On $(a, b; n)$-graceful labeling of path P_n

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Abstract This paper devotes to solving the following conjecture proposed by Gvozdjak: "An (a,b;n)-graceful labeling of P_n exists if and only if the integers a,b,n satisfy (1) $b - a$ has the same parity as $n(n+1)/2$; (2) $0 < |b-a| \leq (n+1)/2$ and (3) $n/2 \leq a + b \leq 3n/2$." Its solving can shed some new light on the solving the famous Oberwolfach problem. It is shown that the conjecture is true for every n if the conjecture is true when $n \leq 4a + 1$ and a is a fixed value. Moreover, we prove that the conjecture is true for $a = 0, 1, 2, 3, 4, 5, 6$.

Key Words: labeling ; $(a, b; n)$-graceful; path P_n; conjecture.

AMS Classification: 05C78

§1 Introduction

A graph $G(V, E)$ is called graceful graph if there exists an non-negative integer $g(v)$ such that the followings are satisfied (1) $\max\{g(v)|v \in V\} = |E(G)|$; (2) If $u \neq v$ for any $u, v \in V$, then $g(u) \neq g(v)$; (3) If $e_1 \neq e_2$ for any $e_1, e_2 \in E(G)$, then $g^*(e_1) \neq g^*(e_2)$ for $g^*(e) = |g(u) - g(v)|$ and $uv = e$.

Let P_n be a path with $n+1$ vertices consecutively denoted by $v_0, v_1, \cdots, v_n$, and a, b be non-negative integers. If graceful labeling g of P_n satisfies $g(v_0) = a$ and $g(v_n) = b$, then g is called an $(a, b; n)$-graceful labeling of P_n, denoted by $g(a, b; n)$, and P_n is called $(a, b; n)$-graceful.

The term "graceful labeling" was introduced by S.Golomb in [1]. A detailed history of the graph labeling problems and relating to results appear

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in Gallian in [2].

(a,b;n)-conjecture The graph P_n is $(a,b;n)$-graceful if and only if all of the following conditions hold:
(1) $b-a$ has the same parity as $\frac{n(n+1)}{2}$;
(2) $0<|b-a|\le\frac{n+1}{2}$;
(3) $\frac{n}{2}\le a+b\le\frac{3n}{2}$.

Gvzdjak has proved that the conditions of $(a,b;n)$-conjecture are necessary in [3]. For sufficiency, the $(a,b;n)$-conjecture is true when $a=0$ by Lee in [4]. Fan and Liang have shown that the $(a,b;n)$-conjecture is true for $a=1$ and $a=2$ in [5]. We will discuss the equal theory of this conjecture and also show that the conjecture is true for $a=0,1,2,3,4,5,6$.

In this paper, parameters a, b, n are non-negative integers without special statement. We define

$$g(a,b;n)=(g(v_0),g(v_1),\cdots,g(v_n));$$

$$g(a,b;n)\oplus m=(g(v_0)+m,g(v_1)+m,\cdots,g(v_n)+m);$$

$$g^{-1}(a,b;n)=(g(v_n),g(v_{n-1}),\cdots,g(v_0));$$

$$n-g(a,b;n)=(n-g(v_0),n-g(v_1),\cdots,n-g(v_n));$$

$$(x_1,x_2,\cdots,x_k)\wedge(x_{k+1},x_{k+2},\cdots,x_l)=(x_1,x_2,\cdots,x_l).$$

§2 Main results

Theorem 1 Let a, b, n satisfy the conditions of $(a,b;n)$-conjecture. (1) If a is an odd and P_n is $(a,b;n)$-graceful with $n\le 4a$, then P_n is $(a,b;n)$-graceful for all n. (2) If a is a positive even, and P_n is $(a,b;n)$-graceful with $n\le 4a+1$, then P_n is $(a,b;n)$-graceful for all n.

In order to prove Theorem 1, we will give the following Lemmas.

Lemma 1 Let a, b, n satisfy the condition (1) of $(a,b;n)$-conjecture, the possible types are given as follows:
(1) a is an odd, b is an odd, $n\equiv 0\ (mod\ 4)$; (2) a is an even, b is an even, $n\equiv 0\ (mod\ 4)$;
(3) a is an odd, b is an even, $n\equiv 1\ (mod\ 4)$; (4) a is an even, b is an odd, $n\equiv 1\ (mod\ 4)$;
(5) a is an odd, b is an even, $n\equiv 2\ (mod\ 4)$; (6) a is an even, b is an odd, $n\equiv 2\ (mod\ 4)$;
(7) a is an odd, b is an odd, $n\equiv 3\ (mod\ 4)$; (8) a is an even, b is an even,

$n \equiv 3\ (mod\ 4)$.

Lemma 2 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture, then a, $b+a+1$, $n+2(a+1)$ satisfy the conditions of $(a, b+a+1; n+2(a+1))$-conjecture.

Proof We first show that a, $b+a+1$, $n+2(a+1)$ satisfy the condition (1) of $(a, b+a+1; n+2(a+1))$-conjecture. When $a \equiv 1\ (mod\ 2)$, we can get $a+1 \equiv 0\ (mod\ 2)$ and $2(a+1) \equiv 0\ (mod\ 4)$. Then $n \equiv n+2(a+1)\ (mod\ 4)$. So $n(n+1)/2$ has the same parity as $(n+2(a+1))(n+2(a+1)+1)/2$, and $b-a$ has the same parity as $b+(a+1)-a = b+1$, thus the result is true.

When $a \equiv 0\ (mod\ 2)$, $a+1 \equiv 1\ (mod\ 2)$ and $2(a+1) \equiv 2\ (mod\ 4)$. According to Lemma 1, there are four types of a, b, n that satisfy the condition (1) of $(a, b; n)$-conjecture.

(1) When $n \equiv 0\ (mod\ 4)$, b is an even, then $(n+2(a+1)) \equiv 2\ (mod\ 4)$, $b+a+1$ is an odd, $b+a+1-a = b+1$ is an odd, $(n+2(a+1))(n+2(a+1)+1)/2$ is an odd;

(2) When $n \equiv 1\ (mod\ 4)$, b is an odd, then $(n+2(a+1)) \equiv 3\ (mod\ 4)$, $b+a+1$ is an even, $b+a+1-a = b+1$ is an even, $(n+2(a+1))(n+2(a+1)+1)/2$ is an even;

(3) When $n \equiv 2\ (mod\ 4)$, b is an odd, then $(n+2(a+1)) \equiv 0\ (mod\ 4)$, $b+a+1$ is an even, $b+a+1-a = b+1$ is an even, $(n+2(a+1))(n+2(a+1)+1)/2$ is an even;

(4) When $n \equiv 3\ (mod\ 4)$, b is an even, then $(n+2(a+1)) \equiv 1\ (mod\ 4)$, $b+a+1$ is an odd, $b+a+1-a = b+1$ is an odd, $(n+2(a+1))(n+2(a+1)+1)/2$ is an odd.

So when $a \equiv 0\ (mod\ 2)$, then the conclusion is true.

Next we shall show that $a, a+b+1, n+2(a+1)$ satisfy condition (2) of $(a, b+a+1; n+2(a+1))$-conjecture.

When $n \equiv 0\ (mod\ 2)$, $0 < |b-a| \le (n+1)/2$ if and only if $b \ne a$ and $a-n/2 \le b \le a+n/2$. So $b+(a+1) \ne a$ and $a-n/2+(a+1) \le b+(a+1) \le a+n/2+(a+1)$. Since $a-n/2-(a+1) < a-n/2+(a+1)$, we have $b+(a+1) \ne a$, then $a-(n+2(a+1))/2 \le b+(a+1) \le a+(n+2(a+1))/2$, $a-n/2-(a+1) < a-n/2+(a+1)$, $0 < |b+(a+1)-a| \le (n+2(a+1))/2$.

When $n \equiv 1\ (mod\ 2)$, $0 < |b-a| \le (n+1)/2$ if and only if $b \ne a$ and $a-(n+1)/2 \le b \le a+(n+1)/2$. So $b+(a+1) \ne a$ and $a-(n+1)/2+(a+1) \le b+(a+1) \le a+(n+1)/2+(a+1)$. As $a-(n+1)/2-(a+1) < a-(n+1)/2+(a+1)$, we obtain $b+(a+1) \ne a$, then $a-(n+1)/2-(a+1) \le b+(a+1) \le a+(n+1)/2+(a+1), 0 < |b+(a+1)-a| \le (n+1+2(a+1))/2$.

Finally, we shall prove that $a, b+a+1, n+2(a+1)$ satisfy the condition (3) of $(a, b+a+1; n+2(a+1))$-conjecture.

When $n \equiv 0\ (mod\ 2)$, $n/2 \le a+b \le 3n/2$, we know $n/2+(a+1) \le$

$a+b+(a+1) \leq 3n/2+(a+1)$. As $3n/2+(a+1) \leq a+b \leq 3n/2+3(a+1)$, we can get that $(n+2(a+1))/2 \leq a+b+(a+1) \leq 3(n+2(a+1))/2$.

When $n \equiv 1\ (mod\ 2)$, $n+2(a+1) \equiv 1\ (mod\ 2)$, $n/2 \leq a+b \leq 3n/2$ if and only if $(n+1)/2 \leq a+b \leq (3n-1)/2$, then $(n+1)/2+(a+1) \leq a+b+(a+1) \leq (3n-1)/2+(a+1)$, So $(n+1+2(a+1))/2 \leq a+b+(a+1) \leq 3(n+2(a+1))/2$. □

Lemma 3 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture. If graceful labeling of path P_n is $g(a, b; n)$, then the path $P_{n+2(a+1)}$ satisfies conditions of $(a, b+(a+1); n+2(a+1))$-conjecture and the graceful labeling is $g(a, b+(a+1); n+2(a+1))$.

Proof According to Lemma 1, we have a, $b+a+1$, $n+2(a+1)$ satisfy the conditions of $(a, b+a+1; n+2(a+1))$-conjecture. Let $v_0, v_1, \cdots, v_{n+2(a+1)}$ be the vertices of path $P_{n+2(a+1)}$. Put the vertices labeling f as follows:

$$f(a, b+a+1; n+2(a+1))$$
$$= (a, n+2(a+1)-a, a-1, n+2(a+1)-(a-1), \cdots, 1,$$
$$n+2(a+1)-1, 0, n+2(a+1)) \wedge (g(a, b; n) \oplus (a+1)).$$

Hence $f(v_0) = a$, $f(v_{n+2a+2}) = b+(a+1)$. Considering the vertices labeling set A of $P_{n+2(a+1)}$.

$A=\{f(v_i) \mid i = 0, 1, \cdots, 2a+1\} \cup \{f(v_i) \mid i = 2a+1, 2a+2, \cdots, n+2(a+1)\}$

$=\{0, 1, \cdots, a\} \cup \{n+a+2, n+a+3, \cdots, n+2(a+1)\} \cup \{a+1, a+2, \cdots, a+n+1\}$

$=\{0, 1, \cdots, n+2(a+1)\}$.

There are $n+2a+3$ different numbers of set A which are the same as the number of vertices of $P_{n+2(a+1)}$, so the vertices labelings are different.

We denote B as the set of edges labeling of $P_{n+2(a+1)}$. Then

$B=\{f^*(v_{i-1}v_i) \mid i = 1, 2, \cdots, n+2(a+1)\}$

$=\{|f(v_{i-1}) - f(v_i)| \mid i = 1, 2, \cdots, n+2(a+1)\}$

$=\{|f(v_{i-1}) - f(v_i)| \mid i = 1, 2, \cdots, 2a+1\} \cup \{|f(v_{2a+1}) - f(v_{2a+2})|\} \cup \{|f(v_{i-1}) - f(v_i)| \mid i = 2a+3, 2a+4, \cdots, 2a+2+n\}$

$=\{n+2, n+3, \cdots, n+2a+2\} \cup \{n+2a+2-(2a+1)\} \cup \{1, 2, \cdots, n\}$

$=\{1, 2, \cdots, n+2a+2\}$.

The number of the edges of $P_{n+2(a+1)}$ is the same as the number of the set B, thus f^* is a one to one mapping. Therefore the Lemma is obtained. □

By using induction arguments, we can get the following Lemma 4 .

Lemma 4 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture. If graceful labeling of path P_n is $g(a, b; n)$, then the path $P_{n+2m(a+1)}$ satisfies conditions of $(a, b+m(a+1); n+2m(a+1))$-conjecture and the gracefull labeling is $g(a, b+m(a+1); n+2m(a+1))$.

Lemma 5 Let a, n be non-negative integers and $n \geq 2a$, then the possibility of b satisfying the conditions of $(a, b; n)$-conjecture are as follows:
(1) If $n \equiv 0 \ (mod\ 4)$, then $b = \{\frac{n+4(k-1)}{2} - a \mid 0 \leq k \leq a+1, k \neq a+1-\frac{n}{4}\}$;
(2) If $n \equiv 1 \ (mod\ 4)$, then $b = \{\frac{n+1+4(k-1)}{2} - a \mid 0 \leq k \leq a+1\}$;
(3) If $n \equiv 2 \ (mod\ 4)$, then $b = \{\frac{n+4(k-1)}{2} - a \mid 0 \leq k \leq a+1\}$;
(4) If $n \equiv 3 \ (mod\ 4)$, then $b = \{\frac{n+1+4(k-1)}{2} - a \mid 0 \leq k \leq a+1, k \neq a+1-\frac{n}{4}\}$.

Proof From the condition(2) and condition(3) of $(a, b; n)$-conjecture, we have $0 < |b-a| \leq \frac{n+1}{2}$ and $\frac{n}{2} \leq a+b \leq \frac{3n}{2}$, which is equal to $b \neq a$, $a - \frac{n+1}{2} \leq b \leq a + \frac{n+1}{2}$ and $\frac{n}{2} - a \leq b \leq \frac{3n}{2} - a$.

Case(a) n is an even.

Since $n \geq 2a$, we have $\frac{n}{2} - a - (a - \frac{n}{2}) = n - 2a \geq 0$ and $\frac{3n}{2} - a - (a + \frac{n}{2}) = n - 2a \geq 0$. Then $b \neq a$, $a - \frac{n}{2} \leq b \leq a + \frac{n}{2}$, $\frac{n}{2} - a \leq b \leq \frac{3n}{2} - a$. So $b \neq a$, $\frac{n}{2} - a \leq b \leq \frac{n}{2} + a$. If $k = a + 1 - \frac{n}{4}$, then $\frac{n+4(k-1)}{2} - a = a$. According to Lemma 1 and condition(1) of conjecture, we will get the results as follows:
If $n \equiv 0 \ (mod\ 4)$ and a is an even, then b is an even and $b = \{\frac{n+4(k-1)}{2} - a \mid 1 \leq k \leq a+1, k \neq a+1-\frac{n}{4}\}$;
If $n \equiv 0 \ (mod\ 4)$ and a is an odd, then b is an odd and $b = \{\frac{n+4(k-1)}{2} - a \mid 1 \leq k \leq a+1, k \neq a+1-\frac{n}{4}\}$;
If $n \equiv 2 \ (mod\ 4)$ and a is an odd, then b is an even and $b = \{\frac{n+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$;
If $n \equiv 2 \ (mod\ 4)$ and a is an even, then b is an odd and $b = \{\frac{n+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$.

Case(b) n is an odd.

It is clear that $n \geq 2a$ if and only if $n \geq 2a+1$. Since a, b are integers, we have $b \neq a$, $a - \frac{n+1}{2} \leq b \leq a + \frac{n+1}{2}$, $\frac{n+1}{2} - a \leq b \leq \frac{3n}{2} - a$. So $b \neq a$, $\frac{n+1}{2} - a \leq b \leq \frac{n+1}{2} + a$. If $k = a + 1 - \frac{n+1}{4}$, then $\frac{n+1+4(k-1)}{2} - a = a$. According to Lemma 1 and condition (1) of conjecture, we will get the results as follows:
If $n \equiv 1 \ (mod\ 4)$ and a is an odd, then b is an even and $b = \{\frac{n+1+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$;
If $n \equiv 1 \ (mod\ 4)$ and a is an even, then b is an odd and $b = \{\frac{n+1+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$;
If $n \equiv 3 \ (mod\ 4)$ and a is an odd, then b is an odd and $b = \{\frac{n+1+4(k-1)}{2} - a \mid 1 \leq k \leq a+1, k \neq a+1-\frac{n+1}{4}\}$;
If $n \equiv 3 \ (mod\ 4)$ and a is an even, then b is an even and $b = \{\frac{n+1+4(k-1)}{2} - a \mid 1 \leq k \leq a+1, k \neq a+1-\frac{n+1}{4}\}$. □

Lemma 6 Let a, n be non-negative integers and $n \geq 4a+1$, then the number of the b satisfying the conditions of $(a,b;n)$-conjecture is $a+1$ and the types of b are as follows:
(1) If $n \equiv 0$ $(mod\ 2)$, then $b = \{\frac{n+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$;
(2) If $n \equiv 1$ $(mod\ 2)$, then $b = \{\frac{n+1+4(k-1)}{2} - a \mid 1 \leq k \leq a+1\}$

Lemma 7 Let a be a fixed value. If P_n is $(a,b;n)$-graceful with $n \leq 6a+2$, then P_s is $(a,t;s)$-graceful with $s \leq 6a+2$.

Proof Suppose that $s > 6a+2$ and a, t, s satisfy the conditions of $(a,t;s)$-conjecture, there exists $n_0 \in \{4a+1, 4a+2, \cdots, 6a+2\}$ and $m \in Z^+$ such that $s = n_0 + 2m(a+1)$.

Case (a) $s \equiv 0$ $(mod\ 2)$.

According to Lemma 6, there exists $k_0 \in \{0, 2, \cdots, a+1\}$ such that $t = \frac{s+4(k_0-1)}{2} - a$. We define $b_0 = \frac{n_0+4(k_0-1)}{2} - a$, then $t = b_0 + m(a+1)$. It follows from Lemma 6 and the contents of subject that the labeling $g(a, b_0; n_0)$ of P_{n_0} exists. In view of Lemma 4, we obtain that there is a labeling $g(a,t;s)$ of P_s.

Case (b) $s \equiv 1$ $(mod\ 2)$.

According to Lemma 6, there exists $k_0 \in \{0, 2, \cdots, a+1\}$ such that $t = \frac{s+1+4(k_0-1)}{2} - a$. We define $b_0 = \frac{n_0+1+4(k_0-1)}{2} - a$, then $t = b_0 + m(a+1)$. It follows from Lemma 6 and the contents of subject that the labeling $g(a, b_0; n_0)$ of P_{n_0} exists. In view of Lemma 4, we obtain that there is a labeling $g(a,t;s)$ of P_s. □

Lemma 8 Let a, b, n satisfy the conditions of $(a,b;n)$-conjecture and $b = 2a+1$. Then a, $n-(a+b)$, $n-b$ satisfy the conditions of $(a, n-(a+b); n-b)$-conjecture with $4a+1 < n \leq 6a+2$.

Proof We define $m = n-(a+b)-a = n-4a-1$, $k = n-b = n-2a-1$ and $s = \frac{k(k+1)}{2}$. Since $b = 2a+1$ is an odd, then the possible cases of a, b, n satisfying (a,b,n)-conjecture are as follows:

(1) a is an odd, $n \equiv 0$ $(mod\ 4)$.

It is easy to get m, k and $s = k(\frac{n}{2} - a)$ are odds. Since $4a+1 < n \leq 6a+2$, we have $m \neq 0$. So $0 < |n-(a+b)-a| \leq \frac{n-b+1}{2}$ and $\frac{n-b}{2} \leq n-(a+b)+a \leq \frac{3(n-b)}{2}$.

(2) a is an odd, $n \equiv 3$ $(mod\ 4)$.

It is easy to get m, k and s are evens. Since $4a+1 < n \leq 6a+2$, we have $m \neq 0$. So $0 < |n-(a+b)-a| \leq \frac{n-b+1}{2}$ and $\frac{n-b}{2} \leq n-(a+b)+a \leq \frac{3(n-b)}{2}$.

(3) a is an even, $n \equiv 1$ $(mod\ 4)$.

It is easy to get m, k and s are evens. Since $4a+1 < n \leq 6a+2$, we have

$m \neq 0$. So $0 < |n-(a+b)-a| \leq \frac{n-b+1}{2}$ and $\frac{n-b}{2} \leq n-(a+b)+a \leq \frac{3(n-b)}{2}$.

(4) a is an even, $n \equiv 2$ $(mod\ 4)$.

It is easy to get m, k and s are odds. Since $4a+1 < n \leq 6a+2$, we have $m \neq 0$. So $0 < |n-(a+b)-a| \leq \frac{n-b+1}{2}$ and $\frac{n-b}{2} \leq n-(a+b)+a \leq \frac{3(n-b)}{2}$.□

Lemma 9 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture. If $n \geq 4a+1$, then a, $b-(a+1)$, $n-2(a+1)$ satisfy the condition(1) of $(a, b-(a+1); n-2(a+1))$-conjecture.

Proof According to Lemma 1, considering the following cases:

(1) When a is an odd, $n \equiv 0$ $(mod\ 4)$, b is an odd, $b-(a+1)$ is an odd, $b-(a+1)-a$ is an even, as $n-2(a+1) \equiv 0$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an even.

(2) When a is an odd, $n \equiv 1$ $(mod\ 4)$, b is an even, $b-(a+1)$ is an even, $b-(a+1)-a$ is an odd, as $n-2(a+1) \equiv 1$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an odd.

(3) When a is an odd, $n \equiv 2$ $(mod\ 4)$, b is an even, $b-(a+1)$ is an even, $b-(a+1)-a$ is an odd, as $n-2(a+1) \equiv 2$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an odd.

(4) When a is an odd, $n \equiv 3$ $(mod\ 4)$, b is an odd, $b-(a+1)$ is an odd, $b-(a+1)-a$ is an even, as $n-2(a+1) \equiv 0$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an even.

(5) When a is an even, $n \equiv 0$ $(mod\ 4)$, b is an even, $b-(a+1)$ is an odd, $b-(a+1)-a$ is an odd, as $n-2(a+1) \equiv 2$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an odd.

(6) When a is an even, $n \equiv 1$ $(mod\ 4)$, b is an odd, $b-(a+1)$ is an even, $b-(a+1)-a$ is an even, as $n-2(a+1) \equiv 3$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an even.

(7) When a is an even, $n \equiv 2$ $(mod\ 4)$, b is an odd, $b-(a+1)$ is an even, $b-(a+1)-a$ is an even, as $n-2(a+1) \equiv 0$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an even.

(8) When a is an even, $n \equiv 3$ $(mod\ 4)$, b is an even, $b-(a+1)$ is an odd, $b-(a+1)-a$ is an odd, as $n-2(a+1) \equiv 1$ $(mod\ 4)$, we have $\frac{(n-2(a+1))(n-2(a+1)+1)}{2}$ is an odd. □

Lemma 10 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture, $a > 0$ and $b \neq 2a+1$. Then a, $b-(a+1)$, $n-2(a+1)$ satisfy the conditions of $(a, b-(a+1); n-2(a+1))$-conjecture for $n \geq 4a+2$.

Proof We have a, $b-(a+1)$, $n-2(a+1)$ satisfy the condition(1) of $(a, b-(a+1); n-2(a+1))$-conjecture from Lemma 9.

Case (a) n is an even.

Because of $n \geq 4a+2$, we get $n-2(a+1) \geq 2a$. It is clear that $0 < |b-a| \leq \frac{n+1}{2}$ is equal to $0 < |b-a| \leq \frac{n}{2}$. So $a-\frac{n}{2} \leq b \leq a+\frac{n}{2}$ and $\frac{n}{2}-a \leq b \leq \frac{3n}{2}-a$. Since $n > 4a+1$, we have $a-\frac{n}{2} \leq \frac{n}{2}-a$ and $a+\frac{n}{2} \leq \frac{3n}{2}-a$. Hence $\frac{n}{2}-a \leq b \leq \frac{n}{2}+a$ and $\frac{n-2(a+1)}{2}-a \leq b-(a+1) \leq \frac{n-2(a+1)}{2}+a$. Since $n-2(a+1) \geq 2a$, we have $a-\frac{n-2(a+1)}{2} \leq \frac{n-2(a+1)}{2}-a$ and $a+\frac{n-2(a+1)}{2} \leq \frac{3(n-2(a+1))}{2}-a$. Hence $a-\frac{n-2(a+1)}{2} \leq b-(a+1) \leq a+\frac{n-2(a+1)}{2}$ and $\frac{n-2(a+1)}{2}-a \leq b-(a+1) \leq \frac{3(n-2(a+1))}{2}-a$. Since $2(a+1)$ is an even, we have $n-2(a+1)$ is an even. Hence $0 \leq |b-(a+1)-a| \leq \frac{n-2(a+1)+1}{2}$ and $\frac{n-2(a+1)}{2} \leq b-(a+1)+a \leq \frac{3(n-2(a+1))}{2}$. Since $b \neq 2a+1$, we have $0 < |b-(a+1)-a| \leq \frac{n-2(a+1)+1}{2}$ and $\frac{n-2(a+1)}{2} \leq b-(a+1)+a \leq \frac{3(n-2(a+1))}{2}$. Thus the condition (2) and condition(3) are obtained.

Case (b) n is an odd.

It is clear that $\frac{n}{2} \leq a+b \leq \frac{3n}{2}$ is equal to $\frac{n+1}{2} \leq a+b \leq \frac{3n}{2}$. According to $0 < |b-a| \leq \frac{n+1}{2}$ and $\frac{n+1}{2} \leq a+b \leq \frac{3n}{2}$, then $a-\frac{n+1}{2} \leq b \leq a+\frac{n+1}{2}$ and $\frac{n+1}{2}-a \leq b \leq \frac{3n}{2}-a$. Since $n > 4a+1$, we have $a-\frac{n+1}{2} \leq \frac{n+1}{2}-a$ and $a+\frac{n+1}{2} \leq \frac{3(n+1)}{2}-a$. Hence $\frac{n+1}{2}-a \leq b \leq \frac{n+1}{2}+a$ and $\frac{n-2(a+1)+1}{2}-a \leq b-(a+1) \leq \frac{n-2(a+1)+1}{2}+a$. Since $n-2(a+1) \geq 2a$, we have $a-\frac{n-2(a+1)+1}{2} \leq \frac{n-2(a+1)+1}{2}-a$ and $a+\frac{n-2(a+1)}{2} \leq \frac{3(n-2(a+1))}{2}-a$. Hence $a-\frac{n-2(a+1)+1}{2} \leq b-(a+1) \leq a+\frac{n-2(a+1)+1}{2}$ and $\frac{n-2(a+1)+1}{2}-a \leq b-(a+1) \leq \frac{3(n-2(a+1)+1)}{2}-a$. Since $b \neq 2a+1$, we have $0 < |b-(a+1)-a| \leq \frac{n-2(a+1)+1}{2}$ and $\frac{n-2(a+1)}{2} \leq b-(a+1)+a \leq \frac{3(n-2(a+1))}{2}$. Thus the condition(2) and condition(3) are obtained. □

Lemma 11 Let a, b, n satisfy the conditions of $(a,b;n)$-conjecture. If P_n is $(a,b;n)$-graceful with $a > 0$ and $n \leq 4a$, then P_s is $(a,t;s)$-graceful with $4a+1 < s \leq 6a+2$ and a, t, s satisfy the conditions of $(a,t;s)$-conjecture.

Proof Case (a) $t = 2a+1$.

Since $4a+1 < s \leq 6a+2$, it follows from Lemma 8 and the contents of subject that the labeling $g^{-1}(a,s-(a+t);s-t)$ exists. Let $v_0, v_1, \cdots, v_s$ be vertices of path P_s. Put the vertices labeling f as follows:

$$(a, s-a+1, a-1, s-a+2, \cdots, s, 0) \wedge (g^{-1}(a, s-(a+t); s-t) \oplus (a+1)).$$

Hence $f(v_0) = a$, $f(v_s) = a+(a+1) = b$. Considering vertices labeling set A of P_s.

$A=\{f(v_i) \mid i = 0,1,\cdots,2a\} \cup \{f(v_i) \mid i = 2a+1, 2a+2, \cdots, s\}$

$=\{f(v_i) \mid i = 0,1,\cdots,2a\} \cup (\{g^{-1}(v_{i+2a+1}) \mid i = 0,1,\cdot,s-2a-1\} \oplus (a+1))$

$=\{0,1,\cdots,a\} \cup \{s-a+1, s-a+2, \cdots, s\} \cup \{a+1, a+2, \cdots, s-b+(a+1)\}$

$=\{0,1,\cdots,a\}\cup\{s-a+1,s-a+2,\cdots,s\}\cup\{a+1,a+2,\cdots,s-a\}$
$=\{0,1,\cdots,s\}$.
Because there are $s+1$ different elements in set A, which are the same as the vertices of P_s, we have the vertices labeling are different. We denote B as the set of edges labeling of P_s. Then
$B=\{f^*(v_{i-1}v_i)\mid i=1,2,\cdots,s\}$
$=\{|f(v_{i-1})-f(v_i)|\mid i=1,2,\cdots,2a\}\cup\{|f(v_{2a})-f(v_{2a+1})|\}\cup\{|f(v_{i-1})-f(v_i)|\mid i=2a+2,2a+3,\cdots,s\}$
$=\{s-2a+1,s-2a+2,\cdots,s\}\cup\{|0-(s-(a+t)+(a+1))|\}\cup\{1,2,\cdots,s-t\}$
$=\{s-2a+1,s-2a+2,\cdots,s\}\cup\{s-2a\}\cup\{1,2,\cdots,s-2a-1\}$
$=\{1,2,\cdots,s\}$.
The number of the edges of P_s is the same as the number of elements in set B, thus f^* is a one to one mapping.

Case (b) $t\neq 2a+1$.

Since $4a+1<s\leq 6a+2$, it follows from Lemma 9, Lemma 10 and the contents of subject that the labeling $g(a,t-(a+1);s-2(a+1))$ exists. Let $v_0,v_1,\cdots,v_s$ be vertices of path P_s. Put the vertices labeling f as follows:

$$(a,s-a,a-1,s-a+1,\cdots,0,s)\wedge(g(a,t-(a+1);s-2(a+1))\oplus(a+1))$$

Hence $f(v_0)=a$, $f(v_s)=t-(a+1)+(a+1)=t$. Considering vertices labeling set A of P_s.
$A=\{f(v_i)\mid i=0,1,\cdots,2a+1\}\cup\{f(v_i)\mid i=2a+2,2a+3,\cdots,s\}$
$=\{0,1,\cdots,a\}\cup\{s-a,s-a+1,\cdots,s\}\cup\{0+(a+1),1+(a+1),\cdots,s-2(a+1)+(a+1)\}$
$=\{0,1,\cdots,s\}$
The number of the vertices of P_s is the same as the number of elements in set A, thus f is a one to one mapping from vertices onto the set of $\{0,1,\cdots,s\}$.
We denote B as the set of edges labeling of P_s. Then
$B=\{f^*(v_{i-1}v_i)\mid i=1,2,\cdots,s\}$
$=\{|f(v_{i-1})-f(v_i)|\mid i=1,2,\cdots,2a+1\}\cup\{|f(v_{2a+1})-f(v_{2a+2})|\}\cup\{|f(v_{i-1})-f(v_i)|\mid i=2a+3,2a+4,\cdots,s\}$
$=\{s-2a,s-2a+1,\cdots,s\}\cup\{|s-(2a+1)|\}\cup\{1,2,\cdots,s-2(a+1)\}$
$=\{1,2,\cdots,s\}$
The number of the edges of P_s is the same as the number of elements in set B, thus f^* is a one to one mapping. We obtain f is an $(a,t;s)$-graceful labeling of P_s from case(a) and case(b). □

Lemma 12 Let a,b,n be non-negative integers satisfying the conditions of $(a,b;n)$-conjecture and $3a+1\leq n\leq 4a+1$. If $z\leq 2a$, there exists a graceful labeling $f(x,y,z)$ of P_z. If P_z satisfies the conditions of $f(x,y,z)$-conjecture, then there exists an $(a,b;n)$-graceful labeling $g(a,b;n)$ of P_n

with $b \neq n-2a$, $\frac{6a+3-n}{2} \leq b \leq \frac{3(n-2a}{2}$.

Proof First, we will prove that $n-(3a+1), b-(a+1)$ and $n-(2a+1)$ satisfy the conditions (2), (3) of $(n-3(a+1), b-(a+1); n-(2a+1))$−conjecture.

(1) It is easy to prove that $n-(3a+1)-(b-(a+1)) = n-2a-b$ has the same parity as $\frac{(n-(2a+1))(n-2a)}{2}$.

(2) we will prove that $n-(3a+1), b-(a+1), n-(2a+1)$ satisfy the conditions (2), (3) of $(n-(3a+1), b-(a+1); n-(2a+1))$-conjecture.

When n is an even, as a, b, n satisfy the conditions of $(a, b; n)$-conjecture and $2a \leq 3a+1 \leq n$, we will obtain $\frac{n}{2}-a \leq b \leq \frac{n}{2}+a$ by Lemma 5. Since $3a+1 \leq n \leq 4a+1$, we know that $\frac{6a+3-n}{2} \geq \frac{n}{2}-a$ and $\frac{3(n-2a)}{2} \leq \frac{n}{2}+a$. So $\frac{n}{2}-a \leq b \leq \frac{3(n-2a)}{2}$ and $\frac{6a+3-n}{2} \leq b \leq \frac{n}{2}+a$. Thus $|n-(3a+1)-(b-(a+1))| \leq \frac{n-(2a+1)+1}{2}$ and $\frac{n-(2a+1)}{2} \leq n-(3a+1)+b-(a+1) \leq \frac{3(n-(2a+1))}{2}$. Since $b \neq n-2a$, we obtain that the condition (2) and condition (3) of $(n-(3a+1), b-(a+1); n-(2a+1))$are true.

When n is an odd, as a, b, n satisfy the conditions of $(a, b; n)$-conjecture and $2a \leq 3a+1 \leq n$, we will get $\frac{n+1}{2}-a \leq b \leq \frac{n+1}{2}+a$ by Lemma 5. Since $3a+1 \leq n \leq 4a+1$, we have $\frac{6a+3-n}{2} \geq \frac{n+1}{2}-a$ and $\frac{3(n-2a)}{2} \leq \frac{n+1}{2}+a$, so $\frac{n+1}{2}-a \leq b \leq \frac{3(n-2a)}{2}$ and $\frac{6a+3-n}{2} \leq b \leq \frac{n+1}{2}+a$. Thus $|n-(3a+1)-(b-(a+1))| \leq \frac{n-(2a+1)+1}{2}$ and $\frac{n-(2a+1)}{2} \leq n-(3a+1)+b-(a+1) \leq \frac{3(n-(2a+1))}{2}$. Since $b \neq n-2a$, we know that the conditions (2), (3) of $(n-(3a+1), b-(a+1); n-(2a+1))$-conjecture are true. Since $3a+1 \leq n \leq 4a+1$, we know that $n-(3a+1) \leq a, n-(2a+1) \leq 2a$. Because there exists the $(n-(3a+1), b-(a+1); n-(2a+1))$-graceful labeling $f(n-(3a+1), b-(a+1), n-(2a+1))$ of $P_{n-(2a+1)}$, we have the $(a, b; n)$-graceful labeling $g(a, b, n)$ of P_n: $g(a, b, n) = (a, n-a+1, a-1, n-a+2, \cdots, n-1, 1, n, 0) \wedge (f(n-(3a+1), b-(a+1), n-(2a+1)) \oplus (a+1))$. It's easy to prove that $g(a, b, n)$ is an $(a, b; n)$-graceful labeling of P_n. □

Corollary 13 Let a be odd with $a \geq 1$ and $z \leq 2a$. If P_z satisfies the conditions $(x, y; z)$-conjecture and graceful labeling is $f(x, y, z)$, then $a, t, 4a+1$ satisfy the conditions of $(a, t; 4a+1)$-conjecture and P_{4n+1} is $(a, t; 4a+1)$-graceful.

Proof According to Lemma 6, $t \in \{a+1, a+2, \cdots, 3a+1\}$, we get $\frac{6a+3-(4a+1)}{2} \leq t \leq \frac{3(4a+1-2a)}{2}$. As $n = 4a+1 \equiv 1 \ (mod\ 4a)$, we have t is an even from Lemma 1. So $t \neq n-2a$, otherwise, contradicting to $n-2a = 2a+1$ is an odd. According to Lemma 12, there exists an $(a, t; 4a+1)$-graceful labeling $g(a, t, 4a+1)$ of P_{4n+1}. □

According to Lemma 7, Lemma 11, Lemma 12, and Corollary 13, we will obtain Theorem 1.

§3 The $(a, b; n)$-gracefulness of P_n

In this section, we will discuss $(a, b; n)$-gracefulness of P_n in the case of a is $1, 2, 3, 4, 5$ and 6.

Lemma 14 If a, b, n satisfy the conditions of $(a, b; n)$-conjecture, then $n-a$, $n-b$, n satisfy the conditions of $(n-a, n-b; n)$-conjecture.

Proof Since $n-a-(n-b)$ has the same parity as $b-a$, $|n-a-(n-b)| = |b-a|$ and $b \neq a$, it is easy to get that $n-a$, $n-b$, n satisfy the condition (1) and (2) of $(n-a, n-b; n)$-conjecture.

It remains to prove condition (3). Since $\frac{n}{2} \leq a+b \leq \frac{3n}{2}$, we have $2n - \frac{n}{2} \leq 2n-(a+b) \leq 2n - \frac{3n}{2}$, so $\frac{n}{2} \leq n-a+n-b \leq \frac{3n}{2}$. The conclusion is obtained. □

Lemma 15 If x, y, m satisfy the conditions of $(x, y; m)$-conjecture and the labeling of path P_m is $g(x, y; m)$ for $x < a$, then
(1)The labeling of P_n is $g(a, b; n)$ for $b < a$ or $n-b < a$;
(2)the labeling of P_n is $g(a, b; n)$ for $n < 2a$.

Proof (1) If $b < a$, we have $g(a, b; n) = n - g(b, a; n) = n - g^{-1}(a, b; n)$. If $n-b < a$, we have $g(a, b; n) = n - g^{-1}(n-b, n-a; n)$.

(2) If $n < 2a$, we have $b < a$ or $a < b \leq n \leq 2a$. If $a < b \leq n \leq 2a$, then $n-b < n-a \leq 2a-a = a$. In view of (1), the conclusion is obtained.□

Lemma 16 Let a, b, n satisfy the conditions of $(a, b; n)$-conjecture. If P_z satisfy $(x, y; z)$-conjecture for $z \leq 2a$ and graceful labeling of P_z is $f(x, y, z)$, then there exists the $(a, b; n)$-graceful labeling $g(a, b, n)$ of P_n for $a \geq 3, 3a+1+\frac{a}{3} \leq n \leq 4a, b = n-2a$.

Proof It's easy to prove that $a, n-(3a+1), n-2(a+1)$ satisfy the conditions of $(a, n-(3a+1); n-2(a+1))$-graceful conjecture. As $3a+1 \leq n \leq 4a+1$, we have $n-2(a+1) \leq 2a$. Since there exists the $(a, n-(3a+1); n-2(a+1))$-graceful labeling $f(a, n-(3a+1), n-2(a+1))$ of $P_{n-2(a+1)}$, we will obtain the $(a, b; n)$-graceful labeling $g(a, b, n)$ of P_n: $g(a, b, n) = (a, n-a, a-1, n-a+1, \cdots, 1, n-1, 0, n), \wedge(f(a, n-(3a+1), n-2(a+1)) \oplus (a+1))$. It's easy to prove that $g(a, b, n)$ is the graceful labeling of P_n. □

Next the conjectures will be obtained when $a = 0, 1, 2, 3, 4, 5, 6$ by recursion, b which can be transformed into a known number via Lemma 12 is marked by $[b]$, via Lemma 15 is marked by (b) and via Lemma 16 is marked by $\{b\}$. For fixed $a \neq 0$, from Lemma 15 and Theorem 1, if a is an even, the case $2a+1 \leq n \leq 4a+1$ can only be discussed; if a is an odd, the case $2a+1 \leq n \leq 4a$ can only be discussed.

Theorem 2 ([4]) P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 0$.

Proof We have $4a + 1 = 1$ and $6a + 2 = 2$ for $a = 0$. According to lemma 7, we can only consider the cases when $n = 1$ and $n = 2$. If $n = 1$, then $b = 1$ and $g(0, 1; 1) = (0, 1)$. If $n = 2$, then $b = 1$ and $g(0, 1; 2) = (0, 2, 1)$. □

Theorem 3 ([5]) P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b, n)$-graceful for $a = 1$.

Proof Since $a = 1$, we have $2a + 1 = 3, 4a = 4, n = 3, b = (3); n = 4, b = [3], g(1, 3, 4) = (1, 4, 0) \wedge (g(, 1, 1) \oplus 2)$. □

Theorem 4([5]) P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 2$.

Proof We have $2a + 1 = 5, 4a + 1 = 9$ for $a = 2$. According to Lemma 12, Lemma 14, Lemma 15 and Theorem 1, we have
$n = 5, b = (1), 3, (5), g(2, 3, 5) = (2, 1, 4, 0, 5, 3)$.
$n = 6, b = (1), 3, (5), g(2, 3, 6) = (2, 6, 0, 5, 4, 1, 3)$.
$n = 7, b = [4], (6)$.
$n = 8, b = \{4\}, [6]$.
$n = 9, b = (1), [3], 5, [7], g(2, 5, 9) = (2, 4, 7, 3, 9, 0, 8, 1, 6, 5)$. □

Theorem 5 P_n satisfying conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 3$.

Proof If $a = 3$, then $2a + 1 = 7$ and $4a = 12$, so we have
$n = 7, b = (1), (5), (7)$.
$n = 8, b = (1), 5, (7), g(3, 5, 8) = (3, 8, 0, 7, 1, 4, 2, 6, 5)$.
$n = 9, b = (2), 4, 6, (8), g(3, 4, 9) = (3, 5, 2, 6, 7, 1, 8, 0, 9, 4), g(3, 6, 9) = (3, 7, 1, 8, 0, 9, 4, 2, 5, 6)$.
$n = 10, b = (2), 4, [6], (8), g(3, 4, 10) = (3, 5, 10, 0, 9, 1, 8, 2, 6, 7, 4)$.
$n = 11, b = \{5\}, [7], (9)$.

$n = 12, b = [5], [7], [9]$. □

Theorem 6 P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 4$.

Proof If $a = 4$, then $2a + 1 = 9$ and $4a + 1 = 17$, so we have
$n = 9, b = (1), (3), 5, (7), (9), g(4, 5, 9) = (5, 9, 0, 8, 1, 7, 2, 3, 6, 4)$.
$n = 10, b = (1), (3), 5, (7), (9), g(4, 5, 10) = (4, 10, 0, 9, 1, 8, 7, 2, 6, 3, 5)$.
$n = 11, b = (2), 6, (8), (10), g(4, 6, 11) = (4, 3, 7, 5, 8, 2, 9, 1, 10, 0, 11, 6)$.
$n = 12, b = (2), 6, 8, (10), g(4, 6, 12) = (4, 5, 8, 3, 7, 9, 2, 10, 1, 11, 0, 12, 6)$,
$g(4, 8, 12) = (4, 12, 0, 11, 1, 10, 3, 9, 5, 2, 7, 6, 8)$.
$n = 13, b = (3), 5, [7], 9, (11)$,
$g(4, 5, 13) = (4, 13, 0, 12, 1, 11, 3, 10, 6, 9, 8, 2, 7, 5)$,
$g(4, 9, 13) = (4, 13, 0, 12, 1, 11, 3, 10, 5, 7, 8, 2, 6, 9)$.
$n = 14, b = (3), 5, [7], [9], (11)$,
$g(4, 5, 14) = (4, 7, 11, 6, 8, 9, 3, 10, 2, 12, 1, 13, 0, 14, 5)$.
$n = 15, b = [6], [8], [10], (12)$.
$n = 16, b = [6], \{8\}, [10], [12]$.
$n = 17, b = [5], [7], 9, [11], [13]$,
$g(4, 9, 17) = (4, 12, 3, 14, 2, 15, 1, 16, 0, 17, 7, 13, 6, 8, 11, 10, 5, 9)$. □

Theorem 7 P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 5$.

Proof If $a = 5$, then $2a + 1 = 11$ and $4a = 20$, so we have
$n = 11, b = (1), (3), (7), (9), (11)$.
$n = 12, b = (1), (3), 7, (9), (11)$,
$g(5, 7, 12) = (5, 3, 6, 12, 0, 11, 1, 10, 2, 9, 4, 8, 7)$.
$n = 13, b = (2), (4), 6, 8, (10), (12)$,
$g(5, 6, 13) = (5, 4, 8, 3, 10, 2, 11, 1, 12, 0, 13, 7, 9, 6)$,
$g(5, 8, 13) = (5, 7, 3, 10, 4, 12, 0, 13, 2, 11, 1, 6, 9, 8)$.
$n = 14, b = (2), (4), 6, 8, (10), (12)$,
$g(5, 6, 14) = (5, 9, 14, 0, 13, 1, 12, 2, 11, 3, 10, 4, 7, 8, 6)$,
$g(5, 8, 14) = (5, 9, 4, 6, 7, 10, 3, 11, 2, 12, 1, 13, 0, 14, 8)$.
$n = 15, b = (3), 7, 9, (11), (13)$,
$g(5, 7, 15) = (5, 10, 6, 8, 9, 15, 0, 14, 1, 13, 2, 12, 3, 11, 4, 7)$,
$g(5, 9, 15) = (5, 10, 6, 8, 7, 4, 11, 3, 12, 2, 13, 1, 14, 0, 15, 9)$.
$n = 16, b = (3), 7, [9], 11, (13)$,
$g(5, 7, 16) = (5, 9, 8, 6, 11, 4, 12, 3, 13, 2, 14, 1, 15, 0, 16, 10, 7)$,
$g(5, 11, 16) = (5, 12, 4, 13, 2, 14, 1, 15, 0, 16, 6, 8, 3, 9, 10, 7, 11)$.
$n = 17, b = (4), 6, [8], [10], 12, (14)$,
$g(5, 6, 17) = (5, 10, 9, 13, 7, 14, 4, 16, 0, 17, 2, 15, 1, 12, 3, 11, 8, 6)$,
$g(5, 12, 17) = (5, 13, 4, 14, 2, 15, 1, 16, 0, 17, 6, 11, 7, 8, 10, 3, 9, 12)$.

$n = 18, b = (4), 6, \{8\}, [10], [12], (14)$,
$g(5, 6, 18) = (5, 11, 10, 7, 9, 18, 0, 17, 1, 16, 2, 15, 3, 14, 4, 12, 8, 13, 6)$.
$n = 19, b = [7], \{9\}, [11], [13], (15)$.
$n = 20, b = [7], [9], [11], [13], [15]$. □

Theorem 8 P_n satisfying the conditions of $(a, b; n)$-conjecture is $g(a, b; n)$-graceful for $a = 6$.

Proof If $a = 6$, then $2a + 1 = 113$ and $4a + 1 = 25$, so we have
$n = 13, b = (1), (3), (5), 7, (9), (11), (13)$,
$g(6, 7, 13) = (6, 10, 8, 5, 4, 9, 3, 11, 2, 12, 1, 13, 0, 7)$.
$n = 14, b = (1), (3), (5), 7, (9), (11), (13)$,
$g(6, 7, 14) = (6, 10, 8, 5, 4, 9, 3, 11, 2, 12, 1, 13, 0, 14, 7)$.
$n = 15, b = (2), (4), 8, (10), (12), (14)$,
$g(6, 8, 15) = (6, 4, 9, 10, 7, 11, 5, 12, 3, 13, 2, 14, 1, 15, 0, 8)$.
$n = 16, b = (2), (4), 8, 10, (12), (14)$,
$g(6, 8, 16) = (6, 7, 9, 5, 11, 4, 12, 2, 16, 0, 15, 3, 14, 1, 10, 13, 8)$,
$g(6, 10, 16) = (6, 16, 0, 15, 1, 14, 2, 13, 4, 12, 5, 11, 9, 8, 3, 7, 10)$.
$n = 17, b = (3), (5), 7, 9, 11, (13), (15)$,
$g(6, 7, 17) = (6, 11, 17, 0, 16, 1, 15, 2, 14, 3, 13, 4, 12, 5, 9, 8, 10, 7)$,
$g(6, 9, 17) = (6, 10, 7, 8, 13, 11, 5, 12, 4, 14, 3, 15, 2, 16, 1, 17, 0, 9)$,
$g(6, 11, 17) = (6, 17, 0, 16, 1, 15, 2, 14, 4, 13, 5, 12, 8, 3, 9, 7, 10, 11)$.
$n = 18, b = (3), (5), 7, 9, 11, (13), (15)$,
$g(6, 7, 18) = (6, 10, 8, 9, 14, 11, 5, 12, 4, 13, 3, 15, 2, 16, 1, 17, 0, 18, 7)$,
$g(6, 9, 18) = (6, 10, 7, 8, 13, 11, 5, 12, 4, 14, 3, 15, 2, 16, 1, 17, 0, 18, 9)$,
$g(6, 11, 18) = (6, 18, 0, 17, 1, 16, 2, 15, 4, 14, 5, 13, 9, 3, 10, 12, 7, 8, 11)$.
$n = 19, b = (4), 8, [10], 12, (14), (16)$,
$g(6, 8, 19) = (6, 11, 10, 12, 9, 13, 7, 14, 5, 15, 4, 16, 3, 17, 2, 18, 1, 19, 0, 8)$,
$g(6, 12, 19) = (6, 19, 0, 18, 1, 17, 2, 16, 4, 15, 5, 14, 11, 3, 10, 9, 7, 13, 8, 12)$.
$n = 20, b = (4), 8, [10], [12], 14, (16)$,
$g(6, 8, 20) = (6, 11, 9, 10, 14, 7, 13, 4, 12, 15, 5, 16, 3, 17, 2, 18, 1, 19, 0, 20, 8)$,
$g(6, 14, 20) = (6, 20, 0, 19, 1, 18, 2, 17, 4, 16, 5, 15, 11, 3, 12, 9, 10, 8, 13, 7, 14)$.
$n = 21, b = (5), 7, \{9\}, [11], [13], 15, (17)$,
$g(6, 7, 21) = (6, 11, 9, 17, 10, 13, 12, 8, 14, 5, 15, 4, 16, 3, 18, 2, 19, 1, 20, 0, 21, 7)$,
$g(6, 15, 21) = (6, 21, 0, 20, 1, 19, 2, 18, 4, 17, 5, 16, 11, 9, 10, 13, 3, 12, 8, 14, 7, 15)$.
$n = 22, b = (5), 7, [9], [11], [13], [15], (17)$,
$g(6, 7, 22) = (6, 22, 0, 21, 1, 20, 2, 19, 4, 18, 5, 17, 12, 10, 11, 14, 3, 13, 9, 15, 8, 16, 7)$.
$n = 23, b = [8], [10], [12], [14], [16], (18)$.
$n = 24, b = [8], [10], 12, [14], [16], [18]$.
$n = 25, b = [7], [9], [11], 13, [15], [17], [19]$,

$g(6,13,25) = (6,25,0,24,1,23,2,22,4,21,5,20,12,7,19,8,18,9,15,16,3,17,10,14,11,13)$. □

Acknowledgment

This work is supported by NSFC (No. 61172094). The authors would like to thank the referees for their illuminating comments which greatly improve the presentation of the paper.

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