The truncated determinants of combinatorial rectangular arrays *

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Abstract

There are many rectangular arrays whose n^{th} column is the n-fold convolution of the 0^{th} column in combinatorics. For this type of rectangular arrays, we prove a formula for evaluating the determinant of certain submatrices, which was conjectured by Hoggatt and Bicknell. Our result unifies the determinant evaluation of submatrices of the rectangular arrays consisting of binomial coefficients, multinomial coefficients, Fibonacci numbers, Catalan numbers, generalized Catalan and Motzkin numbers.

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1 Introduction

There have been extensive study and application of determinant evaluations in combinatorics, algebra and mathematical physics. See [7, 8] and the references therein. Our concern in this paper is the determinant evaluation of submatrices of combinatorial rectangular arrays.

Let $\{x_i\}_{i\geq 0}$ be a sequence of real numbers. The n-fold convolution of the sequence $\{x_i\}$ with itself n times can be recursively defined by $\{y_i^{(n)}\}_{i\geq 0}$, where $y_i^{(n)} = \sum_j x_j y_{i-j}^{(n-1)}$ and $y_i^{(1)} = \sum_j x_j x_{i-j}$. There are many combinatorial rectangular arrays whose n^{th} column is the n-fold convolution of the 0^{th} column (the leftmost column), which are also named *convolution*

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arrays (see [4, 6]). For example, Pascal rectangular array (the rectangular array consisting of binomial coefficients) [13, A007318]

$$\left(\binom{i+j}{j}\right)_{i,j\geq 0}=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the convolution array for the sequence $\{1, 1, 1, \ldots\}$.

Convolution arrays can be described in the context of Riordan array. A *(proper) Riordan array* is an infinite lower triangular matrix with the k^{th} column generating function is $d(t)h(t)^k$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$ and $h(0)$ is not equal to zero. Denote a Riordan array by $D = (d_{nk})_{n,k\in\mathbb{N}} = (d(t), h(t))$, where $d_{nk} = [t^n]d(t)h(t)^k$. The Riordan arrays satisfying $h(t) = td(t)$ are called *Bell-type arrays* or *renewal arrays* (see [2, 3, 12]). Clearly, convolution arrays are just Bell-type arrays written in the rectangular form since the high convolution can be stated in terms of the powers of a generating function.

In what follows, we consider the determinant of submatrices of convolution arrays. Let $C = (c_{ij})_{i,j\geq 0}$ be a convolution array. Consider two submatrices $M_0(n,p) = (c_{ij})_{0\leq i\leq n-1,\ p-1\leq j\leq n+p-2}$ and $M_1(n,p) = (c_{ij})_{1\leq i\leq n,\ p-1\leq j\leq n+p-2}$. That is, $M_0(n,p)$ (resp. $M_1(n,p)$) is any $n\times n$ submatrix of C containing consecutive rows and columns, with its first row along the 0^{th} row of C (resp. the 1^{st} row of C) and its first column along the $(p-1)^{th}$ column of C. Take Pascal rectangular array for example,

$$M_0(4,2)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \end{pmatrix},\quad M_1(4,3)=\begin{pmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{pmatrix}.$$

Hoggatt and Bicknell *et al* [1, 5, 6, 9] showed that $\det(M_0(n,p)) = 1$ and $\det(M_1(n,p)) = \binom{p+n-1}{n}$ for Pascal rectangular array and other combinatorial rectangular arrays which are included in Section 3 of this paper. Furthermore, they conjectured that the previous determinant evaluation can be generalized for an arbitrary convolution array (see [6, p.401]).

The object of this paper is to prove a unified formula for evaluating the determinant of submatrices of convolution arrays, which confirms the conjecture of Hoggatt and Bicknell. Given a sequence $\{1, a_1, a_2, \ldots\}$ whose first term is assumed to be 1 for convenience, consider its corresponding convolution array

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ a_1 & 2a_1 & 3a_1 & 4a_1 & \cdots \\ a_2 & 2a_2 + a_1^2 & 3a_2 + 3a_1^2 & 4a_2 + 6a_1^2 & \cdots \\ a_3 & 2a_3 + 2a_1a_2 & 3a_3 + 6a_1a_2 + a_1^3 & 4a_3 + 12a_1a_2 + 4a_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Our main result is the following theorem.

Theorem 1. *Let $C = (c_{ij})_{i,j\geq 0}$ be the convolution array for the sequence $\{1, a_1, a_2, \ldots\}$. Let $M_0(n,p) = (c_{ij})_{0\leq i\leq n-1,\ p-1\leq j\leq n+p-2}$ and $M_1(n,p) = (c_{ij})_{1\leq i\leq n,\ p-1\leq j\leq n+p-2}$. Then*

(i) $\det(M_0(n,p)) = a_1^{\binom{n}{2}}$;

(ii) $\det(M_1(n,p)) = \binom{p+n-1}{n} a_1^{\binom{n+1}{2}}$.

In Section 2, we give the proof of Theorem 1 by using the difference operator, which is more natural and general than the proof given by Hoggatt and Bicknell for the determinant evaluation relating to specific convolution arrays. In Section 3, we apply Theorem 1 to evaluate the determinants of submatrices of some well-known combinatorial rectangular arrays,

2 Proof of Theorem 1

Let c_{ij} be the (i,j)-entry of C (the entry in the i^{th} row and j^{th} column of C). Clearly, the j^{th} column generating function is given by

$$\sum_{i\geq 0} c_{ij} t^i = (1 + a_1 t + a_2 t^2 + \cdots)^{j+1}.$$

Then

$$\begin{aligned} c_{0j} &= 1, \\ c_{1j} &= \binom{j+1}{1} a_1, \\ c_{2j} &= \binom{j+1}{1} a_2 + \binom{j+1}{2} a_1^2, \\ c_{3j} &= \binom{j+1}{1} a_3 + \binom{j+1}{2} 2a_1a_2 + \binom{j+1}{3} a_1^3, \\ c_{4j} &= \binom{j+1}{1} a_4 + \binom{j+1}{2} (2a_1a_3 + a_2^2) + \binom{j+1}{3} 3a_1^2a_2 + \binom{j+1}{4} a_1^4. \end{aligned}$$

In general, for $i \geq 1$,

$$\begin{aligned} c_{ij} &= \sum_{\pi(i)} \binom{j+1}{k} \frac{k!}{k_1! \cdots k_i!} a_1^{k_1} \cdots a_i^{k_i} \\ &= \sum_{k=1}^{i} \binom{j+1}{k} \bar{c}_{ik}(a_1, a_2, \ldots), \end{aligned}$$

where

$$\bar{c}_{ik}(a_1, a_2, \ldots) = \sum_{\pi(i)} \frac{k!}{k_1! \cdots k_i!} a_1^{k_1} \cdots a_i^{k_i},$$

$\pi(i)$ is a partition of the set $\{1, 2, \ldots, i\}$ and $k = k_1 + k_2 + \cdots + k_i$ (see, e.g., [11]). In the rest of this paper, denote that $\bar{c}_{ik} = \bar{c}_{ik}(a_1, a_2, \ldots)$ for brevity. Specifically, $\bar{c}_{ii} = a_1^i$ and $\bar{c}_{i,i-1} = (i-1)a_1^{i-2}a_2$ for $i \geq 1$.

For any function $f : \mathbb{Z} \longrightarrow \mathbb{C}$, the backward difference operator is defined by $\Delta f(m) = f(m) - f(m-1)$ and the d^{th} difference operator by $\Delta^d f(m) = \Delta(\Delta^{d-1} f(m))$. By induction on d, we have

$$\Delta^d f(m) = \sum_{k=0}^{d} (-1)^k \binom{d}{k} f(m-k). \tag{1}$$

If $f(m) = \binom{m}{i}$, then

$$\Delta^d \binom{m}{i} = \binom{m-d}{i-d}. \tag{2}$$

Combining (1) and (2), we get the following identity.

Lemma 1.

$$\sum_{k=0}^{d} (-1)^k \binom{d}{k} \binom{m-k}{i} = \binom{m-d}{i-d}.$$

Another lemma will be needed for the proof of Theorem 1.

Lemma 2.

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{p}{n-i} \binom{p+i-1}{i} = \delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker symbol which is 1 if $n = 0$ and 0 otherwise.

Proof. Using the trick of generating functions, i.e.,

$$[t^n](1+t)^p \left[\frac{1}{(1+u)^p} \bigg|_{u=t} \right] = \delta_{n,0},$$

we obtain

$$\sum_{i=0}^{n}(-1)^i\binom{p}{n-i}\binom{p+i-1}{i}=\delta_{n,0}.$$

Since $(-1)^{-i}=(-1)^i$ and $(-1)^n\delta_{n,0}=\delta_{n,o}$, the desired identity follows. □

We are now in a position to prove Theorem 1. Throughout the proof, "the k^{th} column" of $M_0(n,p)$ or $M_1(n,p)$ is the column having the generating function $(1+a_1t+a_2t^2+\cdots)^{k+1}$.

The proof of Theorem 1. (i) We will transform $M_0(n,p)$ into a lower triangular matrix by performing a sequence of the difference operations on its rows. At first, subtract the $(j-1)^{th}$ column from the j^{th} column successively, where $j=p,p+1,\dots,p+n-2$. It is just a difference operator on each row. Consequently, the i^{th} row of the resulting matrix translates into

$$(c_{i,p-1},c_{i,p}-c_{i,p-1},c_{i,p+1}-c_{i,p},\dots,c_{i,p+n-2}-c_{i,p+n-3}).$$

Next, for the resulting matrix, we proceed to subtract the $(j-1)^{th}$ column from the j^{th} column successively, where $j=p+1,p+2,\dots,p+n-2$. The i^{th} row turns to be

$$(c_{i,p-1},c_{i,p}-c_{i,p-1},c_{i,p+1}+c_{i,p-1}-2c_{i,p},\dots,c_{i,p+n-2}+c_{i,p+n-4}-2c_{i,p+n-3}).$$

Generally, in the d^{th} step, we subtract the $(j-1)^{th}$ column from the j^{th} column successively, where $j=p+d-1,p+d,\dots,p+n-2$. Continuing the similar difference process, the i^{th} row is finally transformed to be

$$\left(\sum_{k=0}^{j-p+1}(-1)^k\binom{j-p+1}{k}c_{i,j-k}\right)_j,\quad j=p-1,\dots,p+n-2.$$

Denote the final matrix by $M_0^*(n,p)$. We claim that $M_0^*(n,p)$ is a lower triangular matrix. Indeed, the 0^{th} row of $M_0^*(n,p)$ is trivially $\{1,0,\dots,0\}$. By the expression for c_{ij}, we have

$$c_{i,j-k}=\sum_{\ell=1}^{i}\binom{j+1-k}{\ell}\bar{c}_{i\ell},$$

where $i\geq 1$. Then the (i,j)-entry $(1\leq i\leq n-1)$ of $M_0^*(n,p)$ is further simplified to

$$\begin{aligned}
&\sum_{k=0}^{j-p+1}(-1)^k\binom{j-p+1}{k}c_{i,j-k}\\
=\ &\sum_{k=0}^{j-p+1}(-1)^k\binom{j-p+1}{k}\sum_{\ell=1}^{i}\binom{j+1-k}{\ell}\bar{c}_{i\ell}\\
=\ &\sum_{\ell=1}^{i}\bar{c}_{i\ell}\sum_{k=0}^{j-p+1}(-1)^k\binom{j-p+1}{k}\binom{j+1-k}{\ell}\\
=\ &\sum_{\ell=1}^{i}\binom{p}{\ell-j+p-1}\bar{c}_{i\ell},
\end{aligned}$$

where the last equality follows from Lemma 1. Specifically, for $j-i=p-1$,

$$\begin{aligned}
&\sum_{\ell=1}^{i}\binom{p}{\ell-j+p-1}\bar{c}_{i\ell}\\
=\ &\sum_{\ell=1}^{i}\binom{p}{\ell-i}\bar{c}_{i\ell}\\
=\ &\bar{c}_{ii}\\
=\ &a_1^i,
\end{aligned}$$

which means that the entries on the main diagonal of $M_0^*(n,p)$ are a_1^i $(1\le i\le n-1)$. Moreover, all the entries above the main diagonal are zero since $\sum_{\ell=1}^{i}\binom{p}{\ell-j+p-1}\bar{c}_{i\ell}=0$ for $j-i>p-1$. We summarize that $M_0^*(n,p)$ is a lower triangular matrix with the diagonal entries a_1^i $(0\le i\le n-1)$. Thus, $\det(M_0(n,p))=\det(M_0^*(n,p))=a_1{}^{\binom{n}{2}}$.

(ii) First we apply the similar difference operators on the rows of $M_1(n,p)$ as in the proof of Theorem 1(i). Denote the resulting matrix by $M_1^*(n,p)$. Clearly, the (i,j)-entry $(1\le i\le n)$ of $M_1^*(n,p)$ is given by

$$\sum_{\ell=1}^{i}\binom{p}{\ell-j+p-1}\bar{c}_{i\ell}. \tag{3}$$

Note that the entries on the main diagonal of $M_1^*(n,p)$ are $pa_1^i+(i-1)a_1^{i-2}a_2$

$(1 \le i \le n)$ because for $j - i = p - 2$,

$$\begin{aligned} & \sum_{\ell=1}^{i} \binom{p}{\ell - j + p - 1} \bar{c}_{i\ell} \\ = & \sum_{\ell=1}^{i} \binom{p}{\ell - i + 1} \bar{c}_{i\ell} \\ = & p\bar{c}_{ii} + \bar{c}_{i,i-1} \\ = & pa_1^i + (i-1)a_1^{i-2}a_2. \end{aligned}$$

The entries on the superdiagonal of $M_1^*(n,p)$ (the diagonal immediately above the main diagonal) are a_1^i $(1 \le i \le n-1)$ since for $j - i = p - 1$,

$$\begin{aligned} & \sum_{\ell=1}^{i} \binom{p}{\ell - j + p - 1} \bar{c}_{i\ell} \\ = & \sum_{\ell=1}^{i} \binom{p}{\ell - i} \bar{c}_{i\ell} \\ = & \bar{c}_{ii} \\ = & a_1^i. \end{aligned}$$

The entries above the superdiagonal are zero since $\sum_{\ell=1}^{i} \binom{p}{\ell-j+p-1} \bar{c}_{i\ell} = 0$ for $j - i > p - 1$.

Next we will triangularize $M_1^*(n,p)$ by eliminating the entries on its superdiagonal. From the expression (3), we observe that the entries in the same row of $M_1^*(n,p)$ share the common terms $\bar{c}_{ik}$. Then we use the elementary column operations for the columns of $M_1^*(n,p)$ from left to right to eliminate the terms $\bar{c}_{ik}$ successively until the entries on the superdiagonal turn to be zero. To illustrate, take

$$M_1^*(3,p) = \begin{pmatrix} pa_1 & a_1 & 0 \\ pa_2 + \binom{p}{2}a_1^2 & a_2 + pa_1^2 & a_1^2 \\ pa_3 + 2\binom{p}{2}a_1a_2 + \binom{p}{3}a_1^3 & a_3 + 2pa_1a_2 + \binom{p}{2}a_1^3 & 2a_1a_2 + pa_1^3 \end{pmatrix}$$

for example. Multiplying the $(p-1)^{th}$ column by $-\frac{1}{p}$ and adding it to the p^{th} column, we have

$$\begin{pmatrix} pa_1 & 0 & 0 \\ pa_2 + \binom{p}{2}a_1^2 & \left[p - \frac{1}{p}\binom{p}{2}\right]a_1^2 & a_1^2 \\ pa_3 + 2\binom{p}{2}a_1a_2 + \binom{p}{3}a_1^3 & 2\left[p - \frac{1}{p}\binom{p}{2}\right]a_1a_2 + \left[\binom{p}{2} - \frac{1}{p}\binom{p}{3}\right]a_1^3 & 2a_1a_2 + pa_1^3 \end{pmatrix}.$$

Next, multiplying the p^{th} column by $-\frac{1}{p - \frac{1}{p}\binom{p}{2}}$ and adding it to the $(p+1)^{th}$

column, we obtain the lower triangular matrix

$$\begin{pmatrix} pa_1 & 0 & 0 \\ pa_2+\binom{p}{2}a_1^2 & \left[p-\frac{1}{p}\binom{p}{2}\right]a_1^2 & 0 \\ pa_3+2\binom{p}{2}a_1a_2+\binom{p}{3}a_1^3 & 2\left[p-\frac{1}{p}\binom{p}{2}\right]a_1a_2+\left[\binom{p}{2}-\frac{1}{p}\binom{p}{3}\right]a_1^3 & \left[p-\frac{\binom{p}{2}-\frac{1}{p}\binom{p}{3}}{\binom{p}{1}-\frac{1}{p}\binom{p}{2}}\right]a_1^3 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det(M_1(3,p)) &= \det(M_1^*(3,p)) \\ &= p\left[\binom{p}{1}-\frac{1}{p}\binom{p}{2}\right]\left[\binom{p}{1}-\frac{\binom{p}{2}-\frac{1}{p}\binom{p}{3}}{\binom{p}{1}-\frac{1}{p}\binom{p}{2}}\right]a_1^{1+2+3} \\ &= \binom{p+2}{3}a_1^6, \end{aligned}$$

where we write $p=\binom{p}{1}$ for some p. Meanwhile, we get

$$\begin{aligned} \det(M_1(2,p)) &= p\left[\binom{p}{1}-\frac{1}{p}\binom{p}{2}\right]a_1^{1+2} \\ &= \binom{p+1}{2}a_1^3. \end{aligned}$$

In a similar way,

$$\begin{aligned} &\det(M_1(4,p)) \\ &= p\left[\binom{p}{1}-\frac{1}{p}\binom{p}{2}\right]\left[\binom{p}{1}-\frac{\binom{p}{2}-\frac{1}{p}\binom{p}{3}}{\binom{p}{1}-\frac{1}{p}\binom{p}{2}}\right]\left[\binom{p}{1}-\frac{\binom{p}{2}-\frac{\binom{p}{3}-\frac{1}{p}\binom{p}{4}}{\binom{p}{1}-\frac{1}{p}\binom{p}{2}}}{\binom{p}{1}-\frac{\binom{p}{2}-\frac{1}{p}\binom{p}{3}}{\binom{p}{1}-\frac{1}{p}\binom{p}{2}}}\right]a_1^{10} \\ &= \binom{p+3}{4}a_1^{10}. \end{aligned}$$

Noting that

$$\frac{\binom{p+i-1}{i}}{\binom{p+i-2}{i-1}}=\frac{p+i-1}{i},$$

it follows from the ratios $\frac{\det(M_1(i,p))}{\det(M_1(i-1,p))}$ $(i=2,3,4)$ that

$$\binom{p}{1}-\frac{1}{p}\binom{p}{2}=\frac{p+1}{2},$$

$$\binom{p}{1}-\frac{\binom{p}{2}-\frac{1}{p}\binom{p}{3}}{\frac{p+1}{2}}=\frac{p+2}{3},$$

$$\binom{p}{1} - \frac{\binom{p}{2} - \frac{\binom{p}{3}-\frac{1}{p}\binom{p}{4}}{\frac{p+1}{2}}}{\frac{p+2}{3}} = \frac{p+3}{4}.$$

In what follows, we will prove Theorem 1(ii) by generalizing the identities above.

We proceed by induction on the order n of $M_1(n,p)$. It is trivial that $\det(M_1(1,p)) = pa_1$. Assume that Theorem 1(ii) holds for the integers less than n, namely, $\det(M_1(i,p)) = \binom{p+i-1}{i} a_1^{\binom{i+1}{2}}$ and $\det(M_1(i-1,p)) = \binom{p+i-2}{i-1} a_1^{\binom{i}{2}}$, $2 \le i \le n-1$. Equivalently,

$$\det(M_1(i,p)) = \frac{p+i-1}{i} a_1^i \det(M_1(i-1,p)), \quad 2 \le i \le n-1.$$

Consider the case for n. Our aim is to prove

$$\det(M_1(n,p)) = \frac{p+n-1}{n} a_1^n \det(M_1(n-1,p).$$

By the induction hypothesis and the triangularizing process for $M_1^*(i,p)$ $(2 \le i \le n-1)$, it suffices to show that

$$\binom{p}{1} - \frac{\binom{p}{2} - \frac{\binom{p}{3} - \frac{\vdots\ \frac{\binom{p}{n-1}-\frac{1}{p}\binom{p}{n}}{}}{\frac{p+n-4}{n-3}}}{\frac{p+n-3}{n-2}}}{\frac{p+n-2}{n-1}} = \frac{p+n-1}{n}.$$

After the reduction,

$$\binom{p}{1}\binom{p+n-2}{n-1} - \binom{p}{2}\binom{p+n-3}{n-2} + \cdots + (-1)^{n-1}\binom{p}{n} = \binom{p+n-1}{n},$$

i.e.,

$$0 = \binom{p+n-1}{n} - \binom{p}{1}\binom{p+n-2}{n-1} + \binom{p}{2}\binom{p+n-3}{n-2} - \cdots + (-1)^n\binom{p}{n}.$$

The identity above follows immediately from Lemma 2. This completes the proof of the theorem. □

3 Applications

In this section, we give an application of Theorem 1 to get some results on the determinant evaluation for certain well-known combinatorial rectangular arrays. Fibonacci rectangular array [13, A037027]

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & 5 & 9 & 14 & 20 & \cdots \\ 3 & 10 & 22 & 40 & 65 & \cdots \\ 5 & 20 & 51 & 105 & 190 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the convolution array for the well-known Fibonacci sequence [13, A000045].

Corollary 1 ([6, Theorem 5.1]). *For Fibonacci rectangular array, we have* $\det(M_0(n,p)) = 1$ *and* $\det(M_1(n,p)) = \binom{p+n-1}{n}$.

Obviously, Theorem 1 holds for Pascal rectangular array. More general, Theorem 1 is also valid for the rectangular array of multinomial coefficients. To see this, consider the triangle of trinomial coefficients [13, A027907] written in the top-justified form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & 4 & 5 & \cdots \\ & 1 & 3 & 6 & 10 & 15 & \cdots \\ & & 2 & 7 & 16 & 30 & \cdots \\ & & 1 & 6 & 19 & 45 & \cdots \\ & & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the j^{th} column generating function: $(1+t+t^2)^j$ $(j \geq 0)$. Clearly, the array above except the 0^{th} column can be viewed as a convolution array for the finite sequence $(1,1,1)$ since the j^{th} column is $(j-1)$-fold convolution of the 1^{st} column $(j \geq 2)$. Note that the j^{th} column generating function of the rectangular array for the general m-multinomial coefficient is $(1+t+t^2+\cdots+t^{m-1})^j$.

Corollary 2 ([1, Theorem 3.1 and Theorem 3.2]). *For the triangle of multinomial coefficients written in the top-justified form except the leftmost column,* $\det(M_0(n,p)) = 1$ *and* $\det(M_1(n,p)) = \binom{p+n-1}{n}$.

He [2] gave a generalization of Catalan and Motzkin triangles associated with two parameters c and r based on the sequence characterization of Bell-type arrays. The two triangles written in the rectangular form are just two convolution arrays. More precisely, the convolution array of (c,r)-Catalan numbers is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ c & 2c & 3c & \cdots \\ c^2+cr & 3c^2+2cr & 6c^2+3cr & \cdots \\ c^3+3c^2r+cr^2 & 4c^3+8c^2r+2cr^2 & 10c^3+15c^2r+3cr^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the 0^{th} column generating function

$$d_{c,r}(t) = \frac{1-(c-r)t-\sqrt{1-2(c+r)t+(c-r)^2t^2}}{2rt}.$$

The convolution array of (c,r)-Motzkin numbers

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ c & 2c & 3c & \cdots \\ c^2+cr & 3c^2+2cr & 6c^2+3cr & \cdots \\ c^3+3c^2r & 4c^3+8c^2r & 10c^3+4c^2r+11c^2r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the 0^{th} column generating function

$$\tilde{d}_{c,r}(t) = \frac{1-ct-\sqrt{1-2ct+c(c-4r)t^2}}{2crt^2}.$$

Corollary 3. *For the convolution array of (c,r)-Catalan or (c,r)-Motzkin numbers,* $\det(M_0(n,p)) = c^{\binom{n}{2}}$ *and* $\det(M_1(n,p)) = \binom{p+n-1}{n}c^{\binom{n+1}{2}}$.

Catalan triangle [13, A039598] written in the rectangular form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 2 & 4 & 6 & 8 & \cdots \\ 5 & 14 & 27 & 44 & \cdots \\ 14 & 48 & 110 & 208 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the convolution array with the 0^{th} column generating function: $\frac{1-2t-\sqrt{1-4t}}{2t^2}$.

Corollary 4 (conjectured by Miana and Romero [10]). *The $n \times n$ leading principal minor of Catalan triangle in the rectangular form equals to* $2^{\binom{n}{2}}$.

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