Paired domination number of generalized Petersen graphs $P(n, 2)$

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Abstract

Let $G = (V, E)$ be a simple graph. A paired-dominating set of a graph G is a dominating set whose induced subgraph contains a perfect matching. The paired domination number of a graph G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set in G. In this paper, we study the paired domination number of generalized Petersen graphs $P(n, 2)$ and prove that for any integer $n \geq 6, \gamma_p(P(n, 2)) = 2\left(\lfloor \frac{n}{3} \rfloor + n(\text{mod}3)\right)$.

Keywords: generalized Petersen graph, paired domination set, paired domination number
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1. Introduction

Throughout this paper all graphs are finite and simple. Readers are suggested to refer to [1] for graph theoretical terminologies not specified here.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $v \in V$, the open neighborhood of v, $N(v)$, is the set of all vertices adjacent to v in G and the closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d(v) = |N(v)|$. The distance $d(x, y)$ between two vertices x and y in G is the length of the shortest path from x to y. If $S \subseteq V$, then $\langle S \rangle$ is the subgraph induced by S and $N[S] = \bigcup_{v \in S} N[v]$. For $S_1, S_2 \subseteq V$, the distance from $\langle S_1 \rangle$ to $\langle S_2 \rangle$ in G is $d(\langle S_1 \rangle, \langle S_2 \rangle) = min\{d(x, y) | x \in S_1, y \in S_2\}$. S is a dominating set of G if $N[S] = V$. If S is a dominating set of G and for each $v \in V$, $N(v) \cap S \neq \emptyset$, then S is a total dominating set of G. The total domination number, $\gamma_t(G)$, is the minimum cardinality of the total dominating sets of G. If S is a dominating set of G and $\langle S \rangle$ contains a perfect matching,

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then S is a paired dominating set of G. A paired dominating set of G is also a total dominating set of G. The paired domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of paired dominating sets of G. The concept of paired domination number is given by Haynes and Slater[6] and there have been many results concerning the paired domination number of graphs, see, for example [3, 7, 9].

The generalized Petersen graph $P(n,k)$ is the graph with vertex set $V = U \cup W$, where $U = \{u_0, u_1, u_2, ..., u_{n-1}\}$ and $W = \{w_0, w_1, w_2, ..., w_{n-1}\}$, and edge set $E = \{w_i w_{i+k}, u_i u_{i+1}, w_i u_i | 0 \le i \le n-1$, subscripts modulo $n\}$. We call the vertices in U as vertices in the inner circle and the vertices in W as the vertices in the outer circle. The generalized Petersen graphs are widely studied by researchers[4, 10, 5, 8]. In this paper, we study the paired domination number of $P(n,2)$ and prove that for any integer $n \ge 6$, $\gamma_p(P(n,2)) = 2\left(\lfloor \frac{n}{3} \rfloor + n(\text{mod}3)\right)$.

2. Paired domination number of generalized Petersen graphs $P(n,2)$

Lemma 1. ([6]) *For any connected graph G with order $n(n \ge 2)$, $\gamma_p(G) \ge \gamma_t(G)$.*

Lemma 2. ([2]) *For any generalized Petersen graphs $P(n,2)(n \ge 6)$, $\gamma_t(P(n,2)) = 2\lceil \frac{n}{3} \rceil$.*

Lemma 3. *If $n = 3k (k \ge 2)$, then $\gamma_p(P(n,2)) = 2k$.*

proof. Let $T = \{w_i, u_i | i = 3t, 0 \le t \le k-1\}$, then T is a paired dominating set of $P(n,2)$ and $|T| = 2k$. Thus $\gamma_p(P(n,2)) \le 2k$. On the other hand, by Lemma 1 and 2, $\gamma_p(P(n,2)) \ge 2k$. Therefore, $\gamma_p(P(n,2)) = 2k$.□

Lemma 4. *If $n = 3k+1 (k \ge 2)$, then $\gamma_p(P(n,2)) = 2k+2$.*

proof. Let $T = \{w_i, u_i | i = 3t, 0 \le t \le k-1\} \cup \{w_{3k-2}, u_{3k-2}\}$, then T is a paired dominating set of $P(n,2)$ and $|T| = 2k+2$. Thus $\gamma_p(P(n,2)) \le 2k+2$. On the other hand, by Lemma 1 and 2, $\gamma_p(P(n,2)) \ge 2k+2$. Therefore, $\gamma_p(P(n,2)) = 2k+2$.□

Lemma 5. *If $n = 3k+2 (k \ge 2)$, then $\gamma_p(P(n,2)) = 2k+4$.*

proof. The order of $P(n,2)$ is $2n = 6k+4$. By Lemma 1 and 2, $\gamma_p(P(n,2)) \ge 2k+2$. Let $T = \{w_i, u_i | i = 3t, 0 \le t \le k-1\} \cup \{w_{3k-2}, u_{3k-2}, w_{3k+1}, u_{3k+1}\}$, then T is a paired dominating set of $P(n,2)$ and $|T| = 2k+4$. Thus $\gamma_p(P(n,2)) \le$

$2k+4$. Therefore, $2k+2 \leq \gamma_p(P(n,2)) \leq 2k+4$. In the following, we will prove that $\gamma_p(P(n,2)) \neq 2k+2$.

We assume that $\gamma_p(P(n,2)) = 2k+2$. Let S be a paired dominating set of $P(n,2)$ and $|S| = 2k+2$.

Claim 1. *The components of $\langle S \rangle$ are P_2 or P_4, and there is at most one component that is P_4.*

proof. We assume that there is a component of $\langle S \rangle$, H, with $|V(H)| \geq 5$. Since $P(n,2)$ is a 3-regular graph, there are at most $3|V(H)| - 3$ vertices of $P(n,2)$ dominated by $V(H)$. S is also a total dominating set of $P(n,2)$, so $S - V(H)$ dominates at most $3|S - V(H)|$ vertices of $P(n,2)$. Thus S dominates at most $3|S| - 3 = 6k+3 < 6k+4$ vertices of $P(n,2)$, which contradicts the fact that S is a dominating set of $P(n,2)$. Therefore, the order of each component of $\langle S \rangle$ is less than five. Since, the minimum circle in $P(n,2)$ is 5-circle, $\langle S \rangle$ doesn't contain circle. Further, $\langle S \rangle$ has perfect matching, thus $\langle S \rangle$ doesn't contain $K_{1,3}$, P_3 or K_1. Therefore, the components of $\langle S \rangle$ are P_2 or P_4. We assume that the number of P_4 in $\langle S \rangle$ is $t (t \geq 2)$. Since $V(P_4)$ dominates at most 10 vertices of $P(n,2)$ and $V(P_2)$ dominates at most 6 vertices of $P(n,2)$, S dominates at most $10t + 6(|S| - 4t)/2 = 6k + 6 - 2t \leq 6k + 2 < 6k + 4$ vertices of $P(n,2)$, a contradiction. Therefore $t \leq 1$, i.e. $\langle S \rangle$ contains at most one P_4. $\diamond$

Claim 2. *If $\langle S \rangle$ contains a P_4, then the distance of each two components of $\langle S \rangle$ in $P(n,2)$ is at least three.*

proof. We assume that there are two components, H_1 and H_2, of $\langle S \rangle$ with $d(H_1, H_2) \leq 2$. Then, $|N(H_1) \cap N(H_2)| \geq 1$. Thus S dominates at most $10 + 6(|S| - 4)/2 - 1 = 6k+3 < 6k+4$ vertices of $P(n,2)$, a contradiction to the assumption that S is a dominating set of $P(n,2)$.$\diamond$

Claim 3. *$\langle S \rangle$ contains only P_2 as its components.*

proof. We assume that $\langle S \rangle$ contains a P_4 as its component. By Claim 1, $\langle S \rangle$ contains only one P_4 and other components are all P_2. According to the symmetry of $P(n,2)$, the cases of P_4 in $P(n,2)$ are illustrated in Fig. 1.

For cases (b),(d) and (g) in Fig.1, $V(P_4)$ dominates nine vertices of $P(n,2)$, then S dominates at most $9+6(|S|-4)/2 = 6k+3 < 6k+4$ vertices of $P(n,2)$, a contradiction.

According to the symmetry of $P(n,2)$, let the subscript of the left vertex of P_4 in Fig.1 to be 0.

For case (a) as illustrated in Fig.2, since S dominates w_4, by Claim 2, $w_6 \in S$. In the same way, S dominatesu_5, thus $u_6 \in S$. By Claim 1, $\langle \{w_6, u_6\} \rangle$ is a

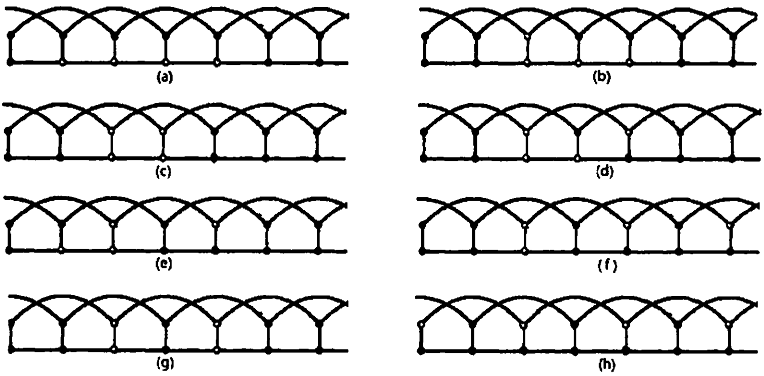

Fig.1: the cases of P_4 in $P(n, 2)$

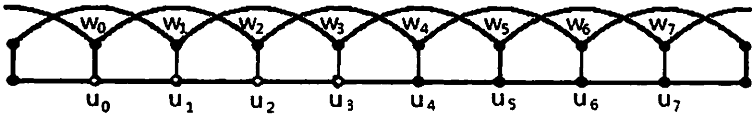

Fig.2: The case (a) in Fig.1

component of $\langle S \rangle$. By Claim 2, $w_3, w_5, w_7, u_5 \notin S$, which contradicts to the fact that S dominates w_5.

For case (c) as illustrated in Fig.3, since S dominates u_3, by Claim 2, $u_4 \in S$. By Claim 1 and Claim 2, $u_5 \in S$. Further, since S dominates w_6 and u_7, by Claim 2, $w_8, u_8 \in S$. By Claim 2, $w_5, w_7, u_7, w_9 \notin S$, which contradicts to the fact that S dominates w_7.

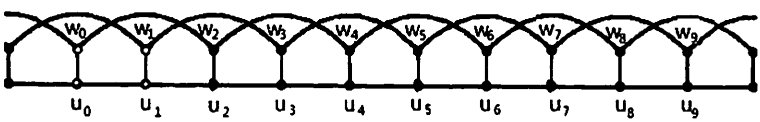

Fig.3: The case (c) in Fig.1

For case (e) as illustrated in Fig.4, since S dominates w_2, by Claim 1 and Claim 2, $w_4, w_6 \in S$. By Claim 2, $u_4, u_5, u_6, w_5 \notin S$, which contradicts to the fact that S dominates u_5.

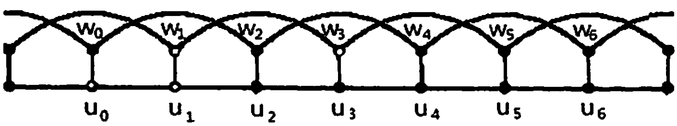

Fig.4: The case (e) in Fig.1

For case (f) as illustrated in Fig.5, by Claim 2, $w_1, u_2, u_3, u_4 \notin S$. Since S dominates u_3, by Claim 1, $w_3, w_5 \in S$. By Claim 2, $w_6, u_5, u_6, u_7 \notin S$, which contradicts to the fact that S dominates u_6.

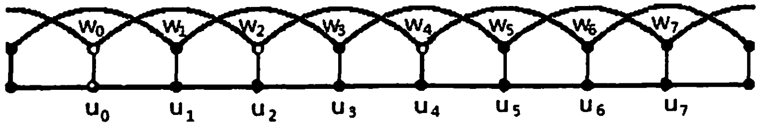

Fig.5: The case (f) in Fig.1

For case (h) as illustrated in Fig.6, since S dominates u_1 and u_3, by Claim 2, $w_1, w_3 \in S$. Thus $w_5, u_4, u_5, u_6 \notin S$, which contradicts to the fact that S dominates u_5.

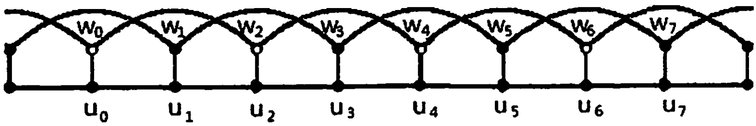

Fig.6: The case (h) in Fig.1

From above, $\langle S \rangle$ contains only P_2 as its components. ◇

Claim 4. *Let* $F = \{v | v \in V - S$ and $|N(v) \cap S| \geq 2\}$, *then* $|F| \leq 2$.

proof. We assume that $|F| \geq 3$, i.e. there are at least three vertices of $V - S$ respectively dominated by two different vertices of S. Since S is also a total dominating set of $P(n,2)$, S dominates at most $3|S| - 3 = 6k + 3 < 6k + 4$ vertices of $P(n,2)$, a contradiction.◇

Claim 5. *If* $\langle\{x, y\}\rangle$ *is a component of* $\langle S \rangle$, *then there is at most one vertex of* x *and* y *belonging to inner circle* U.

proof. We assume that both x and y belongs to inner circle U and $\{x, y\} = \{u_0, u_1\}$ without loss of generality. By Claim 3, $w_0, w_1, u_2 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates w_2, we have $w_4 \in S$. By Claim 3, $w_2, w_4 \in S$, $w_4, u_4 \in S$ or $w_4, w_6 \in S$.

If $w_2, w_4 \in S$ as illustrated in Fig. 7, by Claim 3, we have $u_4 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_3, thus $w_3 \in S$. Therefore w_0, w_1 and u_2 are respectively dominated by two different vertices of S, a contradiction to Claim 4.

If $w_4, u_4 \in S$ as illustrated in Fig.8, by Claim 3, we have $u_3, u_5 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates w_3, thus $w_5 \in S$. By Claim 3, we have $w_3, w_5 \in S$ or $w_5, w_7 \in S$. If $w_3, w_5 \in S$, then w_1, u_3, u_5 are

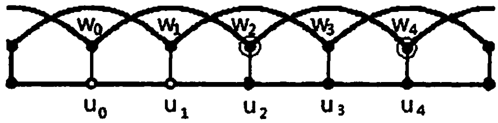

Fig.7: $w_2, w_4 \in S$

respectively dominated by two different vertices of S, a contradiction to Claim 4. If $w_5, w_7 \in S$, then $u_5, w_6, u_7 \notin S$, which contradicts to the fact that S total dominates u_6.

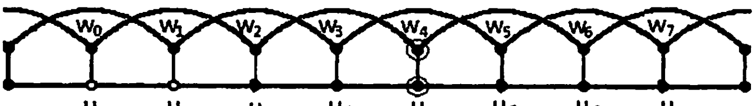

Fig.8: $w_4, u_4 \in S$

If $w_4, w_6 \in S$ as illustrated in Fig. 9, by Claim 3, we have $u_4, u_6, w_8 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_3, we have $w_3 \in S$. By Claim 3, $w_3, u_3 \in S$ or $w_3, w_5 \in S$. If $w_3, u_3 \in S$, then $w_1, u_2, u_4$are respectively dominated by two different vertices of S, a contradiction to Claim 4. If $w_3, w_5 \in S$, then $w_7 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_7, we have $u_8 \in S$. Thus w_1 and w_8 are respectively dominated by two different vertices of S. According to the symmetry, w_0, the symmetric point of w_1 about $\langle\{u_0, u_1\}\rangle$, is also dominated by two different vertices of S,a contradiction to Claim 4.

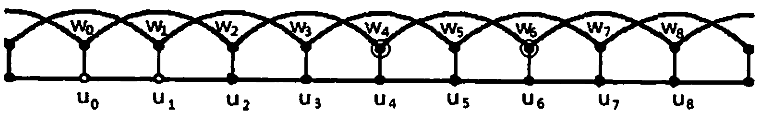

Fig.9: $w_4, w_6 \in S$

In conclusion, Claim 5 is proved.◇

Claim 6. *If $\langle\{x, y\}\rangle$ is a component of $\langle S\rangle$, then there is at most one vertex of x and y belonging to the outer circle W.*

proof. We assume that both x and y belongs to outer circle W and $\{x, y\} = \{w_0, w_2\}$. By Claim 3, $u_0, u_2, w_4 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_1, we have $w_1 \in S$. By Claim 3, $w_1, u_1 \in S$, $w_1, w_3 \in S$ or $w_1, w_{n-1} \in S$.

If $w_1, u_1 \in S$ as illustrated in Fig.10, by Claim 3, we have $w_3 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_3, we have $u_4 \in S$.

Therefore, u_0, u_2 and $w_4$are respectively dominated by two different vertices of S, a contradiction to Claim 4.

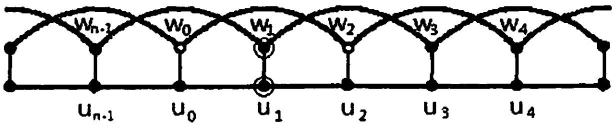

Fig.10: $w_1, u_1 \in S$

If $w_1, w_3 \in S$ as illustrated in Fig.11, by Claim 3, $u_1, u_3, w_5 \notin S$. Since S is a total dominating set of $P(n,2)$ and S dominates u_4, we have $u_5 \in S$. By Claim 3, $u_4, u_5 \in S$ or $u_5, u_6 \in S$, a contradiction to Claim 5.

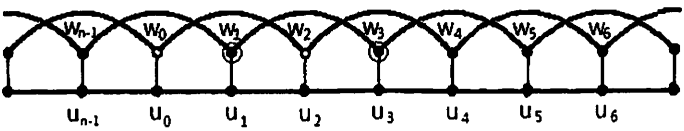

Fig.11: $w_1, w_3 \in S$

Since w_{n-1} is symmetric to w_3 about w_1 and $\langle\{w_0, w_2\}\rangle, w_1, w_{n-1} \in S$ also conflict in the same way to the case $w_1, w_3 \in S$ as illustrated in Fig. 11.

In conclusion, Claim 6 is obtained.◇

By Claim 3, Claim 5 and Claim 6, we assume $\langle\{w_i, u_i\}\rangle$ and $\langle\{w_j, u_j\}\rangle$ are two components of $\langle S\rangle$, then $min\{(i-j)(\text{mod}n), (j-i)(\text{mod}n)\} \geq 3$. Therefore, the number of P_2 in $\langle S\rangle$ is at most $\frac{n}{3} = k + \frac{2}{3}$. On the other hand, since $\gamma_p(P(n,2)) = 2k+2$, by Claim 3, there are $k+1$ P_2 in $\langle S\rangle$, a contradiction.

From above, $\gamma_p(P(n,2)) = 2k+4$.□

By Lemma 3, Lemma 4 and Lemma 5, we obtain the following theorem.

Theorem 6. *For any integer* $n \geq 6, \gamma_p(P(n,2)) = 2\left(\lfloor\frac{n}{3}\rfloor + n(\text{mod}3)\right)$.

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