AN INCOMPARABLE UPPER BOUND FOR THE LARGEST LAPLACIAN GRAPH EIGENVALUE

SEZER SORGUN

ABSTRACT. In this paper, we obtain the following upper bounds for the largest Laplacian graph eigenvalue:

$$\mu_1 \leq \max_i \left\{ \sqrt{2d_i\left(m_i + d_i\right) + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j|} \right\}$$

where d_i and m_i are the degree of vertex i and the average degree of vertex i, respectively; $|N_i \cap N_j|$ is the number of common neighbors of i and j vertices. We also compare this bound with the some known upper bounds.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph on vertex set $V = \{1, 2, ..., n\}$ and edge set E. The terms of order and size refer to the numbers $n = |V|$ of vertices and $m = |E|$ of edges of G, respectively. We use d_i and m_i to denote the degree of vertex i and the average degree of vertex i, respectively. We also use N_i and $|N_i \cap N_j|$ to denote the set of neighbors of i and the number of common neighbors of i and j, respectively. Let $\Delta = \max_i \{d_i\}$ and $\delta = \min_i \{d_i\}$. Let $D = D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees and let $A = A(G)$ be the adjacency matrix of G. The Laplacian matrix of G is denoted by $L(G) = D(G) - A(G)$. Since $L(G)$ is a positive semidefinite matrix, therefore all eigenvalues are nonnegative. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$ with the all ones vector as eigenvector. The eigenvalues of $L(G)$ are denoted by $\mu_1 \geq \mu_2 \geq ... \geq \mu_n = 0$.

In the literature, there are a great number studies about the bounds (upper or lower) of the largest Laplacian eigenvalue. Some known results for the upper bounds of μ_1 are given as below.

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$$[3] \quad \mu_1 \leq \max_i \{d_i + m_i\} \tag{1.1}$$

$$[9] \quad \mu_1 \leq \max_i \left\{d_i + \sqrt{d_i m_i}\right\} \tag{1.2}$$

$$[6] \quad \mu_1 \leq \max_i \left\{\sqrt{2d_i\,(d_i + m_i)}\right\} \tag{1.3}$$

$$[12] \quad \mu_1 \leq \max_i \left\{\frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2}\right\} \tag{1.4}$$

$$[2] \quad \mu_1 \leq \max_{i\sim j} \{d_i + d_j\} \tag{1.5}$$

$$[4] \quad \mu_1 \leq \max_{i\sim j} \left\{\frac{d_i\,(d_i + m_i) + d_j\,(d_j + m_j)}{d_i + d_j}\right\} \tag{1.6}$$

$$[9] \quad \mu_1 \leq \max_{i\sim j} \left\{\sqrt{d_i\,(d_i + m_i) + d_j\,(d_j + m_j)}\right\} \tag{1.7}$$

$$[9] \quad \mu_1 \leq \max_{i\sim j} \left\{2 + \sqrt{d_i\,(d_i + m_i - 4) + d_j\,(d_j + m_j - 4) + 4}\right\} \tag{1.8}$$

$$[10] \quad \mu_1 \leq \max_{i\sim j} \left\{\frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2}\right\} \tag{1.9}$$

$$[5] \quad \mu_1 \leq \max_{i\sim j} \{d_i + d_j - |N_i \cap N_j|\} \tag{1.10}$$

$$[14] \quad \mu_1 \leq \max_i \left\{\sqrt{2d_i^2 + 2d_i m_i - 2\sum_{j:j\sim i} |N_i \cap N_j|}\right\} \tag{1.11}$$

$$[13] \quad \mu_1 \leq \delta + \frac{1}{2} + \sqrt{\left(\delta - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - \delta)} \tag{1.12}$$

$$[7] \quad \mu_1 \leq \frac{\delta - 1 + \sqrt{(\delta - 1)^2 + 8\,(\Delta^2 + 2m - \delta(n-1))}}{2} \tag{1.13}$$

$$[8] \quad \mu_1 \leq \Delta + \sqrt{2m + \Delta\,(\delta - 1) - \delta(n-1)} \tag{1.14}$$

$$[11] \quad \mu_1 \leq \frac{\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 4\,(4m - 2\delta\,(n-1))}}{2} \tag{1.15}$$

Until now, it has been aimed to find to better (sharper) bound for the largest Laplacian eigenvalues. Generally, it has been done the numerical comparison among the bounds which is looked for sharper one. Sometimes we cannot compare the bounds in terms of graph invariants as theoretically, except extremal graphs. Even, some bounds are incomparable. In this paper, our motivation is to present this situation. Also, we give a new upper bounds on the largest Laplacian eigenvalues for graphs. In some case, our bound is much better than the known upper bounds. We also compare these bounds with the known bounds which are mentioned above by helping some graph samples.

2. Main Results

Lemma 1. *(Bapat, [1]). Let G be a graph with $V(G) = \{1, 2, ..., n\}$. Let the eigenvalues of $L(G)$ be $\mu_1 \geq \mu_2 \geq ... \geq \mu_n = 0$. Then the eigenvalues of $L(G) + aJ$ matrix where J denotes the square matrix with all entries equal to 1 are $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1}$ and na.*

Theorem 1. *(Das, [9]). If G is a graph on vertex set $V = \{1, 2, 3, ..., n\}$, then*

$$\sum_{j \neq i} |N_i \cap N_j| = \sum_{j \sim i} \{d_j - 1\} \tag{2.1}$$

where d_i is the degree of the vertex i and $|N_i \cap N_j|$ is the number of common neighbors of i and j.

Theorem 2. *Let G be simple connected graph of order n. Then*

$$\mu_1 \leq \max_i \left\{ \sqrt{2d_i (m_i + d_i) + n - 2d_i - 2 \sum_{j:j \sim i} |N_i \cap N_j|} \right\} \tag{2.2}$$

where d_i is the degree of the vertex i ; m_i is the average degree of the vertex i; $|N_i \cap N_j|$ is the number of common neighbors of i and j. Equality in (2.2) holds iff G is a bipartite $\frac{n}{2}$−regular graph.

Proof. Let us consider $L^2(G) + J$ matrix such that J is a matrix with all entries are one. Then its elements are

$$\begin{cases} d_i^2 + d_i + 1 & ; \quad \text{if } i = j, \\ -d_i - d_j + |N_i \cap N_j| + 1 & ; \quad \text{if } i \sim j, \\ |N_i \cap N_j| + 1 & ; \quad \text{otherwise.} \end{cases}$$

Let $X = (x_1, x_2, ..., x_n)^T$ be an eigenvector corresponding to the largest eigenvalue of $L^2(G) + J$ matrix. Let us denote the largest eigenvalue of $L^2(G) + J$ as λ_1. We can assume that one eigencomponent (say x_i) is equal to 1 and the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $|x_k| \leq 1$ for all k. We have

$$(L^2(G) + J)X = \lambda_1 X \tag{2.3}$$

From the i th equation of (2.3), we get

$$\begin{aligned}\lambda_1 x_i &= d_i^2 x_i + d_i x_i + x_i + \sum_{j:j\sim i} \{-d_i - d_j + |N_i \cap N_j| + 1\} x_j \\ &\quad + \sum_{j:j\nsim i} \{|N_i \cap N_j| + 1\} x_j\end{aligned}$$

Taking modulus on both sides, we get

$$\begin{aligned}\lambda_1 &= \left| \begin{array}{c} d_i^2 x_i + d_i x_i + x_i + \sum_{j:j\sim i} \{-d_i - d_j + |N_i \cap N_j| + 1\} x_j \\ + \sum_{j:j\nsim i} \{|N_i \cap N_j| + 1\} x_j \end{array} \right| \\ &\leq d_i^2 x_i + d_i x_i + x_i + \sum_{j:j\sim i} |\{-d_i - d_j + |N_i \cap N_j| + 1\}| \, |x_j| \\ &\quad + \sum_{j:j\nsim i} \{|N_i \cap N_j| + 1\} |x_j| \\ &\leq d_i^2 x_i + d_i x_i + x_i + \sum_{j:j\sim i} \{d_i + d_j - |N_i \cap N_j| - 1\} |x_j| \\ &\quad + \sum_{j:j\nsim i} \{|N_i \cap N_j| + 1\} |x_j| \\ &\leq d_i^2 + d_i + 1 + \sum_{j:j\sim i} \{d_i + d_j - |N_i \cap N_j| - 1\} \\ &\quad + \sum_{j:j\nsim i} \{|N_i \cap N_j| + 1\} \qquad (2.4) \\ &= d_i^2 + d_i + 1 + \sum_{j:j\sim i} d_i + \sum_{j:j\sim i} d_j - \sum_{j:j\sim i} |N_i \cap N_j| - \sum_{j:j\sim i} 1 \\ &\quad + \sum_{j:j\nsim i} |N_i \cap N_j| + \sum_{j:j\nsim i} 1 \\ &= 2d_i^2 + d_i + 1 + d_i m_i - \sum_{j:j\sim i} |N_i \cap N_j| - d_i \\ &\quad + \sum_{j:j\nsim i} |N_i \cap N_j| + \sum_{j:j\nsim i} 1 \\ &= 2d_i^2 + 2d_i m_i + 1 - 2\sum_{j:j\sim i} |N_i \cap N_j| - d_i + \sum_{j:j\nsim i} 1 \qquad \text{(from (2.1))}\end{aligned}$$

It is easy to see that

$$\sum_{\substack{j:j\sim i \\ i\neq j}} 1 + \sum_{\substack{j:j\nsim i \\ i\neq j}} 1 = n - 1$$

i.e.

$$\sum_{\substack{j:j\nsim i \\ i\neq j}} 1 = n - d_i - 1$$

Hence we get

$$\lambda_1 \leq 2d_i^2 + d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j| \tag{2.5}$$

i.e.

$$\lambda_1 \leq \max_i \left\{ 2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j| \right\} \tag{2.6}$$

Now, we analyze the eigenvalues of $L^2(G) + J$ and $L^2(G)$ matrices. Let $X' = (1, 1, ..., 1)^T$ be an eigenvector of $L^2(G)$ corresponding to 0. Since

$$\left(L^2(G) + J\right) X' = JX' = nX'$$

then, n is an eigenvalue of $L^2(G) + J$. So let λ be any non-zero eigenvalue corresponding eigenvector $X' = (x_1, x_2, ..., x_n)^T$ of $L^2(G)$. Therefore $\sum_{i=1}^{n} x_i = 0$. We have

$$\begin{aligned} \left(L^2(G) + J\right) X' &= L^2 X' \\ &= \lambda X' \end{aligned}$$

Thus, eigenvalue λ is also an eigenvalue of $L^2(G) + J$ matrix.

Consequently, the eigenvalues of $L^2(G)$ and $L^2(G) + J$ are $\mu_1^2 \geq \mu_2^2 \geq ... \geq \mu_n^2 = 0$ and $\mu_1^2 \geq \mu_2^2 \geq ... \geq \mu_{n-1}^2$ and n, respectively.

There are two cases:

Case 1. Assume that $\mu_1^2 \geq n$. In that case we can say μ_1^2 is the largest eigenvalue for both matrices. So, we get

$$\lambda_1 = \mu_1^2 \leq \max_i \left\{ 2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j| \right\}$$

i.e.

$$\mu_1 \leq \max_i \left\{ \sqrt{2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j|} \right\}$$

Case 2. Assume that $n \geq \mu_1^2$. Then we get

$$\mu_1^2 \leq n \leq \max_i \left\{ 2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j| \right\}$$

from (2.4). Thus we also get

$$\mu_1 \le \max_i \left\{ \sqrt{2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j|} \right\}$$

Therefore we complete the first part of the proof.

Assume that G is bipartite $\frac{n}{2}-$regular graph. Then the equality in (2.2) holds by a simply calculation.

Conversely, assume that the equality in (2.2) holds. Hence all inequality above must be equalities. Then $x_i = 1$ and $|x_k| = 1$ such that $X = (x_1, x_2, ..., x_n)^T$ eigenvector corresponding to the largest eigenvalue of $L^2(G) + J$ matrix. X is also eigenvector of $L(G)$ corresponding to μ_1. Since $\sum_{i=1}^{n} x_i = 0$, G must be graph with order even. Let $V_1 = \{k : x_k = x_i = 1\}$ for every $i \sim k$ and $V_2 = \{k : x_k = -x_i = -1\}$ for every $i \sim k$. Then V_1 and V_2 are partition of V. We have

$$L(G)X = \mu_1 X$$

and for every $i \in V$ we get

$$\mu_1 = 2d_i \tag{2.7}$$

On the other hand, from inequality in (1.11), we get

$$\begin{aligned} &\sqrt{2d_i^2 + 2d_i m_i + n - 2d_i - 2\sum_{j:j\sim i} |N_i \cap N_j|} \\ \le \quad &\sqrt{2d_i^2 + 2d_i m_i - 2\sum_{j:j\sim i} |N_i \cap N_j|} \end{aligned}$$

i.e.

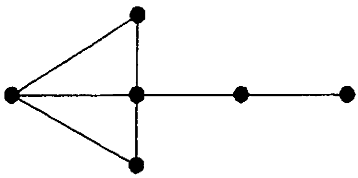
$$n \le 2d_i \tag{2.8}$$

Since $\mu_1 \le n$ for every connected graph, from (2.7) and (2.8) we get

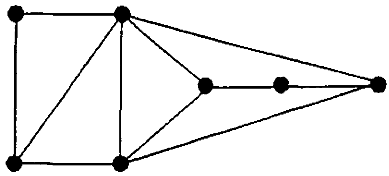
$$n = 2d_i$$

for every $i \in V$. Hence G is a bipartite $\frac{n}{2}-$regular graph. □

Example 1. *Let G and H be as below.*



G



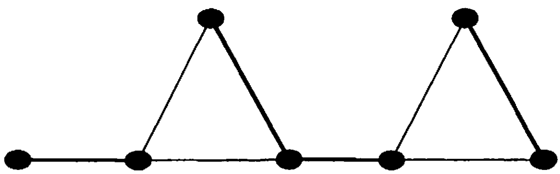
H

For these graphs $\mu_1(G) = 5.08$, $\mu_1(H) = 6.17$ rounded to three decimal places and the mentioned bounds give the following results:

	(1.1)	(1.2)	(1.3)	(1.4)	(1.5)	(1.6)	(1.7)	(1.8)	(1.9)	(1.10)
G	6.25	7.00	7.07	6.69	7.00	6.00	6.48	6.42	6.00	6.00
H	8.00	8.87	8.94	8.52	9.00	7.78	8.37	8.32	7.78	7.00

	(1.11)	(1.12)	(1.13)	(1.14)	(1.15)	(2.2)
G	6.42	6.42	7.07	7.00	6.69	6.32
H	8.00	8.11	8.88	8.87	8.38	7.34

As shown in the table, while (2.2) is better than all bound except (1.6), (1.9) and (1.10) for graph G, it is better almost all bound except only (1.10) for graph H.



L

However, sometimes (2.2) may be better than (1.10). For instance, let L be graph as above. Then we obtain following results:

(1.1)	(1.2)	(1.3)	(1.4)	(1.5)	(1.6)	(1.7)	(1.8)	(1.9)	(1.10)
5.66	5.82	5.83	5.77	6.00	5.50	5.74	5.60	5.49	6.00

(1.11)	(1.12)	(1.13)	(1.14)	(1.15)	(2.2)
5.47	6.42	7.07	6.00	6.00	5.56

As seen, the bound (2.2) is better than (1.10). Therefore our bound is an incomparable bound among the some upper bounds which mentioned above. Hence we can say that the comparison among the bounds can change *according to graph invariants such as the number of edges, the maximum degree, etc.*

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NEVŞEHIR HACI BEKTAŞ VELI UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS, 50300 NEVŞEHIR, TURKEY
E-mail address: srgnrzs@gmail.com