Upper bound on the primitive exponents of a class of three-colored digraphs

Meijin Luo* Xi Li ◇
* Department of Mathematics, Hechi University, Yizhou,Guangxi 546300, P.R. China
◇ Department of Basic Education, Shanxi Yuncheng Vocational College of Agriculture,Yuncheng,Shanxi 044000,P.R. China

Abstract: A three-colored digraph D is primitive if and only if there exist nonnegative integers h, k and v with $h+k+v>0$ such that for each pair (i,j) of vertices there is an (h,k,v)-walk in D from i to j. The exponent of the primitive three-colored digraph D is defined to be the smallest value of $h+k+v$ over all such h,k and v. In the paper, a class of especial primitive three-colored digraphs with n vertices , consisting of one n-cycle and two $(n-1)$-cycles, are considered. For the case $a=c-1$, some primitive conditions, the tight upper bound on the exponents and the characterization of extremal three-colored digraphs are given.

Key words: Exponent; upper bound; three-colored digraph; extremal digraph

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1 Introduction

Let D be a digraph. A *walk* in D of length l is a sequence $v_1v_2\cdots v_lv_{l+1}$ of vertices such that there is an arc in D from v_i to v_{i+1} for $i=1,2,\cdots,l$. The walk is a *path* if the vertices $v_1,v_2,\cdots,v_l,v_{l+1}$ are distinct. The walk is *closed* if $v_1=v_{l+1}$, and a *cycle* is a closed walk in which $v_1,v_2,\cdots,v_l$ are distinct. A *three-colored digraph* is a digraph whose arcs are colored red, yellow and blue. We allow loops and all of red arc, yellow arc or blue arc from i to j for all pairs (i,j) of

[1]Author: Meijin Luo(1981-), female, postgraduate, associate professor, Combinatorial Mathematics. Email: meijin322@163.com.

vertices. The three colored digraph D is *strongly connected* provided for each pair (i, j) of vertices there is a walk in D from i to j. Given a walk ω in D, $r(\omega)$ (respectively, $y(\omega)$ or $b(\omega)$) is the number of red arcs (respectively,yellow arcs or blue arcs) of ω, and the *composition* of ω is the vector $(r(\omega), y(\omega), b(\omega))$ or $(r(\omega), y(\omega), b(\omega))^T$.

A three-colored digraph D is *primitive* if and only if there exist nonnegative integers h, k and v with $h + k + v > 0$ such that for each pair (i, j) of vertices there is an (h, k, v)-walk in D from i to j. The *exponent* of the primitive three-colored digraph D is defined to be the smallest value of $h + k + v$ over all such h, k and v, denoted $exp(D)$.

Let $C = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}$ be the set of cycles of D. Set M is the $3 \times l$ matrix whose ith column is the composition of γ_i. We call *M the cycle matrix* of D. The *content* of M, denoted content(M), is defined to be 0 if the rank of M is less than 3 and the greatest common divisor of all 3×3 minors of M, otherwise.

Lemma 1([2]) Let D be a three-colored digraph having at least one red arc, one yellow arc and one blue arc. Then D is primitive if and only if D is strongly connected and content$(M) = 1$.

It is well known that there is a natural correspondence between two-colored digraphs and nonnegative matrix pairs ([2]). The concept of exponent of nonnegative matrix pair arises in the study of finite Markov chains ([2,3]), and some results have already been obtained ([1-7]). The nonnegative matrix cluster is the extension of nonnegative matrix pairs. Also, there is a natural correspondence between three-colored digraphs and nonnegative matrix cluster, and some upper bound on the exponents of especial primitive three-colored digraphs are given ([8-10]).

In this paper, for $n \geq 3$, we consider the class of three-colored digraphs which have at least one red arc, one yellow arc and one blue arc, and its uncolored digraph is given as in Fig.1.

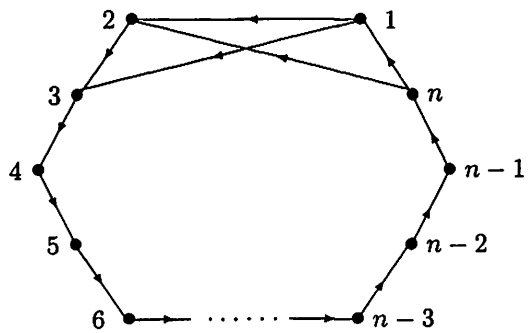

Fig.1 Uncolored digraph of D

Clearly, D has one n-cycle and two $(n-1)$-cycles. Without loss of generality, we may assume $c \geq d$, and the cycle matrix of D is

$$M = \begin{bmatrix} a & c & x \\ b & d & y \\ n-a-b & n-1-c-d & n-1-x-y \end{bmatrix} \qquad (1)$$

for some nonnegative integers a, b, c, d, x, y.

From Fig.1, three cycles in D have the common path of length $(n-3)$ which is $3 \to 4 \to 5 \to \cdots \to n-1 \to n$. So

$$\begin{cases} c-1 \leq a \leq c+2 \\ d-1 \leq b \leq d+2 \\ n-2-c-d \leq n-a-b \leq n+1-c-d \end{cases},$$

and

$$\begin{cases} c-2 \leq x \leq c+2 \\ d-2 \leq y \leq d+2 \\ n-3-c-d \leq n-1-x-y \leq n+1-c-d \end{cases}.$$

2 The primitive conditions

In this section, we will find the primitive conditions of D. We assume $x = c+K, y = d+N (-2 \leq K \leq 2, -2 \leq N \leq 2)$. Since $n-3-c-d \leq n-1-x-y \leq n+1-c-d$, we have $-2 \leq K+N \leq 2$ .Of course, K, N values are not the same, under different circumstances.

In the paper, we only consider $a = c - 1$. When $a = c - 1$, then $d \leq b \leq d + 2$.

Theorem 1 Digraph D is primitive if and only if

$$(ad-bc)(n-1-x-y)+(cy-dx)(n-a-b)+(bx-ay)(n-1-c-d) = \pm 1. \quad (2)$$

Proof From (1), $|M| = (ad - bc)(n - 1 - x - y) + (cy - dx)(n - a - b) + (bx - ay)(n - 1 - c - d)$. By Lemma 1, D is primitive if and only if content$(M) = 1$, that is, $|M| = \pm 1$. Then the theorem holds.

Theorem 2 Let $a = c - 1$, and $b = d$. Then D is primitive if and only if

$$d = 1, K = -1, N = 0$$

Proof Because $a = c - 1, b = d, x = c + K, y = d + N(-2 \leq K \leq 2, -2 \leq N \leq 2, -2 \leq K + N \leq 2)$ and $c \geq d$, from (2), $|M| = (n + c - 1)N - dK$. By Lemma 1, D is primitive if and only if content$(M) = 1$, that is, $(n + c - 1)N = dK \pm 1$. Different situations below are discussed.

(a)If $N = -2$, then $-2(n + c - 1) = dK \pm 1$. Obviously, $-2(n + c - 1) \leq -6, dK \pm 1 \geq -1$. Contradiction.

(b)If $N = -1$, then $-(n + c - 1) = dK \pm 1$. Obviously, $-(n + c - 1) \leq -3, dK \pm 1 \geq -1$. Contradiction.

(c)If $N = 0$, then $dK = \pm 1$. Obviously, $d = 1, K = \pm 1$. But when $N = 0, d = 1, K = 1$, the number of red arcs on the n-cycle is two less than the number of red arcs on the $(n - 1)$-cycle. Contradiction. So if $N = 0$, then $d = 1, K = -1$.

(d)If $N = 1$ or $N = 2$, then $(n+c-1) = dK \pm 1$ or $2(n+c-1) = dK \pm 1$. Obviously, the left of the equation is more than the right of the equation. Contradiction.

Theorem 3 Let $a = c - 1$, and $b = d + 1$. Then D is primitive if and only if

$$c = N = 1, d = 0, K = -1.$$

Proof Because $a = c - 1, b = d + 1$, $x = c + K, y = d + N(-2 \leq K \leq 2, -2 \leq N \leq 2, -2 \leq K + N \leq 2)$ and $c \geq d$, from (2), $|M| = (n + c - 1)N + (n - d - 1)K$. By Lemma 1, D is primitive if and only if content$(M) = 1$, that is, $(n+c-1)N = -(n-d-1)K \pm 1$. Different situations below are discussed.

(a)If $N=-2$, then $-2(n+c-1)=-(n-d-1)K\pm 1$. Obviously, $-2(n+c-1)\leq -6, -(n-d-1)K\pm 1\geq -1$. Contradiction.

(b)If $N=-1$, then $-(n+c-1)=-(n-d-1)K\pm 1$. Obviously, when $K=1, c=1, d=0$, the left of the equation is equal to the right of the equation. But $y=d-1$, that is $y=-1$. Contradiction.

(c)If $N=0$, then $0=-(n-d-1)K\pm 1$. Obviously, $-(n-d-1)K\pm 1\neq 0$. Contradiction.

(d)If $N=1$, then $(n+c-1)=-(n-d-1)K\pm 1$. Obviously, $K=-1, c=1, d=0$, the left of the equation is equal to the right of the equation.

(e)If $N=2$, then $2(n+c-1)=-(n-d-1)K\pm 1$. Obviously, $2(n+c-1)\geq 6, -(n-d-1)K\pm 1\leq 1$. Contradiction.

Theorem 4 Let $a=c-1$, and $b=d+2$. Then D is primitive if and only if

$$c+d=n-2, K=-1, N=1.$$

Proof Because $a=c-1, b=d+2$, $x=c+K, y=d+N(-2\leq K\leq 2, -2\leq N\leq 2, -2\leq K+N\leq 2)$ and $c\geq d$, from (2), $|M|=(n+c-1)N+(2n-d-2)K$. By Lemma 1, D is primitive if and only if content$(M)=1$, that is, $(n+c-1)N=-(2n-d-2)K\pm 1$. Different situations below are discussed.

(a)If $N=-2$, then $-2(n+c-1)=-(2n-d-2)K\pm 1$. Obviously,$-2(n+c-1)\leq -6, -(2n-d-2)K\pm 1\geq -1$. Contradiction.

(b)If $N=-1$, then $-(n+c-1)=-(2n-d-2)K\pm 1$. Obviously, when $K=1, c+d=n-2$,the left of the equation is equal to the right of the equation. But when $N=-1, K=1, c+d=n-2$, the number of red arcs on the n-cycle is two less than the number of red arcs on the $(n-1)$-cycle. Contradiction.

(c)If $N=0$, then $0=-(2n-d-2)K\pm 1$. Obviously, $-(2n-d-2)K\pm 1\neq 0$. Contradiction.

(d)If $N=1$, then $(n+c-1)=-(2n-d-2)K\pm 1$. Obviously, $K=-1, c+d=n-2$, the left of the equation is equal to the right of the equation.

(e)If $N=2$, then $2(n+c-1)=-(2n-d-2)K\pm 1$. Obviously, the left of the equation is more than the right of the equation.

Contradiction.

3 The tight upper bound on the exponents when $a = c - 1$

In this section, we will give the tight upper bound on the exponents when $a = c - 1$. By Theorem 2, we assume that the cycle matrix of D has the form (1), where $n \geq 3, c \geq d$ and a, b, c, d, x, y are nonnegative integers.

Theorem 5 Let $a = c - 1, b = d = 1$, $K = -1, N = 0$ and D be primitive. Then

$$\exp(D) \leq 2n^2 + 2cn - 4n - 2c + 2 \leq 4n^2 - 10n + 6.$$

Proof For any pair (i, j) of vertices of D, let p_{ij} be the shortest path in D from i to j, and denote $r(p_{ij}) = r$, $y(p_{ij}) = y$ and $b(p_{ij}) = b$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes ρ_1 times around the n-cycle, ρ_2 times around one $(n-1)$-cycle, and ρ_3 times around the other $(n-1)$-cycle.

If $d = 1, K = -1, N = 0$, then $|M| = 1$. The cycle matrix is

$$M = \begin{bmatrix} c-1 & c & c-1 \\ 1 & 1 & 1 \\ n-c & n-2-c & n-1-c \end{bmatrix}.$$

Taking $\rho_1 = n - 1 - r + (n-2)y - b$, $\rho_2 = c - r + (c-1)y$ and $\rho_3 = n - 2 + c + 2r - (n + c - 2)y + b$, we see that

$$\begin{bmatrix} r \\ y \\ b \end{bmatrix} + \rho_1 \begin{bmatrix} c-1 \\ 1 \\ n-c \end{bmatrix} + \rho_2 \begin{bmatrix} c \\ 1 \\ n-2-c \end{bmatrix} + \rho_3 \begin{bmatrix} c-1 \\ 1 \\ n-1-c \end{bmatrix}$$
$$= \begin{bmatrix} 2c^2 + 2cn - 2n - 4c + 3 \\ 2n + 2c - 3 \\ 2n^2 - 2c^2 - 4n + 2 \end{bmatrix}.$$

From Fig.1, note that $0 \leq r \leq c$, $0 \leq y \leq 2$, and $0 \leq b \leq n - c$. If $r + b \leq n - 1$, then $y \geq 0$. If $r = c$, then $y \geq 0, b \geq 0$. If $y = 1$, then $r \geq 0, b \geq 0$; if $y = 2$, the way goes around one $(n-1)$-cycle one time at least. Clearly, it is easy to see that $\rho_1 \geq 0$,$\rho_2 \geq 0$ and $\rho_3 \geq 0$. This gives

$$\exp(D) \leq 2c^2 + 2cn - 2n - 4c + 3 + 2n + 2c - 3 + 2n^2 - 2c^2 - 4n + 2$$
$$= 2n^2 + 2cn - 4n - 2c + 2.$$

Denote $f(c) = 2n^2 + 2cn - 4n - 2c + 2$. Clearly, $f'(c) = 2n - 2 \geq 0$, and $f(c)$ is an increasing function of c. Since $1 \leq c \leq n-2$, we have that

$$\exp(D) \leq f(n-2) = 2n^2 + 2(n-2)n - 4n - 2(n-2) + 2 = 4n^2 - 10n + 6.$$

Theorem 6 Let $a = d = 0, b = c = N = 1, K = -1$ and D be primitive. Then

$$\exp(D) \leq 2n^2 - 2n.$$

Proof For any pair (i, j) of vertices of D, let p_{ij} be the shortest path in D from i to j, and denote $r(p_{ij}) = r$, $y(p_{ij}) = y$ and $b(p_{ij}) = b$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes ρ_1 times around the n-cycle, ρ_2 times around one $(n-1)$-cycle, and ρ_3 times around the other $(n-1)$-cycle.

If $a = d = 0, b = c = N = 1, K = -1$, then $|M| = 1$. The cycle matrix is

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ n-1 & n-2 & n-2 \end{bmatrix}.$$

Taking $\rho_1 = n - 1 + (n-2)r + (n-2)y - b$, $\rho_2 = 1 - r$ and $\rho_3 = n - 1 - (n-2)r - (n-1)y + b$, we see that

$$\begin{bmatrix} r \\ y \\ b \end{bmatrix} + \rho_1 \begin{bmatrix} 0 \\ 1 \\ n-1 \end{bmatrix} + \rho_2 \begin{bmatrix} 1 \\ 0 \\ n-2 \end{bmatrix} + \rho_3 \begin{bmatrix} 0 \\ 1 \\ n-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2n-2 \\ 2n^2 - 4n + 1 \end{bmatrix}.$$

From Fig.1, note that $0 \leq r \leq 1$, $0 \leq y \leq 2$ and $0 \leq b \leq n-1$. The one yellow arc must be $n \to 1$ or $1 \to 2$ or $2 \to 3$ on the n-cycle. Consider the following four cases:

(a) If $b = n-1$, then $r \geq 0, y \geq 0$.

(b) If $r = 1, y = 0$, or $r = 0, y = 1$, then $b \geq 0$.

(c) If $r = 1, y = 1$, in this condition, the red arc and the yellow arc must be on two different cycles, and the walk must go around one $(n-1)$-cycle one time at least, then $\rho_3 \geq 0$.

(d) If $r = 1, y = 2$, in this condition, the red arc and two yellow arcs must be on three different cycles, and the walk must go around the n-cycle one time and one $(n-1)$-cycle one time at least, then $\rho_3 \geq 0$. Clearly, it is easy to see that $\rho_1 \geq 0$, $\rho_2 \geq 0$,and $\rho_3 \geq 0$. This gives

$$\exp(D) \leq 1 + 2n - 2 + 2n^2 - 4n + 1 = 2n^2 - 2n.$$

Theorem 7 Let $a = c-1, b = d+2$, $c+d = n-2, K = -1, N = 1$, and D be primitive . Then

$$\exp(D) \leq 2n^2 + 2cn - 4n - 2c + 2 \leq 4n^2 - 10n + 6.$$

Proof For any pair (i,j) of vertices of D, let p_{ij} be the shortest path in D from i to j, and denote $r(p_{ij}) = r$, $y(p_{ij}) = y$ and $b(p_{ij}) = b$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes ρ_1 times around the n-cycle, ρ_2 times around one $(n-1)$-cycle, and ρ_3 times around the other $(n-1)$-cycle.

If $c + d = n - 2, K = -1, N = 1$, then $|M| = -1$. The cycle matrix is

$$M = \begin{bmatrix} c-1 & c & c-1 \\ n-c & n-2-c & n-1-c \\ 1 & 1 & 1 \end{bmatrix}.$$

Taking $\rho_1 = n-1-r-y+(n-2)b$, $\rho_2 = c-r+(c-1)b$ and $\rho_3 = n-2+c+2r+y-(n+c-2)b$, we see that

$$\begin{bmatrix} r \\ y \\ b \end{bmatrix} + \rho_1 \begin{bmatrix} c-1 \\ n-c \\ 1 \end{bmatrix} + \rho_2 \begin{bmatrix} c \\ n-2-c \\ 1 \end{bmatrix} + \rho_3 \begin{bmatrix} c-1 \\ n-1-c \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2c^2 + 2cn - 2n - 4c + 3 \\ 2n^2 - 2c^2 - 4n + 2 \\ 2n + 2c - 3 \end{bmatrix}.$$

Form Fig.1, note that $0 \leq r \leq c$,$0 \leq y \leq n-c$ and $0 \leq b \leq 2$. If $r + y \leq n-1$, then $b \geq 0$. If $r = c$, then $y \geq 0, b \geq 0$. If $b = 1$, then $r \geq 0, y \geq 0$; if $b = 2$, the way goes around the $(n-1)$-cycle one time at least. Clearly, it is easy to see that $\rho_1 \geq 0$,$\rho_2 \geq 0$,and $\rho_3 \geq 0$. This gives

$$\begin{aligned}\exp(D) &\leq 2c^2 + 2cn - 2n - 4c + 3 + 2n^2 - 2c^2 - 4n + 2 + 2n + 2c - 3 \\ &= 2n^2 + 2cn - 4n - 2c + 2.\end{aligned}$$

Denote $f(c) = 2n^2 + 2cn - 4n - 2c + 2$, then $f'(c) = 2n - 2 \geq 0$. Clearly, and $f(c)$ is an increasing function of c. Since $1 \leq c \leq n-2$, we have that

$$\exp(D) \leq f(n-2) = 2n^2 + 2(n-2)n - 4n - 2(n-2) + 2 = 4n^2 - 10n + 6.$$

4 The extremal three-colored digraphs of D

In this section, we obtain the characterization of the three-colored digraphs of D.

Theorem 8 Let $a = c - 1, b = d = 1$, $K = -1, N = 0$ and D be primitive. Then

$$\exp(D) = 2n^2 + 2cn - 4n - 2c + 2$$

if and only if the number of red arcs and blue arcs in the path $3 \to 4 \to \cdots \to n-1 \to n$ are $c-1$ and $n-c-2$, and

(1) $n \to 2$ is red, $n \to 1 \to 2$ are blue, $1 \to 3$ and $2 \to 3$ are yellow; or

(2) $1 \to 3$ is red, $1 \to 2 \to 3$ are blue, $n \to 1$ and $n \to 2$ are yellow.

Proof Combining Theorem 5, we need to prove $\exp(D) \geq 2n^2 + 2cn - 4n - 2c + 2$.

Suppose that h, k, v are nonnegative integers such that for all pairs (i, j) of vertices there is an (h, k, v)-walk from i to j. By considering $i = j = n$, we see that there exist nonnegative integers u_1,u_2 and u_3 with

$$\begin{bmatrix} h \\ k \\ v \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Taking i and j to be the initial and terminal vertices of the yellow arc on the n-cycle or $(n-1)$-cycle, then there is a unique path from i to j, and this path has composition $(0,1,0)$. Hence

$$Mz = \begin{bmatrix} h \\ k-1 \\ v \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1}\begin{bmatrix} h \\ k-1 \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -(n-2) \\ -(c-1) \\ n-2+c \end{bmatrix} \geq 0.$$

So $u_3 \geq n-2+c$. Next take i and j to be the terminal and initial vertices of the yellow arc on the n-cycle or $(n-1)$-cycle, then there is a unique path for each cycle from i to j, and this path has composition $(c-1,0,n-c)$, or $(c,0,n-2-c)$, or $(c-1,0,n-1-c)$. Hence

$$Mz = \begin{bmatrix} h-(c-1) \\ k \\ v-(n-c) \end{bmatrix},$$

or

$$Mz = \begin{bmatrix} h-c \\ k \\ v-(n-2-c) \end{bmatrix}$$

or

$$Mz = \begin{bmatrix} h-(c-1) \\ k \\ v-(n-1-c) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1}\begin{bmatrix} h-(c-1) \\ k \\ v-(n-c) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c-1 \\ 0 \\ n-c \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-1 \\ c-1 \\ -(n-2+c) \end{bmatrix} \geq 0,$$

or

$$z = M^{-1}\begin{bmatrix} h-c \\ k \\ v-(n-2-c) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c \\ 0 \\ n-2-c \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ c \\ -(n-2+c) \end{bmatrix} \geq 0,$$

or

$$z = M^{-1}\begin{bmatrix} h-(c-1) \\ k \\ v-(n-1-c) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c-1 \\ 0 \\ n-1-c \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ c-1 \\ -(n-3+c) \end{bmatrix} \geq 0.$$

So $u_1 \geq n-1, u_2 \geq c$. Thus

$$h+k+v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \begin{bmatrix} n & n-1 & n-1 \end{bmatrix} \begin{bmatrix} n-1 \\ c \\ n-2+c \end{bmatrix}$$

$$= 2n^2 + 2cn - 4n - 2c + 2.$$

Theorem 9 Let $a = d = 0, b = c = N = 1, K = -1$ and D be primitive. Then

$$\exp(D) = 2n^2 - 2n$$

if and only if

(1) $3 \to 4 \to \cdots \to n \to 1 \to 2$ are blue and consecutive, $1 \to 3$ is red, $2 \to 3$ is yellow, and $n \to 2$ is blue; or

(2) $1 \to 2 \to \cdots \to n$ are blue and consecutive, $n \to 2$ is red, $n \to 1$ is yellow, and $1 \to 3$ is blue; or

(3) $2 \to 3 \to \cdots \to n \to 1$ are blue and consecutive, $n \to 2$ (respectively $1 \to 3$) is red, $1 \to 3$ (respectively $n \to 2$) and $1 \to 2$ are yellow.

Proof Combining Theorem 6, we need to prove $\exp(D) \geq 2n^2 - 2n$.

Suppose that h, k, v are nonnegative integers such that for all pairs (i, j) of vertices there is an (h, k, v)-walk from i to j. By considering $i = j = n$, we see that there exist nonnegative integers u_1,u_2 and u_3 with

$$\begin{bmatrix} h \\ k \\ v \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Taking i and j to be the initial and terminal vertices of the yellow arc on the n-cycle or $(n-1)$-cycle, then there is a unique path from i to j, and this path has composition $(0, 1, 0)$. Hence

$$Mz = \begin{bmatrix} h \\ k-1 \\ v \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1} \begin{bmatrix} h \\ k-1 \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -(n-2) \\ 0 \\ n-1 \end{bmatrix} \geq 0.$$

So $u_3 \geq n - 1$.

Next take i and j to be the terminal and initial vertices of the yellow arc on the n-cycle or $(n-1)$-cycle, then there is a unique path for each cycle from i to j, and this path composition $(0, 0, n-1)$ or $(0, 0, n-2)$. Hence

$$Mz = \begin{bmatrix} h \\ k \\ v-(n-1) \end{bmatrix},$$

or

$$Mz = \begin{bmatrix} h \\ k \\ v-(n-2) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1}\begin{bmatrix} h \\ k \\ v-(n-1) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} 0 \\ 0 \\ n-1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-1 \\ 0 \\ -(n-1) \end{bmatrix} \geq 0,$$

or

$$z = M^{-1}\begin{bmatrix} h \\ k \\ v-(n-2) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} 0 \\ 0 \\ n-2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ 0 \\ -(n-2) \end{bmatrix} \geq 0.$$

So $u_1 \geq n-1$.

Taking i and j to be the initial and terminal vertices of the red arc on the $(n-1)$-cycle, then there is a unique path from i to j, and this path has composition $(1,0,0)$. Hence

$$Mz = \begin{bmatrix} h-1 \\ k \\ v \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1}\begin{bmatrix} h-1 \\ k \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -(n-2) \\ 1 \\ n-2 \end{bmatrix} \geq 0.$$

So $u_2 \geq 1, u_3 \geq n-2$.

Next take i and j to be the terminal and initial vertices of the red arc on the $(n-1)$-cycle, then there is a unique path from i to j, and this path has composition $(0, 0, n-2)$. Hence

$$Mz = \begin{bmatrix} h \\ k \\ v-(n-2) \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1} \begin{bmatrix} h \\ k \\ v-(n-2) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1} \begin{bmatrix} 0 \\ 0 \\ (n-2) \end{bmatrix}$$
$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ 0 \\ -(n-2) \end{bmatrix} \geq 0.$$

So $u_1 \geq n-2$. Values of u_1, u_3 is compared,thus $u_1 \geq n-1, u_3 \geq n-1$, and

$$h+k+v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \begin{bmatrix} n & n-1 & n-1 \end{bmatrix} \begin{bmatrix} n-1 \\ 1 \\ n-1 \end{bmatrix}$$
$$= 2n^2 - 2n.$$

Theorem 10 Let $a = c-1, b = d+2$, $c+d = n-2, K = -1, N = 1$, and D be primitive. Then

$$\exp(D) = 2n^2 + 2cn - 4n - 2c + 2$$

if and only if the number of red arcs and yellow arcs in the path $3 \to 4 \to \cdots \to n-1 \to n$ are $c-1$ and $n-c-2$, and

(1)$n \to 2$ is red, $n \to 1 \to 2$ are yellow, $1 \to 3$ and $2 \to 3$ are blue; or

(2) $1 \to 3$ is red, $1 \to 2 \to 3$ are yellow, $n \to 1$ and $n \to 2$ are blue.

Proof Combining Theorem 7, we need to prove $\exp(D) \geq 2n^2 + 2cn - 4n - 2c + 2$.

Suppose that h, k, v are nonnegative integers such that for all pairs (i, j) of vertices there is an (h, k, v)-walk from i to j. By considering $i = j = n$, we see that there exist nonnegative integers u_1,

u_2 and u_3 with

$$\begin{bmatrix} h \\ k \\ v \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Taking i and j to be the initial and terminal vertices of the blue arc on the n-cycle or $(n-1)$-cycle,then there is a unique path from i to j, and this path has composition $(0,0,1)$. Hence

$$Mz = \begin{bmatrix} h \\ k \\ v-1 \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1} \begin{bmatrix} h \\ k \\ v-1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} -(n-2) \\ -(c-1) \\ n-2+c \end{bmatrix} \geq 0.$$

So $u_3 \geq n-2+c$. Next take i and j to be the terminal and initial vertices of the blue arc on the n-cycle or $(n-1)$-cycle, then there is a unique path for each cycle from i to j, and this path has composition $(c-1, n-c, 0)$,or $(c, n-2-c, 0)$,or $(c-1, n-1-c, 0)$. Hence

$$Mz = \begin{bmatrix} h-(c-1) \\ k-(n-c) \\ v \end{bmatrix},$$

or

$$Mz = \begin{bmatrix} h-c \\ k-(n-2-c) \\ v \end{bmatrix},$$

or

$$Mz = \begin{bmatrix} h-(c-1) \\ k-(n-1-c) \\ v \end{bmatrix}$$

has a nonnegative integer solution. Necessarily,

$$z = M^{-1}\begin{bmatrix} h-(c-1) \\ k-(n-c) \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c-1 \\ n-c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-1 \\ c-1 \\ -(n-2+c) \end{bmatrix} \geq 0,$$

or

$$z = M^{-1}\begin{bmatrix} h-c \\ k-(n-2-c) \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c \\ n-2-c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ c \\ -(n-2+c) \end{bmatrix} \geq 0,$$

or

$$z = M^{-1}\begin{bmatrix} h-(c-1) \\ k-(n-1-c) \\ v \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - M^{-1}\begin{bmatrix} c-1 \\ n-1-c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} n-2 \\ c-1 \\ -(n-3+c) \end{bmatrix} \geq 0.$$

So $u_1 \geq n-1, u_2 \geq c$. Thus

$$h+k+v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \begin{bmatrix} n & n-1 & n-1 \end{bmatrix}\begin{bmatrix} n-1 \\ c \\ n-2+c \end{bmatrix}$$

$$= 2n^2 + 2cn - 4n - 2c + 2.$$

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