{(k)} x^n = \frac{-\ln(1-x)}{(1-x)^k}, \tag{1.1}$$

and these numbers can be expressed as

$$H_n^{(k)} = \sum_{j=1}^{n} \binom{n-j+k-1}{n-j} \frac{1}{j}.$$

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Recently, many works have been devoted to the study of harmonic number identities by various methods. For example, in [1-3], S. Cheon and El-Mikkawy presented some harmonic number identities by means of the Riordan array method. In [5], Chu established some harmonic number identities by the theory of hypergeometric series. In [7], Munarini presented a general identity involving the row sums of a Riordan array and harmonic numbers.

In the present paper, we give some general identities involving the weighted row sums of a Riordan array and hyperharmonic numbers. From these general identities, we can establish some identities involving hyperharmonic numbers and some other special combinatorial sequences.

For convenience, we recall some definitions and notations. Throughout the paper, we denote the Stirling numbers of the second kind by $\left\{ {n \atop k} \right\}$, and let F_n, L_n, $\tilde{b}_n$, $\tilde{b}_n^{(k)}$ be the Fibonacci numbers, the Lucas numbers, the ordered Bell numbers and the higher order ordered Bell numbers, respectively. These numbers satisfy the following generating functions (see [1-6]):

$$S^{(k)}(x) = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}, \tag{1.2}$$

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}, \tag{1.3}$$

$$L(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}, \tag{1.4}$$

$$\tilde{B}(x) = \sum_{n=0}^{\infty} \tilde{b}_n \frac{x^n}{n!} = \frac{1}{2 - e^x}, \tag{1.5}$$

$$\tilde{B}^{(k)}(x) = \sum_{n=0}^{\infty} \tilde{b}_n^{(k)} \frac{x^n}{n!} = \left(\frac{1}{2 - e^x} \right)^k. \tag{1.6}$$

Additionally, it is known that $F_n = \frac{\varphi^n - \tilde{\varphi}^n}{\sqrt{5}}$, and $L_n = \varphi^n + \tilde{\varphi}^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\tilde{\varphi} = \frac{1-\sqrt{5}}{2}$.

Let $c(x)$ be the generating function of the sequence $\{c_k\}_{k \in \mathbb{N}}$, then

$$c(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c_n}{n} x^n = \int_0^x \mathcal{R}(c(t)) dt, \tag{1.7}$$

where $\mathcal{R}$ is the operator defined by

$$\mathcal{R}(c(x)) = \frac{c(x) - c_0}{x} \tag{1.8}$$

Due to[9, 11, 12, 13], a Riordan array is a pair of formal power series $(g(x), f(x))$. It defines an infinite, lower triangular array $(d_{n,k})_{n,k\in\mathbf{N}}$ according to the rule

$$d_{n,k} = [x^n]g(x)(f(x))^k.$$

Hence we write $(d_{n,k}) = (g(x), f(x))$.

Let $(g(x), f(x))$ be a Riordan array and $a(x)$ be the generating function of the sequence $\{a_k\}_{k\in\mathbf{N}}$, i.e, $a(x) = \sum_{k=0}^{\infty} a_k x^k$. As in [7], define the operator $\mathcal{T}_{\mathcal{R}}$ by

$$\mathcal{T}_{\mathcal{R}}(a(x)) = g(x)a(f(x)) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_{n,k}a_k\right)x^n. \tag{1.9}$$

We denote the weighted row-sum sequence by $\{d_n^{[k]}\}_{n\in\mathbf{N}}$, and $\{s_n\}_{n\in\mathbf{N}}$, where

$$d_n^{[k]} = \sum_{j=0}^{n}\binom{k+j}{j}d_{n,j} \quad \text{and} \quad s_n = \sum_{j=0}^{n} jd_{n,j}, \tag{1.10}$$

and denote the weighted diagonal sum sequence by $\{l_n^{[k]}\}_{n\in\mathbf{N}}$, where

$$l_n^{[k]} = \sum_{j=0}^{[\frac{n}{2}]}\binom{k+j}{j}d_{n-j,j}. \tag{1.11}$$

These sequences satisfy the following generating functions respectively:

$$\begin{aligned} d^{[k]}(x) = \sum_{n=0}^{\infty} d_n^{[k]}x^n &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{k+j}{j}d_{n,j}\right)x^n \\ &= \frac{g(x)}{(1-f(x))^{k+1}}, \end{aligned} \tag{1.12}$$

$$s(x) = \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} jd_{n,j}\right)x^n = \frac{g(x)f(x)}{(1-f(x))^2}, \tag{1.13}$$

$$\begin{aligned} l^{[k]}(x) = \sum_{n=0}^{\infty} l_n^{[k]}x^n &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\binom{k+j}{j}d_{n-j,j}\right)x^n \\ &= \frac{g(x)}{(1-xf(x))^{k+1}}. \end{aligned} \tag{1.14}$$

2. General identities

In this section, we establish three general identities related to the weighted row sums of a Riordan array and hyperharmonic numbers.

Theorem 2.1. *Let $(d_{n,k}) = (g(x), f(x))$ be a Riordan array with weighted row-sum sequence $\{d_n^{[k]}\}_{n\in\mathbb{N}}$. For any sequence $\{h_n\}_{n\in\mathbb{N}}$ having $h(x) = \sum_{n=0}^{\infty} h_n x^n$ as its ordinary generating function and satisfying the relation*

$$\mathcal{R}(h(x)) = \frac{h(x)-h_0}{x} = \frac{f'(x)}{1-f(x)}, \tag{2.1}$$

we have

$$\sum\nolimits_{j=1}^{n} \frac{h_j d_{n-j}^{[k]}}{j} = \sum_{j=1}^{n} \frac{h_j}{j} \sum_{i=0}^{n-j} \binom{k+i}{k} d_{n-j,i} = \sum_{j=0}^{n} d_{n,j} H_j^{(k+1)}. \tag{2.2}$$

Proof. By (1.7), (2.1) and (1.9), we obtain the generating function:

$$\sum_{n=1}^{\infty} \frac{h_n}{n} x^n = \int_0^x \mathcal{R}(h(t))dt = \int_0^x \frac{f'(t)}{1-f(t)} dt = \frac{1}{\ln(1-f(x))}.$$

Hence

$$\sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{h_j d_{n-j}^{[k]}}{j} \right) x^n = \sum_{n=0}^{\infty} d_n^{[k]} x^n \sum_{n=1}^{\infty} \frac{h_n}{n} x^n$$
$$= \frac{g(x)}{(1-f(x))^{k+1}} \frac{1}{\ln(1-f(x))} = g(x) H^{(k+1)}(f(x)) = \mathcal{T}_{\mathcal{R}}(H^{(k+1)}(x)).$$

Then identity (2.2) holds. Note that it reduces to the conclusion of [7] when $k = 0$. □

Similarly, we obtain the following theorem.

Theorem 2.2. *Let $(d_{n,k}) = (g(x), f(x))$ be a Riordan array. For any sequence $\{h_n\}_{n\in\mathbb{N}}$ having $h(x) = \sum_{n=0}^{\infty} h_n x^n$ as its ordinary generating function and satisfying the relation*

$$\mathcal{R}(h(x)) = \frac{h(x)-h_0}{x} = \frac{f'(x)}{1-f(x)}, \tag{2.3}$$

we have

$$\sum\nolimits_{j=1}^{n} \frac{h_j s_{n-j}}{j} = \sum_{j=0}^{n} \frac{h_j}{j} \sum_{i=0}^{n-j} i d_{n-j,i} = \sum_{j=0}^{n} d_{n,j} (H_j^{(2)} - H_j^{(1)}). \tag{2.4}$$

Moreover, the next theorem can be established.

Theorem 2.3. *Let* $(d_{n,k}) = (g(x), f(x))$ *be a Riordan array. For any sequence* $\{h_n\}_{n\in\mathbb{N}}$ *having* $h(x) = \sum_{n=0}^{\infty} h_n x^n$ *as its ordinary generating function and satisfying the relation*

$$\mathcal{R}(h(x)) = \frac{h(x) - h_0}{x} = \frac{(xf(x))'}{1 - xf(x)}, \tag{2.5}$$

we have

$$\sum_{j=1}^{n} \frac{h_j l_{n-j}^{[k]}}{j} = \sum_{j=0}^{n} \frac{h_j}{j} \sum_{i=0}^{n-j} \binom{k+i}{k} d_{n-j,i} = \sum_{j=0}^{[\frac{n}{2}]} H_j^{(k+1)} d_{n-j,j}. \tag{2.6}$$

Proof. From (1.7), (2.5) and (1.1), we have

$$\sum_{n=1}^{\infty} \frac{h_n}{n} x^n = \int_0^x \mathcal{R}(h(t))dt = \int_0^x \frac{(tf(t))'}{1 - tf(t)} dt = \frac{1}{\ln(1 - xf(x))}.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{h_j l_{n-j}^{[k]}}{j} \right) x^n &= \sum_{n=0}^{\infty} l_n^{[k]} x^n \sum_{n=1}^{\infty} \frac{h_n}{n} x^n \\
&= \frac{g(x)}{(1 - xf(x))^{k+1}} \frac{1}{\ln(1 - xf(x))} = g(x) H^{(k+1)}(xf(x)) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[\frac{n}{2}]} H_j^{(k+1)} d_{n-j,j} \right) x^n.
\end{aligned}$$

Then (2.6) holds. □

3. Identities involving special combinatorial sequences

From the general identity (2.2), we obtain some identities involving the hyperharmonic numbers, and some other combinatorial sequences, such as the ordered Bell numbers, the higher order ordered Bell numbers the Stirling numbers, the Fibonacci numbers, the Lucas numbers and the binomial coefficients.

Theorem 3.1. *Let* $n \geq 1, k \geq 0$ *be any positive integers, then*

$$n\tilde{b}_{n-1}^{(k+1)} + 2\sum_{j=2}^{n} \binom{n}{j} \bar{b}_{j-1} \tilde{b}_{n-j}^{(k+1)} = \sum_{j=1}^{n} \left\{ {n \atop j} \right\} j! H_j^{(k+1)}, \tag{3.1}$$

$$n\tilde{b}_{n-1} + 2\sum_{j=2}^{n}\binom{n}{j}\tilde{b}_{j-1}\tilde{b}_{n-j} = \sum_{j=1}^{n}\left\{ {n \atop j} \right\} j!H_j \,, \tag{3.2}$$

where $\tilde{b}_j^{(1)} = \tilde{b}_j$,

Proof. The coefficients of the Riordan array $(d_{n,k}) = (1, e^x - 1)$ are

$$[x^n](e^x-1)^k = k![x^n]\frac{(e^x-1)^k}{k!} = \left\{ {n \atop k} \right\}\frac{k!}{n!} .$$

Moreover,

$$d^{[k]}(x) = \frac{1}{(2-e^x)^{k+1}}, \quad \mathcal{R}(h(x)) = \frac{f'(x)}{1-f(x)} = \frac{2}{2-e^x} - 1 ,$$

so we have

$$d_n^{[k]} = \frac{\tilde{b}_n^{(k+1)}}{n!} \quad , \quad h_n = \begin{cases} \frac{2\tilde{b}_{n-1}}{(n-1)!}, & n \ge 2 , \\ 1, & n = 1 , \end{cases}$$

and identity (2.2) becomes

$$\frac{\tilde{b}_{n-1}^{(k+1)}}{(n-1)!} + 2\sum_{j=2}^{n}\frac{\tilde{b}_{j-1}}{(j-1)!}\frac{\tilde{b}_{n-j}^{(k+1)}}{(n-j)!} = \sum_{j=1}^{n}\left\{ {n \atop j} \right\}\frac{j!}{n!}H_j^{(k+1)} .$$

By multiplying both sides by $n!$, we obtain (3.1). Setting $k = 0$ in (3.1) further gives us (3.2). □

Theorem 3.2. *Let $q \in \mathbb{C}$, and let r and k be any positive integers, then*

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j}\binom{r+n-j-i-1}{r-1}\binom{k+i}{i}\frac{q^{n-j-i}}{j}$$
$$= \sum_{j=0}^{n}\binom{r+n-j-1}{r-1}H_j^{(k+1)}q^{n-j} . \tag{3.3}$$

Proof. For the Riordan array

$$(d_{n,k}) = \left(\binom{r+n-k-1}{r-1}q^{n-k}\right) = \left(\frac{1}{(1-qx)^r}, x\right),$$

we obtain

$$d(x) = \sum_{j=1}^{\infty} d_n^{[k]}x^n = \frac{1}{(1-qx)^r(1-x)^{k+1}}, \quad \mathcal{R}(h(x)) = \frac{f'(x)}{1-f(x)} = \frac{1}{1-x} .$$

So we have

$$d_n^{[k]} = \sum_{i=0}^{n} \binom{r+n-i-1}{r-1}\binom{k+i}{i} q^{n-i} \quad \text{and} \quad h_n = 1\,, \quad n \geq 1\,.$$

Then Eq. (3.3) holds. □

In Theorem 3.2, setting $q = 2, 1$, and $1/2$, we have

Corollary 3.3. *The following relations hold:*

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j} \binom{r+n-j-i-1}{r-1}\binom{k+i}{i}\frac{1}{j2^{i+j}} = \sum_{j=0}^{n}\binom{r+n-j-1}{r-1}\frac{H_j^{(k+1)}}{2^j}\,,$$

$$\sum_{j=1}^{n}\binom{r+n-j+k}{r+k}\frac{1}{j} = \sum_{j=0}^{n}\binom{r+n-j-1}{r-1}H_j^{(k+1)}\,,$$

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j} \binom{r+n-j-i-1}{r-1}\binom{k+i}{i}\frac{2^{i+j}}{j} = \sum_{j=0}^{n}\binom{r+n-j-1}{r-1}H_j^{(k+1)}2^j\,.$$

Setting further $r = 1$ in Corollary 3.3 yields Corollary 3.4, and setting further $k = 0$ in Corollary 3.4 yields Corollary 3.5.

Corollary 3.4.

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j}\binom{k+i}{i}\frac{1}{j2^{i+j}} = \sum_{j=0}^{n}\frac{H_j^{(k+1)}}{2^j}\,,$$

$$\sum_{j=1}^{n}\binom{r+n-j+1}{r+1}\frac{1}{j} = \sum_{j=0}^{n}H_j^{(k+1)}\,,$$

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j}\binom{k+i}{i}\frac{2^{i+j}}{j} = \sum_{j=0}^{n}H_j^{(k+1)}2^j\,.$$

Corollary 3.5.

$$\sum_{j=0}^{n}\frac{H_j}{2^j} = \sum_{j=1}^{n}\frac{1}{j2^{j-1}} - \frac{H_n}{2^n}\,,$$

$$\sum_{j=1}^{n}H_j = (n+1)H_n - n\,,$$

$$\sum_{j=1}^{n}H_j2^j = 2^{n+1}H_n - \sum_{j=1}^{n}\frac{2^j}{j}\,.$$

Theorem 3.6. *Let r, k be any positive integers, then*

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j}\binom{r+n-j-i-1}{r-1}\binom{k+i}{i}\frac{F_{n-j-i}}{j} = \sum_{j=0}^{n}\binom{r+n-j-1}{r-1}H_j^{(k+1)}F_{n-j}\,. \tag{3.4}$$

Proof. We replace q in identity (3.3) first by φ and then by $\tilde{\varphi}$, do some simplification by the relations $\varphi^2 = \varphi+1$ and $\tilde{\varphi}^2 = \tilde{\varphi}+1$, take the difference of the two identities just obtained, divide both sides by $\sqrt{5}$, and finally simplify by Eq. (1.3). □

In Theorem 3.6, setting $r = 1$, we have

Corollary 3.7.

$$\sum_{j=1}^{n}\frac{1}{j}\sum_{i=0}^{n-j}\binom{k+i}{i}F_{n-j-i} = \sum_{j=0}^{n}H_j^{(k+1)}F_{n-j}\,.$$

Theorem 3.8. *Let $\alpha \in \mathbb{C}$, and let $r \geq 1, k \geq 0$ be integers, then*

$$\sum_{j=1}^{n}\sum_{i=0}^{n-j}\binom{n-j-i}{r}\binom{k+i}{i}\frac{\alpha^{n-j-i}}{j} = \sum_{j=0}^{n-r}\binom{n-j}{r}H_j^{(k+1)}\alpha^{n-j}\,. \tag{3.5}$$

The proof of Theorem 3.8 is similar to that of Theorem 3.2. Setting $\alpha = 1, k = 0$ in Theorem 3.8 gives the next two identities.

Corollary 3.9.

$$\sum_{j=1}^{n}\binom{n-j+k+1}{n-j-r}\frac{1}{j} = \sum_{j=1}^{n-r}\binom{n-j}{r}H_j^{(k+1)}\,,$$

$$\sum_{j=1}^{n}\binom{n-j+1}{n-j-r}\frac{1}{j} = \sum_{j=1}^{n}\binom{n-j}{r}H_j\,.$$

Theorem 3.10. *Let $n \geq 2, k \geq 0$, $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$, then*

$$\sum_{j=1}^{n}\frac{\alpha^j+\beta^j}{j}\sum_{i=0}^{[\frac{n-j}{2}]}\binom{k+n-j-i}{n-j-i}\binom{n-j-i}{i}(\alpha+\beta)^{n-j-2i}(-\alpha\beta)^i$$
$$= \sum_{j=0}^{[\frac{n}{2}]}\binom{n-j}{j}H_{n-j}^{(k+1)}(\alpha+\beta)^{n-2j}(-\alpha\beta)^j \tag{3.6}$$

and

$$\sum_{j=1}^{n}\frac{\alpha^j+\beta^j}{j}\sum_{i=0}^{n-j}\binom{k+i}{i}\binom{k+n-j-i}{n-j-i}\alpha^i\beta^{n-j-i}$$
$$=\sum_{j=0}^{[\frac{n}{2}]}\binom{n-j}{j}H_{n-j}^{(k+1)}(\alpha+\beta)^{n-2j}(-\alpha\beta)^j\,. \tag{3.7}$$

Proof. Let $(d_{n,k})=(1,(\alpha+\beta)x-\alpha\beta x^2)$, then

$$d_{n,j}=[x^n]((\alpha+\beta)x-\alpha\beta x^2)^j=\binom{j}{n-j}(\alpha+\beta)^{2j-n}(-\alpha\beta)^{n-j},$$

and we have

$$d_n^{[k]}=\sum_{j=0}^{n}\binom{k+j}{k}d_{n,j}=\sum_{j=[\frac{n}{2}]}^{n}\binom{k+j}{k}\binom{j}{n-j}(\alpha+\beta)^{2j-n}(-\alpha\beta)^{n-j}$$
$$=\sum_{j=0}^{[\frac{n}{2}]}\binom{k+n-j}{k}\binom{n-j}{j}(\alpha+\beta)^{n-2j}(-\alpha\beta)^j$$
$$=[x^n]\frac{1}{(1-\alpha x)^{k+1}(1-\beta x)^{k+1}}=\beta^n\sum_{i=0}^{n}\binom{k+i}{i}\binom{k+n-i}{n-i}\left(\frac{\alpha}{\beta}\right)^i$$

and

$$\mathcal{R}(h(x))=\frac{f'(x)}{1-f(x)}=\frac{\alpha+\beta-2\alpha\beta x}{(1-\alpha x)(1-\beta x)},\quad h_n=\alpha^n+\beta^n,\quad n\geq 1\,.$$

By (2.2), identities (3.6) and (3.7) can be established. □

In identity (3.6), setting $\alpha=(\frac{1+\sqrt{5}}{2})^m$ and $\beta=(\frac{1-\sqrt{5}}{2})^m$, we have

Corollary 3.11.

$$\sum_{j=1}^{n}\frac{L_{mj}}{j}\sum_{i=0}^{[\frac{n-j}{2}]}\binom{k+n-j-i}{n-j-i}\binom{n-j-i}{i}L_m^{n-j-2i}$$
$$=\sum_{j=0}^{[\frac{n}{2}]}\binom{n-j}{j}H_{n-j}^{(k+1)}L_m^{n-2j}\,.$$

In Corollary 3.11, setting $k=0$ gives the conclusion of [7]. Moreover, by the proof of Theorem 3.10, we have

Corollary 3.12. *Let $m \geq 2, k \geq 0$, $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$, then*

$$\sum_{j=0}^{[\frac{m}{2}]} \binom{k+m-i}{m-i}\binom{m-i}{i}(\alpha+\beta)^{m-2i}(-\alpha\beta)^{i}$$

$$= \sum_{j=0}^{m} \binom{k+i}{i}\binom{k+m-i}{m-i}\alpha^{i}\beta^{m-i}. \qquad (3.8)$$

In (3.8), setting $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$ and $\alpha = \beta = 1$ gives the following relations:

$$\sum_{j=0}^{[\frac{m}{2}]} \binom{k+m-i}{m-i}\binom{m-i}{i}$$

$$= \sum_{j=0}^{m}(-1)^{m-i}\binom{k+i}{i}\binom{k+m-i}{m-i}\left(\frac{3+\sqrt{5}}{2}\right)^{i-\frac{m}{2}},$$

$$\sum_{j=0}^{[\frac{m}{2}]} \binom{k+m-i}{m-i}\binom{m-i}{i}2^{m-2i} = \sum_{j=0}^{m}\binom{k+i}{i}\binom{k+m-i}{m-i}.$$

From the general identities (2.4) and (2.6), we can also establish particular identities involving other special combinatorial sequences, which are left to the readers.

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