On the Semisymmetric Graphs of Order $2p^3$: faithful and primitive Case

LI WANG [†]

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, 454000, China

Abstract

A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. A semisymmetric graph must be bipartite whose automorphism group has two orbits of same size on the vertices. One of our long term goals is to determine all the semisymmetric graphs of order $2p^3$, for any prime p. All these graphs Γ with the automorphism group $\mathrm{Aut}(\Gamma)$ are divided into two subclasses: (I) $\mathrm{Aut}(\Gamma)$ acts unfaithfully on at least one bipart; and (II) $\mathrm{Aut}(\Gamma)$ acts faithfully on both biparts. In [9], [19] and [20], a complete classification was given for Subclass (I). In this paper, a partial classification is given for Subclass (II), when $\mathrm{Aut}(\Gamma)$ acts primitively on one bipart.

Keywords: Permutation group, Primitive group, Semisymmetric graph

1 Introduction

All graphs considered in this paper are finite, connected, simple and undirected. For a graph $\Gamma = (V, E)$ with the vertex set V and edge set E, by $\{u, v\}$ we denote an edge of Γ, and by $\mathrm{Aut}(\Gamma)$ its full automorphism group. A graph Γ is said to be *regular* if all the vertices have the same degree. Set $G = \mathrm{Aut}(\Gamma)$. The graph Γ is said to be *vertex-transitive* and *edge-transitive* if G acts transitively on V and E, respectively. If Γ is bipartite with the bipartition $V = W \cup U$, then we let G^+ be a subgroup of G preserving both W and U. Since Γ is connected, we know that either $|G : G^+| = 2$ or $G = G^+$, depending on whether or not there exists an automorphism which interchanges the two biparts. For $A \leq G^+$, Γ is said to be *A-semitransitive* if A acts transitively on both W and U, while an G^+-semitransitive graph is simply said to be *semitransitive*.

[†] E-mail: wanglimath@hpu.edu.cn

A graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is a semitransitive bipartite graph with two biparts of equal size.

The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed several infinite families of such graphs and proposed eight open problems, see [11]. Afterwards, much work has been done on semisymmetric graphs. In particular, by using group-theoretical methods, Iofinova and Ivanov [15] in 1985 classified cubic semisymmetric graphs whose automorphism group acts primitively on both biparts. This was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups and some methods from graph coverings, some new results of semisymmetric graphs have appeared, see [6, 7, 10, 12, 16, 17, 18] and so on.

In [11], Folkman proved that there are no semisymmetric graphs of order $2p$ and $2p^2$ where p is a prime. In [10], Du and Xu classified semisymmetric graphs of order $2pq$ for two distinct primes p and q. Therefore, a natural question is to determine semisymmetric graphs of order $2p^3$, where p is a prime. Since the smallest semisymmetric graphs have order 20 (see [11]), we let $p \geq 3$. It was proved in [17] that the Gray graph of order 54 is the only cubic semisymmetric graph of order $2p^3$. The classification of all the semisymmetric graphs of order $2p^3$ is still one of the attractive and difficult open problems. These graphs Γ are naturally divided into two subclasses:

Subclass (I): $\mathrm{Aut}(\Gamma)$ acts unfaithfully on at least one bipart;

Subclass (II): $\mathrm{Aut}(\Gamma)$ acts faithfully on both biparts.

A complete classification for Subclass (I) has been given in [9], [19] and [20]. In our further research, we shall concentrate on Subclass (II), which can be divided into two cases:

Case (i): $\mathrm{Aut}(\Gamma)$ acts primitively on at least one bipart;

Case (ii): $\mathrm{Aut}(\Gamma)$ acts imprimitively on both biparts.

In this paper, we shall determine the graphs in Case (i). Remark that Case (ii) will be the most difficult and complicated part. Before giving the main theorem of this paper, we first define four graphs Γ.

Definition 1.1 *Define four bipartite graphs Γ with bipartition $V = W \cup U$, where*

$$W = \{(i,j,k) \mid i,j,k \in Z_3\}, \quad U = \{[x,y,z] \mid x,y,z \in Z_3\},$$

and with the respective edge set:

(1) **Graph $\Gamma_3(3)$:**

$$E = \{\{(i,j,k),[x,y,z]\} \mid x=j, y=k, z=0; \\ x=k, y=i, z=1; x=i, y=j, z=2, \\ i,j,k,x,y,z \in Z_3\},$$

(2) **Graph $\Gamma_3(6)$:**

$$E = \{\{(i,j,k),[x,y,z]\} \mid x=j+m, y=k, z=0; \\ x=k+m, y=i, z=1; x=i+m, y=j, z=2, \\ m \in F_3^*, i,j,k,x,y,z \in Z_3\},$$

(3) **Graph $\Gamma_3(12a)$:**

$$E = \{\{(i,j,k),[x,y,z]\} \mid x=j+m, y=k+s, z=0; \\ x=k+m, y=i+s, z=1; x=i+m, y=j+s, z=2, \\ m,s \in F_3^*, i,j,k,x,y,z \in Z_3\},$$

(4) **Graph $\Gamma_3(12b)$:**

$$E = \{\{(i,j,k),[x,y,z]\} \mid x=j+m, y=k, z=0; \\ x=j, y=k+s, z=0; x=m+k, y=i, z=1; \\ x=k, y=s+i, z=1; x=i+m, y=j, z=2; \\ x=i, y=s+j, z=2, m,s \in F_3^*, i,j,k,x,y,z \in Z_3\},$$

Remark 1.2 *For the four graphs in Definition 1.1, one may get the following properties from Section 3:*

(1) **Graph $\Gamma_3(3)$:** $\mathrm{Aut}(\Gamma) = Z_3^3 \rtimes (S_4Z_2)$ *and the valency is* 3;

(2) **Graph $\Gamma_3(6)$:** $\mathrm{Aut}(\Gamma) = Z_3^3 \rtimes (A_4Z_2)$ *and the valency is* 6;

(3) **Graphs $\Gamma_3(12a)$ and $\Gamma_3(12b)$:** $\mathrm{Aut}(\Gamma) = Z_3^3 \rtimes (S_4Z_2)$ *and the valency is* 12.

The main result of this paper is the following theorem, which will be proved in Section 3.

Theorem 1.3 *Let Γ be a semisymmetric graph of order $2p^3$, where p is a prime. Suppose that the automorphism group* $\mathrm{Aut}(\Gamma)$ *acts faithfully on both biparts and primitively on at least one bipart. Then $p = 3$ and $\Gamma \cong \Gamma_3(3), \Gamma_3(6), \Gamma_3(12a)$ or $\Gamma_3(12b)$ as defined in Definition 1.1. Moreover, for the above four graphs,* $\mathrm{Aut}(G)$ *acts primitively on one bipart and imprimitively on other bipart. In particular, the graph $\Gamma_3(3)$ is isomorphic to the Gray graph.*

2 Preliminaries

In this section, we introduce some notation and give some preliminary results.

For group-theoretic concepts and notation, the reader is referred to [1, 4, 14]. Moreover, for a permutation group G on Ω, a subset $\Delta \subset \Omega$ and a subgroup $N \leq G$ preserving Δ, by G_Δ and $G_{(\Delta)}$ we denote the stabilizer of G relative to Δ setwise and pointwise, respectively, and by Δ_N the set of N-orbits on Δ. For a prime p, by $p^i \parallel n$ we mean that $p^i \mid n$ but $p^{i+1} \nmid n$. For a ring S, let S^* be the multiplicative group of all the units in S. For a group G and a subgroup H of G, by $[G : H]$ we denote the set of right cosets of H in G, where the action of G on $[G : H]$ is always assumed to be the right multiplication action. For any α in the n-dimensional row vector space $\mathbf{V} = \mathbf{V}(n,p)$ over $\mathrm{GF}(p)$, we denote by t_α the translation corresponding to α in the affine group $\mathrm{AGL}(n,p)$ and by N the translation subgroup. Then $\mathrm{AGL}(n,p) \cong N \rtimes \mathrm{GL}(n,p)$. We adopt matrix notation for $\mathrm{GL}(n,p)$ and so we have $g^{-1}t_\alpha g = (t_\alpha)^g = t_{\alpha g}$, for any $t_\alpha \in N \leq \mathrm{AGL}(n,p)$ and any $g \in \mathrm{GL}(n,p)$. By $||a,b,c||$, we denote a diagonal matrix of order 3 with entries a, b, c from top left to bottom right.

The following definition is basic for this paper.

Definition 2.1 *Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i Rd_iL$. Define a bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{(Lg, Rdg) \mid g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.*

The following three propositions give some properties for bi-coset graphs.

Proposition 2.2 *[10] The graph $X = \mathbf{B}(G, L, R; D)$ is a well-defined bipartite graph. Under the right multiplication action of G on $V(X)$, the graph X is G-semitransitive. The kernel of the action of G on $V(X)$ is $\mathrm{Core}_G(L) \cap \mathrm{Core}_G(R)$, the intersection of the cores of the subgroups L and R in G. Furthermore, we have*

(i) *X is G-edge-transitive if and only if $D = RdL$ for some $d \in G$;*

(ii) *the degree of any vertex in $[G : L]$ (resp. $[G : R]$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;*

(iii) *X is connected if and only if G is generated by elements of $D^{-1}D$;*

(iv) *$X \cong \mathbf{B}(G, L^a, R^b; D')$ where $D' = \bigcup_i R^b\ (b^{-1}d_ia)L^a$, for any $a, b \in G$;*

(v) $X \cong \mathbf{B}(\hat{G}, L^\sigma, R^\sigma; D^\sigma)$ *where σ is an isomorphism from G to $\hat{G}$ (it does not appear in [10] but it is easy to prove.)*

Proposition 2.3 *[10] Suppose Y is a G-semitransitive graph with bipartition $V(Y) = U(Y) \cup W(Y)$. Take $u \in U(Y)$ and $w \in W(Y)$. Set $D = \{g \in G \mid w^g \in Y_1(u)\}$. Then D is a union of double cosets of G_w and G_u in G, and $Y \cong \mathbf{B}(G, G_u, G_w; D)$.*

Proposition 2.4 *[10] Let $X = \mathbf{B}(G, L, R; D)$. If there exists an involutory automorphism σ of G such that $L^\sigma = R$ and $D^\sigma = D^{-1}$, then X is vertex-transitive. In particular,*

(1) *If G is abelian and acts regularly on both parts of X, then X is vertex-transitive. In other words, bi-Cayley graphs of abelian groups are vertex-transitive.*

(2) *If there exists an involutory automorphism σ of G such that $L^\sigma = R$, and the lengths of the orbits of L on $[G : R]$ (or of the orbit of R on $[G : L]$) are all distinct, then X is vertex-transitive.*

(3) *If the representations of G on the two parts of X are equivalent and all suborbits of G relative to L are self-paired, then X is vertex-transitive.*

Finally, we give some group theoretical results.

Proposition 2.5 *[13] Let T be a nonabelian simple group with a subgroup $H < T$ satisfying $|T : H| = p^a$, for p a prime. Then one of the following holds:*

(i) $T = A_n$ *and* $H = A_{n-1}$ *with* $n = p^a$;

(ii) $T = \mathrm{PSL}(n, q)$ *and H is the stabilizer of a projective point or a hyperplane in* $\mathrm{PG}(n-1, q)$, *and* $|T : H| = (q^n - 1)/(q - 1) = p^a$;

(iii) $T = \mathrm{PSL}(2, 11)$ *and* $H = A_5$, *where T has two subgroups isomorphic to A_5, which are not conjugate in T;*

(iv) $T = M_{11}$ *and* $H = M_{10}$;

(v) $T = M_{23}$ *and* $H = M_{22}$;

(vi) $T = \mathrm{PSU}(4, 2)$ *and H is a subgroup of index* 27.

Proposition 2.6 *[2] For an odd prime p, let $\overline{G} = \mathrm{PSL}(3, p)$ and $\overline{H}$ a proper subgroup of $\overline{G}$. Then one of the following holds:*

(I) *If $\overline{H}$ has no nontrivial normal elementary abelian subgroup, then $\overline{H}$ is conjugate in* $\mathrm{GL}(3, p)/Z(\mathrm{SL}(3, p))$ *to one of the following groups:*

(i) $\mathrm{PSL}(2,7)$, *with* $p^3 \equiv 1 \pmod 7$;

(ii) A_6, *with* $p \equiv 1, 19 \pmod{30}$;

(iii) $\mathrm{PSL}(2,5)$, *with* $p \equiv \pm 1 \pmod{10}$;

(iv) $\mathrm{PSL}(2,p)$ *or* $\mathrm{PGL}(2,p)$ *for* $p \geq 5$.

(II) *If* $\overline{H}$ *has a nontrivial normal elementary abelian subgroup, then* $\overline{H}$ *is conjugate to a subgroup of one of the following subgroups:*

(i) $Z_{(p^2+p+1)/(3,p-1)} \rtimes Z_3$;

(ii) *the subgroup* $\overline{F}$ *of all matrices with only one nonzero entry in each row and column, and* $\overline{F}$ *contains the subgroup* $\overline{D}$ *of all diagonal matrices as a normal subgroup such that* $\overline{F}/\overline{D} \cong S_3$;

(iii) *the point- or line-stabilizer of a given point* $\langle(1,0,0)^T\rangle$ *or the line* $\langle(0,\ \alpha, \beta)^T \mid \alpha, \beta \in F_p\rangle$;

(iv) *the group* $\overline{M}$ *such that* $\overline{M}$ *contains a normal subgroup* $\overline{N} \cong Z_3^2$ *and* $\overline{M}/\overline{N}$ *is isomorphic to* $\mathrm{SL}(2,3)$ *if* $p \equiv 1 \pmod 9$ *or to* Q_8 *if* $p \equiv 4, 7 \pmod 9$.

Proposition 2.7 *[19] For an odd prime p, let G be a primitive group on* Ω, *where* $|\Omega| = p^2$. *Suppose that G has a faithful transitive representation of degree* p^3. *Then G is isomorphic to one of the following groups:*

(1) $\mathrm{P\Gamma L}(2,8)$, *for* $p = 3$;

(2) $Z_3^2 \rtimes H$, *where* $H = \mathrm{SL}(2,3)$ *or* $\mathrm{GL}(2,3)$, *for* $p = 3$;

(3) $Z_5^2 \rtimes H$, *where* $H = \mathrm{SL}(2,5)$ *or* $\mathrm{GL}(2,5)$, *for* $p = 5$;

(4) $Z_7^2 \rtimes \mathrm{SL}(2,7)$, *for* $p = 7$;

(5) $Z_{11}^2 \rtimes \mathrm{SL}(2,11)$, *for* $p = 11$.

All these representations are imprimitive.

3 Proof of Theorem 1.3

To show Theorem 1.3, let Γ be a semisymmetric graph of order $2p^3$ with the bipartition $V = W \cup U$, where p is an odd prime and set $G = \mathrm{Aut}(\Gamma)$. Suppose that G acts faithfully on both biparts and G acts primitively on one bipart, say U. Then the proof of Theorem 1.3 consists of the following two lemmas.

Lemma 3.1 *Under the above hypothesis, G is an affine group.*

Proof Suppose that $G = \text{Aut}(\Gamma)$ acts primitively on U. Then G has a faithful primitive representation of degree p^3. Checking the well-known O'Nan-Scott Theorem [4], G is of almost simple type, product type or affine group type. In what follows, we shall show that the first two cases cannot occur, via showing that the graph is vertex-transitive.

(1) G is of almost simple type.

Set $T = \text{soc}(G)$, a nonabelian simple group. Then by checking Proposition 2.5, we get that soc(G) is A_{p^3}, PSL(n, q) or $PSU(4, 2)$, where $\frac{q^n-1}{q-1} = p^3$. We divide the proof into the following three cases.

Case 1. $T = A_{p^3}$.

Suppose that T is primitive on W. Then two representations T on U and W are equivalent. In other words, T_w and T_u are conjugate in T for some $w \in W$ and $u \in U$. Consider the action of T_u on $[T : T_w]$. Then there are two orbits with different lengths 1 and $p^3 - 1$, respectively. By Proposition 2.4(3), the graph Γ is vertex-transitive.

Suppose that T is imprimitive on W, with a block system, say $\mathcal{B}_W$, where $|\mathcal{B}_W| = p$ or p^2. Since T is a simple group, T acts faithfully on $\mathcal{B}_W$. The first case $|\mathcal{B}_W| = p$ is clearly impossible. Suppose that $|\mathcal{B}_W| = p^2$. By Proposition 2.7, T is imprimitive on $\mathcal{B}_W$, with a block system, say $\overline{\mathcal{B}_W}$, where $|\overline{\mathcal{B}_W}| = p$. This case is impossible too.

Case 2. $T = \text{PSL}(n, q)$, where $\frac{q^n-1}{q-1} = p^3$.

Suppose that T is primitive on W. In this case, T has two equivalent representations on U and W. Consider the action of T_u on $[T : T_w]$. Then there are two orbits with different lengths $q^{n-1}/q - 1$ and q^n, respectively. By Proposition 2.4(3), the graph Γ is vertex-transitive.

Suppose that T is imprimitive on W, with a block system, say $\mathcal{B}_W$. Since T is a simple group, T acts faithfully on $\mathcal{B}_W$. By the same argument as the above case, it is impossible.

Case 3. $T = \text{PSU}(4, 2)$.

Suppose that T is primitive on W, note that in this case, $p = 3$. Then two representations T on U and W are equivalent. In other words, T_w and T_u are conjugate in T for some $w \in W$ and $u \in U$. Consider the action of T_u on $[T : T_w]$. Since $|T_u| = 2^4|A_5|$, we get $13 \nmid |T_u|$. Hence, there are three suborbits of T_u acting on $[T : T_w]$, with length, say $1, r$, and s, respectively, where $r, s \neq 1$. Since $13 \nmid |T_u|$ and $r + s = 26$, we know $r \neq s$. Thus the three orbits have the distinct length. By Proposition 2.4(3), the graph Γ is vertex-transitive.

For the case when T is imprimitive on W, by using the same arguments as in Case 1, we know that this case cannot occur.

(2) G is of product type.

In this case, $G = (M \times M \times M) \rtimes Z_3$ or $(M \times M \times M) \rtimes S_3$, where M is a nonabelian simple group of S_p and $p \geq 5$. Let P be the Sylow p-subgroup of G. Then $P = Z_p^3$. Since G is transitive on both U and W, we get $p^3 \parallel |G|$ and thus P is transitive on both U and W. From Proposition 2.4, the graph is vertex-transitive. □

Lemma 3.2 $\Gamma \cong \Gamma_3(3)$, $\Gamma_3(6)$, $\Gamma_3(12a)$ *or* $\Gamma_3(12b)$, *as defined in Definition 1.1, and moreover, for these four graphs, G acts primitively on one bipart and imprimtively on other bipart.*

Proof By Lemma 3.1, G is an affine group on U. Set $G = T \rtimes H$, where $T \cong Z_p^3$ is the socle of G and H is a subgroup of $\mathrm{GL}(3,p)$.

Suppose that G acts primitively on both U and W. Then T acts transitively and then regularly on both biparts. From Proposition 2.4, the graph Γ is vertex transitive, a contradiction.

From now on, suppose that G acts primitively on U and imprimitively on W, that is, H is an irreducible subgroup of $\mathrm{GL}(3,p)$. Let $\mathbf{V}$ be the 3-dimensional row vector space over $\mathrm{GF}(p)$ and set

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1) \in \mathbf{V}.$$

We shall identify T with the translation subgroup of $\mathrm{AGL}(3,p)$, where by t_v we denote the translation corresponding to $v \in \mathbf{V}$, so that

$$T = \langle t_{v_1} \rangle \times \langle t_{v_2} \rangle \times \langle t_{v_3} \rangle.$$

By the above argument, Γ is isomorphic to one of bi-coset graphs $X = \mathbf{B}(G, G_w, G_u; D)$ for some $w \in W$ and $u \in U$. The proof is divided into the following three steps:

Step (1): Determination of groups G, G_w and G_u.

Recall that T acts regularly on U and intransitively on W. Let $\mathcal{B}_W := \{B_i | i \in Z_{p^3/m}\}$ be a complete m-block system induced by T, where $m = p$ or p^2. Let K be the kernel of the action of G on $\mathcal{B}_W$.

Let $H_1 = H \cap \mathrm{SL}(3,p)$ and $\overline{H_1} = H_1 Z(\mathrm{SL}(3,p))/Z(\mathrm{SL}(3,p)) \leq \mathrm{PSL}(3,p)$. If $m = p$, then $|\mathcal{B}_W| = p^2$. Since H is transitive on $\mathcal{B}_W$, it follows that $p^2 \mid |H|$. Checking Proposition 2.6, we get that either $H = \mathrm{GL}(3,p)$ or $H \cong \mathrm{SL}(3,p)$. However, it is easy to know that $\mathrm{GL}(3,p)$ and $\mathrm{SL}(3,p)$ have no permutation representation of degree p^2. Then $m = p^2$ and $|\mathcal{B}_W| = p$.

It follows that H has a permutation representation of degree p. By Propositions 2.5 and 2.6, H_1 is either the case (iv) of (I) where $p = 5, 7, 11$ or the case (ii) of (II) where $p = 3$, in Proposition 2.6.

Suppose that H_1 is case (iv) of (I) for $p = 5, 7, 11$ and $p \parallel |H|$. Consider the action of $T = Z_p^3$ on B. Recalling that $|\mathcal{B}_W| = p$ and $|B| = p^2$, where $B \in \mathcal{B}_W$ contains $w = v_1 = (1,0,0)$.It is easy to get that for any $w \in B$, $T_w \cong Z_p$ and $T_w \lhd G_w$. Since $(|T_w|, |G_w/T_w|) = 1$, there exists a normal p-complement subgroup M of T_w in G_w. Observe that $G_w = T_w \rtimes M$. Let $\overline{M_1} = (M \cap \mathrm{SL}(3,p))Z(\mathrm{SL}(3,p))/Z(\mathrm{SL}(3,p))$. Note that for three cases, that is, $p = 5, 7, 11$, $\overline{M_1}$ contains subgroup of isomorphic to A_4. Consider the action of H on $\mathcal{B}_W$. Then $H_B < G_B$. Since $T < G_B, T \cap H_B = 1$ and $|T||H_B| = |G_B|$, it follows that $G_B = T \rtimes H_B$. Since $M \leq G_w \leq G_B$ and $|M| = |H_B|$, it follows that M and H_B are conjugated in G_B. Without loss of any generality, we may assume $M = H_B$. Then $M \leq H$. Since M normalizes T_w, we get

$$\overline{M_1} \leq \overline{\begin{pmatrix} |A|^{-1} & 0 \quad 0 \\ a & \\ & A \\ b & \end{pmatrix}},$$

where $a, b \in \mathrm{GF}(p), A \in \mathrm{GL}(2,p)$. Since $p \nmid |M|$, it follows $M \leq \mathrm{GL}(2,p)$. But $\mathrm{GL}(2,p)$ cannot contain a subgroup A_4, see [8], a contradiction.

Suppose that H_1 is case (ii) of (II) for $p = 3$ in Proposition 2.6. By the same argument as the above case, we have $M = H_B = H_w$ and M normalizes T_w. Then

$$M \leq \begin{pmatrix} \alpha & 0 \quad 0 \\ \beta & \\ & A \\ \gamma & \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathrm{GF}(3), A \in \mathrm{GL}(2,3)$. Since $3 \nmid |M|$, it follows that $M \leq \mathrm{GL}(2,3) \times Z_2$. Since $\overline{H \cap \mathrm{SL}(3,3)} \lesssim \overline{F}$, where $\overline{F}/\overline{D} \cong S_3$, and H is an irreducible subgroup of $\mathrm{GL}(3,p)$, we can get $\overline{H \cap \mathrm{SL}(3,3)} \cong A_4$ or S_4. Then H is isomorphic to one of the following groups : $A_4, S_4, \langle A_4, d\rangle$ and $\langle S_4, d\rangle$, where

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let

$$a = t_{(1,0,0)},\ b = t_{(0,1,0)},\ c = t_{(0,0,1)}$$

and

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \pi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mu = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

More clearly, these elements have the following relations:

$$a^\tau = a, b^\tau = b^2, c^\tau = c^2, a^\pi = a^2, b^\pi = b, c^\pi = c^2,$$

$$a^\rho = b, b^\rho = c, c^\rho = a, a^\mu = a^2, b^\mu = c^2, c^\mu = b^2.$$

Step (2): Determination of the possible bi-coset graphs X.

From the last step, there are four cases for group H, that is $A_4, S_4, \langle A_4, d\rangle$ and $\langle S_4, d\rangle$. We discuss them separately.

Case 1: $H \cong A_4$

No loss of any generarlity, let

$$H = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \rangle \rtimes \langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle = \langle \tau, \pi\rangle \rtimes \langle \rho\rangle \cong D_4 \rtimes Z_3,$$

Then

$$G = T \rtimes H = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle) \rtimes (\langle \tau, \pi\rangle \rtimes \langle \rho\rangle),$$

$$G_u = H = \langle \tau, \pi\rangle \rtimes \langle \rho\rangle, G_w = \langle a\rangle \rtimes \langle \tau, \pi\rangle.$$

Then Suppose that our graph Γ is isomorphic to a bi-coset graph $X = \mathbf{B}(G; G_w, G_u, D)$, where $D = G_u g_1 G_w$ for some $g_1 \in G$. Set

$$\begin{array}{rcl} U(X) & = & [G : G_u] = \{G_u a^i b^j c^k \mid i, j, k \in Z_3\}, \\ W(X) & = & [G : G_w] = \{G_w b^x c^y \rho^z \mid x, y, z \in Z_3\}. \end{array}$$

In the following, we shall determine the possible double cosets. Since H is transitive on $\mathcal{B}_W$, there exist edges leading $u = G_u$ to every block in W. Every points in $[G : G_w]$ has the form $G_w b^x c^y \rho^z$ for $x, y, z \in Z_3$ and every block has the form $\{G_w b^x c^y \rho^z \mid x, y \in Z_3\}$ for $z \in Z_3$. Then we just need to consider the orbit of G_u on $[G : G_w]$ containing $G_w b^x c^y$, which corresponds to a double coset $D(x, y) := G_w b^x c^y G_u$, where $x, y \in Z_3$.

Assume $x, y \neq 0$. Under the conjugacy action, τ, μ and π fixes G_w and G_u. Moreover, τ maps $D(1,0)$ to $D(2,0)$; $D(0,1)$ to $D(0,2)$; $D(1,1)$ to $D(2,2)$; $D(1,2)$ to $D(2,1)$, respectively. By the same argument, μ maps $D(1,0)$ to $D(0,1)$ and π maps $D(1,1)$ to $D(1,2)$. Therefore, $\langle \tau, \mu, \pi\rangle$ induces an isomorphism between those corresponding bi-coset graphs. By Proposition 2.2, we just need to consider three cases $D(0,0)$, $D(1,0)$ and $D(1,1)$, separately.

By computing, we get that

$$D(0,0) = G_w G_u = \{G_u \rho^l | l \in Z_3\}, \text{ where } |D(0,0)|/|G_w| = 3;$$

$$D(1,0) = G_w b G_u = \{G_w b^m \rho^l | m \in F_3^*, l \in Z_3\},$$

where $|D(0,0)|/|G_w| = 6$;

$$D(1,1) = G_w bc G_u = \{G_w b^m c^s \rho^l | m, s \in F_3^*, l \in Z_3\},$$

where $|D(0,0)|/|G_w| = 12$.

Correspondingly, for any $i, j, k \in Z_3$, the neighborhood of $G_u a^i b^j c^k$ in the bi-coset graph is

$\{G_w b^j c^k, G_w b^k c^i \rho, G_w b^i c^j \rho^2\}$, so the valency is 3;

$\{G_w b^{j+m} c^k, G_w b^{k+m} c^i \rho, G_w b^{i+m} c^j \rho^2 | m \in F_3^*\}$, so the valency is 6;

$\{G_w b^{j+m} c^{k+s}, G_w b^{k+m} c^{i+s} \rho, G_w b^{i+m} c^{j+s} \rho^2 | m, s \in F_3^*\}$, so the valency is 12.

Now, relabel the vertex $G_u a^i b^j c^k \in U$ and $G_w b^x c^y \rho^z \in W$ by (i, j, k) and $[x, y, z]$, respectively. Then the above three graphs are precisely the graphs $\Gamma_3(3)$, $\Gamma_3(6)$ and $\Gamma_3(12a)$, that is

(1) Graph $\Gamma_3(3)$:

$$E = \{\{(i,j,k),[x,y,z]\} \mid x = j, y = k, z = 0; x = k, y = i, z = 1; x = i, y = j, z = 2; i, j, k, x, y, z \in Z_3\}.$$

(2) Graph $\Gamma_3(6)$:

$$E = \{\{(i,j,k),[x,y,z]\} \mid x = j+m, y = k, z = 0; x = k+m, y = i, z = 1; x = i+m, y = j, z = 2; i, j, k, x, y, z \in Z_3, m \in F_3^*\}.$$

(3) Graph $\Gamma_3(12a)$:

$$E = \{\{(i,j,k),[x,y,z]\} \mid x = j+m, y = k+s, z = 0; x = k+m, y = i+s, z = 1; x = i+m, y = j+s, z = 2; i, j, k, x, y, z \in Z_3, m, s \in F_3^*\}.$$

Case 2: $H = S_4$.

In this case,

$$G_u = \langle \tau, \pi \rangle \rtimes (\langle \rho \rangle \rtimes \langle \mu \rangle), G_w = \langle a \rangle \rtimes (\langle \tau, \pi \rangle \rtimes \langle \mu \rangle),$$

Similarly, we just need to consider two cases $D(0,0)$, $D(1,0)$ and $D(1,1)$. By computing, the neighborhood of $G_u a^i b^j c^k$ in the bi-coset graph is

$\{G_w b^j c^k, G_w b^k c^i \rho, G_w b^i c^j \rho^2\}$, so the valency is 3;

$\{G_w b^{j+m} c^k, G_w b^j c^{k+s}, G_w b^{k+m} c^i \rho, G_w b^k c^{s+i} \rho, G_w b^{i+m} c^j \rho^2, G_w b^i c^{s+j} \rho^2 | m \in F_3^*\}$, so the valency is 12;

$\{G_w b^{j+m} c^{k+s}, G_w b^{k+m} c^{i+s} \rho, G_w b^{i+m} c^{j+s} \rho^2 | m, s \in F_3^*\}$, so the valency is 12.

Relabel the vertex $G_u a^i b^j c^k \in U$ and $G_w b^x c^y \rho^z \in W$ by (i, j, k) and $[x, y, z]$, respectively. Then from the first and third cases again, we get the graphs as same as $\Gamma_3(3)$ and $\Gamma_3(6)$, respectively. Therefore, the second case gives us a new graph, that is $\Gamma_3(12b)$ as defined in Defnition 1.1, that is

(4) Graph $\Gamma_3(12b)$:

$$\begin{aligned} E \;=\; & \{\{(i,j,k),[x,y,z]\} \mid x = j+m, y = k, z = 0; \\ & x = j, y = k+s, z = 0; x = m+k, y = i, z = 1; \\ & x = k, y = s+i, z = 1; x = i+m, y = j, z = 2; \\ & x = i, y = s+j, z = 2; i,j,k,x,y,z \in Z_3, m, s \in F_3^*\}. \end{aligned}$$

Case 3: $H = \langle A_4, d \rangle$.

In this case,

$$G_u = H = (\langle \tau, \pi \rangle \rtimes \langle \rho \rangle) \rtimes \langle d \rangle,\ G_w = (\langle a \rangle \rtimes \langle \tau, \pi \rangle) \rtimes \langle d \rangle,$$

and we can get the same graphs as in the case $H = A_4$.

Case 4: $H = \langle S_4, d \rangle$.

In this case,

$$G_u = H = (\langle \tau, \pi \rangle \rtimes (\langle \rho \rangle \rtimes \langle \mu \rangle)) \rtimes \langle d \rangle,\ G_w = (\langle a \rangle \rtimes (\langle \tau, \pi \rangle \rtimes \langle \mu \rangle)) \rtimes \langle d \rangle,$$

and we can get the same graphs as in the case $H = S_4$.

Step (3): Determination of the automorphism groups, isomorphism classes and semisymmetry for the four graphs

we already know that for the graphs $\Gamma_3(3), \Gamma_3(12a)$ and $\Gamma_3(12b)$, their automorphism groups contains a subgroup isomorphic to $Z_3^3 \rtimes (S_4 Z_2)$, and for the graph $\Gamma_3(6)$, its automorphism group contains a subgroup $Z_3^3 \rtimes (A_4 Z_2)$. We can get that $|\Gamma_3(3)| = |\Gamma_3(12a)| = |\Gamma_3(12b)| = 1296$ and $|\Gamma_3(6)| = 648$ by Magma [3]. Therefore, $\mathrm{Aut}(\Gamma_3(3))$, $\mathrm{Aut}(\Gamma_3(12a))$ and $\mathrm{Aut}(\Gamma_3(12b))$ are isomorphic to $Z_3^3 \rtimes (S_4 Z_2)$, and $\mathrm{Aut}(\Gamma_3(6))$ is isomorphic to $Z_3^3 \rtimes (A_4 Z_2)$. Consequently, all of them are semisymmetric. Moreover, by Magma again, we know $\Gamma_3(12a) \ncong \Gamma_3(12b)$. □

Acknowledgments: This work was supported by Mathematical Tianyuan Foundation of China (No. 11426093), National Natural Science Foundation of China (No. 11501172), Natural Science Foundation of the Education Department of Henan Province(No. 14B110002) and Doctor Foundation of

Henan Polytechnic University (No. B2013-058). The authors thank the anonymous referees for the valuable comments and constructive suggestions.

References

[1] N.L. Biggs, A.T. White, *Permuation groups and combinatorial structures*, Cambrige University Press, 1979.

[2] D.M. Bloom, The subgroups of PSL$(3, q)$ for odd q, *Trans. Amer. Math. Soc.*, **127**(1967), 150-178.

[3] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: The user language. *J. Symbolic Comput.*, **24**(1997), 235-265.

[4] J. D. Dixon, B. Mortimer, *Permutation Groups*, Springer-Verlag, New York/berlin, 1996.

[5] S.F. Du, J.H. Kwak, Simple groups PSL$(3, p)$ and nonorientable regular maps, *J. Algebra*, **321**(2009), 1367-1382.

[6] S.F. Du, Construction of Semisymmetric Graphs, *Graph Theory Notes of New York*, **XXIX**(1995), 47-49.

[7] S.F. Du, D. Marušič, An infinite family of biprimitive semisymmetric graphs, *J. Graph Theory*, **32**(1999), 217-228.

[8] S.F. Du, D. Marušič and A.O. Waller, On 2-arc-transitive covers of complete graphs, *J. Comb. Theory B*, **74**(1998), 276–290.

[9] S.F. Du, L. Wang, A classification of semisymmtric graphs of order $2p^3$: unfaithful case, *J. Algebraic Combin.*41(2015), 275-302.

[10] S.F. Du, M.Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Comm. in Algebra* **28**(2000), 2685-2715.

[11] J. Folkman, Regular line-symmetric graphs, *J. Combin. Theory Ser. B*, **3**(1967), 215-232.

[12] Y.Q. Feng, J.H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin. Theory Ser. B*, **94** (2007), 627-646.

[13] R.M. Guralnick, Subgroups of prime power index in a simple group, *J. Algebra*, **81**(1983), 304-311.

[14] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, 1967.

[15] M. E. Iofinova, A. A. Ivanov, Biprimitive cubic graphs (Russian), in *Investigation in Algebric Theory of Combinatorial Objects*, Proceedings of the seminar, Institute for System Studies, Moscow, 1985, 124-134.

[16] F. Lazebnik, R. Viglione, An infinite series of regular edge-but not vertex-transitive graphs, *J. Graph Theory*, **41**(2002), 249-258.

[17] A. Malnič, D. Marušič, C.Q. Wang, Cubic edge-transitive graphs of order $2p^3$, *Discrete Math.*, **274**(2004), 187-198.

[18] C.W. Parker, Semisymmetric cubic graphs of twice odd order, *European J. Combin.*, **28**(2007), 572-591.

[19] L. Wang, S.F. Du, Semisymmtric graphs of order $2p^3$, *European J. Combin.*, **36**(2014), 393-405.

[20] L. Wang, S.F. Du, X.W. Li, A class of semisymmetric graphs, *Ars Mathmatica Contemporanea*, **7**(2014), 40-53.