# Limited packing vs tuple domination in graphs

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**Abstract**

In this paper we investigate the concepts of $k$-limited packing and $k$-tuple domination in graphs and give several bounds on the size of them. This bounds involve many well known parameters of graphs. Also, we establish a connection between these concepts that implies some new results in this area. Finally, we improve many bounds in literatures.

**Keywords**: Double domination, $k$-limited packing, $k$-tuple domination
**2000 AMS subject classification**: 05C69.

# 1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n$ and edge set $E = E(G)$. The minimum and maximum degrees of $G$ are $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For a vertex $v \in V$, $N(v)$ is the open neighborhood of $v$, which is the set of vertices adjacent to $v$ and $N[v] = N(v) \cup \{v\}$ is

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the closed neighborhood of $v$. A set $S \subseteq V$ is a dominating set if each vertex in $V \backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A subset $S \subseteq V$ is a 2-packing if for every pair of vertices $u, v \in S$, $d(u,v) > 2$. The 2-packing number $\rho(G)$, is the maximum cardinality of a 2-packing in $G$. In [2], Harary and Haynes introduced the concept of tuple domination. A set $D \subseteq V$ is a $k$-tuple dominating set for $G$ if $|N[v] \cap D| \geq k$ for all $v \in V(G)$. The $k$-tuple domination number, denoted $\gamma_{\times k}(G)$, is the smallest number of vertices in a $k$-tuple dominating set. When $k = 2$, $D$ is called a double dominating set and the 2-tuple domination number is called the double domination number and is denoted by $dd(G)$. In fact the authors showed that every graph $G$ with $\delta \geq k-1$ has a $k$-tuple dominating set and hence a $k$-tuple domination number. Gallant et al. [1] introduced the concept of limited packing in graphs. They exhibited some real-world applications of it to network security, NIMBY, market saturation and codes. In fact, a set of vertices $B \subseteq V$ is called a $k$-limited packing in $G$ provided that for all $v \in V(G)$, we have $|N[v] \cap B| \leq k$. The $k$-limited packing number, denoted $L_k(G)$, is the largest number of vertices in a $k$-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. In fact $k$-limited packing is a generalization of 2-packing in a graph. In this paper we obtain some new lower and upper bounds on these parameters in graphs, that some of them improve some results in [1] and [2]. Also we give a connection between these two concepts that leads to some new bounds on them that they involve domination number, 2-packing number, $k$-limited packing number, $k$-tuple domination number and some other parameters. The reader can find comprehensive information about many domination parameters until 1998 in [3].

## 2 Bounds on $L_k(G)$

In [1], it has been proved the following theorem.

**Theorem 1** [1] *Let $G$ be a connected graph of order $n$, and $k$ be a positive integer and $\delta \geq k$. Then $L_k(G) \leq (\frac{k}{k+1})n$ (hence, $L_k(G) \leq \lfloor \frac{k}{k+1} n \rfloor$).*

We can improve this theorem as follows:

**Theorem 2** *Let $G$ be a connected graph of order $n$, $k$ be a positive integer* *and $\delta \geq k-1$. Then $L_k(G) \leq \lfloor \frac{k}{\delta+1} n \rfloor$ and the bound is sharp.*

**Proof.** Let $B$ be a maximum $k$-limited packing set in $G$. We count the number $|[B, V-B]|$, of edges with endpoints in $B$ and $V-B$. Since $B$ is a $k$-limited packing set, the induced subgraph $G[B]$ has maximum degree at most $k-1$. Therefore every vertex in $B$ has at least $\delta - k + 1$ neighbors in $V-B$. Hence $(\delta - k + 1)|B| \leq |[B, V-B]|$. On the other hand every vertex in $V-B$ has at most $k$ neighbors in $B$. Hence $|[B, V-B]| \leq k(n-|B|)$. These two inequalities imply $|B| \leq \frac{kn}{\delta+1}$. Now we show that the bound is sharp. Consider the complete graph $K_n$ and let $k \leq n$. Then $L_k(K_n) = k = \lfloor \frac{k}{n} n \rfloor = \lfloor \frac{k}{\delta+1} n \rfloor$. □

**Theorem 3** *Let $G$ be a connected graph of order $n$, and $k \leq \Delta(G)$. Then $L_{k+1}(G) \geq L_k(G) + 1$. Moreover, $L_k(G) \geq \rho(G) + k - 1$, and this bound is sharp.*

**Proof.** Let $B$ be a maximum $k$-limited packing set in $G$. Then $|N[v] \cap B| \leq k$ for all $v \in V(G)$. Obviously, $L_k(G) \leq L_{k+1}(G)$. We claim that $B \neq V$. If $B = V$ and $u \in V$ such that $deg(u) = \Delta$, then $\Delta + 1 = |N[u]| = |N[u] \cap B| \leq k \leq \Delta$, a contradiction. Now let $u \in V - B$. It is easy to check that $|N[v] \cap (B \cup \{u\}) \leq k+1$ for all $v \in V(G)$. Therefore $B \cup \{u\}$ is a $k+1$-limited packing set in $G$. Hence:

$$L_{k+1}(G) \geq |B \cup \{u\}| = |B| + 1 = L_k(G) + 1$$

Repeating these inequalities, we have $L_k(G) \geq L_{k-1}(G) \geq ... \geq L_1(G) + k - 1 = \rho(G) + k - 1$. For sharpness we consider the graph $K_n$, when $k \leq n$. Then, $L_k(K_n) = k = 1 + k - 1 = \rho(K_n) + k - 1$. □

# 3 Bounds on $\gamma_{\times k}(G)$

Harary and Haynes in [2] obtained the following theorem.

**Theorem 4** [2] *Let $G$ be a graph of order $n$ and with no isolated vertex. Then $dd(G) \geq \frac{2n}{\Delta+1}$ (hence, $dd(G) \geq \lceil \frac{2n}{\Delta+1} \rceil$), and this bound is sharp.*

Then they generalized it by the following theorem.

**Theorem 5** [2] *Let $G$ be a graph of order $n$ and $\delta(G) \geq k-1$. Then $\gamma_{\times k}(G) \geq \frac{kn}{\Delta+1}$ ( hence, $\gamma_{\times k}(G) \geq \lceil \frac{kn}{\Delta+1} \rceil$), and this bound is sharp.*

Now we are going to improve these results.

**Theorem 6** *Let $G$ be a connected graph of order $n$ and $\delta(G) \geq k-1$. Then $\gamma_{\times k}(G) \geq \lceil \frac{kn+n_{k-1}(\Delta-k+1)}{\Delta+1} \rceil$, and this bound is sharp, where $n_{k-1}$ is the number of vertices with degree $k-1$.*

**Proof.** Let $D$ be a minimum $k$-tuple dominating set in $G$. Every vertex in $V-D$ has at least $k$ neighbors in $D$, hence all vertices with degree $k-1$ belong to $D$. Let $S = \{v \in V(G) | deg(v) = k-1\}$ and $|S| = n_{k-1}$. Every vertex in $D$ has at least $k-1$ neighbors in $D$, therefore every vertex in $D$ has at most $\Delta - k + 1$ neighbors in $V-D$, exception vertices in $S$ who have no adjacent in $V-D$. Hence, $||[D, V-D]|| \leq (|D| - n_{k-1})(\Delta - k+1)$. On the other hand every vertex in $V-D$ has at least $k$ neighbors in $D$. Therefore, $k(n-|D|) \leq ||[D, V-D]||$. Together these two inequalities imply $|D| \geq \frac{kn+n_{k-1}(\Delta-k+1)}{\Delta+1}$. Moreover, this bound is sharp. Indeed, $\gamma_{\times k}(K_{k,k-1}) = 2k-1 = \lceil 2(k-1) + \frac{2}{k+1} \rceil = \lceil \frac{k(2k-1)+k}{k+1} \rceil = \lceil \frac{kn+n_{k-1}(\Delta-k+1)}{\Delta+1} \rceil$. □

**Corollary 7** *If $G$ has no isolated vertices, then $dd(G) \geq \lceil \frac{2n+l(\Delta-1)}{\Delta+1} \rceil$, and this bound is sharp, where $l$ is the number of vertices of degree 1 of $G$.*

Obviously, in general the lower bounds in Corollary 7 and Theorem 6 are better than the analogous lower bound in Theorem 4 and Theorem 5, respectively. Of course, they are the same when $\delta \geq 2$ and $\delta \geq k$, respectively. In the process of proof of Theorem 6 we counted the vertices of degree $k-1$ belong to $D$ and have no adjacent in $V-D$. Therefore, we have,

**Proposition 8** *Let $G$ be a graph with $\delta \geq k-1$. If $\gamma_{\times k}(G) \neq n$, then $k + n_{k-1} \leq \gamma_{\times k}(G)$, and this bound is sharp.*

That improves the following theorem, when $\gamma_{\times k}(G) \neq n$.

**Theorem 9** [2] *Let $G$ be a graph with $\delta \geq k-1$. Then, $k \leq \gamma_{\times k}(G) \leq n$, and these bounds are sharp.*

Considering the graph $K_{k,k-1}$ we can check that the lower bound in Proposition 8 is sharp. In [2] the authors obtained the following theorem.

**Theorem 10** [2] *If $\Delta(G) \geq k \geq 2$, then $\gamma_{\times k}(G) \geq \gamma(G) + k - 2$.*

We can show this result can be improved. In fact, we can omit the condition $\Delta \geq k \geq 2$ and will show that this lower bound is not sharp.

**Theorem 11** *Let $G$ be a graph with $k \leq \Delta$. Then $\gamma_{\times k}(G)+1 \leq \gamma_{\times(k+1)}(G)$. Moreover, if $\delta \geq k-1$, then $\gamma_{\times k}(G) \geq \gamma(G)+k-1$, and this bound is sharp.*

**Proof.** Let $D$ be a minimum $k+1$-tuple dominating set in $G$. Then $|N[v] \cap D| \geq k+1$, for all $v \in V(G)$. Let $u \in D$. It is easy to see that $|N[v] \cap (D - \{u\})| \geq k$, for all $v \in V(G)$. Therefore $D - \{u\}$ is a $k$-tuple dominating set in $G$. Hence, $\gamma_{\times k}(G) \leq |D-\{u\}| = |D|-1 \leq \gamma_{\times(k+1)}(G) - 1$. Repeating these inequalities, we have $\gamma_{\times k}(G) \geq \gamma_{\times(k-1)}(G) \geq \dots \geq \gamma_{\times 1}(G)+k-1 = \gamma(G)+k-1$. For sharpness it is sufficient to consider the graph $K_n$ when $k \leq n$. Then $\gamma_{\times k}(K_n) = k = 1+k-1 = \gamma(K_n)+k-1$.

□

# 4 Bound on $L_k(G)$ and $\gamma_{\times k}(G)$ by their relationships

In this section we establish a link between concepts of limited packing and tuple domination. By this connection we will be able to obtain some new sharp bounds. First, we need the following useful lemma.

**Lemma 12** *Let $G$ be a graph. Then the following statements hold. (i) Let $\delta \geq k-1$. If $B \subseteq V$ is a $k$-limited packing set, then $V-B$ is a $\delta-k+1$-tuple dominating set in $G$.*

*(ii) Let $\delta \geq k$. If $D \subseteq V$ is a $k$-tuple dominating set, then $V - D$ is a $\Delta - k + 1$-limited packing set in $G$.*

**Proof.** We only prove (i). Let $B$ be a $k$-limited packing set in $G$. Every vertex in $B$ has at most $k - 1$ neighbors in $B$. Therefore it has at least $\delta - k + 1$ neighbors in $V - B$. On the other hand, every vertex in $V - B$ has at most $k$ neighbors in $B$, hence it has at least $\delta - k$ neighbors in $V - B$. This imply that $V - B$ is a $\delta - k + 1$-tuple dominating set in $G$. □

At this point we are able to obtain a sharp upper bound on $L_k(G)$ that involves the domination number of $G$.

**Theorem 13** *Let $G$ be a graph with $\delta \geq k$. Then $L_k(G) \leq n - \gamma(G) - \delta + k$, and this bound is sharp.*

**Proof.** Let $B$ be a maximum $k$-limited packing set. By Lemma 12, $V - B$ is a $\delta - k + 1$-tuple dominating set in $G$. Therefore $\gamma_{\times(\delta-k+1)}(G) \leq n - |B|$. Since $\delta \geq \delta - k + 1 - 1$, Theorem 11 implies that, $n - |B| \geq \gamma(G) + \delta - k + 1 - 1$. Hence $|B| \leq n - \gamma(G) - \delta + k$. Applying the graph $K_n$ when $n \geq k + 1$ we have, $L_k(K_n) = k = n - 1 - (n - 1) + k = n - \gamma(K_n) - \delta + k$. Therefore this bound is sharp. □

One can directly use tuple domination and limited packing number to obtain upper and lower bounds on each other, respectively. In fact, the authors in [1] showed that for a graph $G$, $L_k(G) \leq k\gamma(G)$. But we can generalize this result and show that $tL_k(G) \leq k\gamma_{\times t}(G)$, for a graph with $\delta \geq t - 1$. In fact, when $t = 1$ we have the previous bound and when $t = k$ we have $L_k(G) \leq \gamma_{\times k}(G)$. Indeed, we have the following theorem.

**Theorem 14** *Let $G$ be a graph and $k, t$ be positive integers such that $\delta \geq t - 1$. Then $L_k(G) \leq \frac{k}{t}\gamma_{\times t}(G)$.*

**Proof.** Let $B$ be a maximum $k$-limited packing and $D$ be a minimum $t$-tuple dominating set in $G$. Let $A$ be the set of ordered pairs $\{(b, d)|\ b \in B,\ d \in D \text{ and } b \in N[d]\}$. Since $B$ is a $k$-limited packing, for every vertex $d \in D$, we have $|N[d] \cap B| \leq k$. Therefore $|A| \leq k|D|$.
On the other hand, since $D$ is a $t$-tuple dominating set, for every vertex $b \in B$, we have $|N[b] \cap D| \geq t$. Therefore $t|B| \leq |A|$. These inequalities imply that $tL_k(G) \leq k\gamma_{\times t}(G)$. □

Finally, we finish this section with a short discussion about regular graphs.

In [1] the authors obtained the following two propositions.

**Proposition 15** [1] *If $G$ is an $r$-regular graph, and $k \leq r-1$, then* $L_{r-k}(G) + \gamma_{\times(k+1)}(G) = n$.

**Proposition 16** [1] *Let $G$ be a cubic graph. Then,* $\frac{1}{4}n \leq L_2(G) \leq \frac{1}{2}n$.

Putting $r = 3$ and $k = 1$ in Proposition 15, we have $L_2(G) + \gamma_{\times 2}(G) = n$. Now Proposition 16 shows that:

**Corollary 17** *Let $G$ be a cubic graph. Then* $\frac{1}{2}n \leq dd(G) \leq \frac{3}{4}n$.

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