The chromatic number of the square of a Halin graph with maximum degree five is six

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Abstract

This article proves that the square of a Halin graph G with $\Delta = 5$ has the chromatic number 6. This gives a positive answer to an open problem in [Y. Wang, Distance two labelling of Halin graphs, Ars Combin. 114 (2014), 331-343].

Keywords. Halin graph; Square; Chromatic number; Maximum degree

1 Introduction

Let G be a simple graph with vertex set $V(G)$, edge set $E(G)$, order $|G|$, and maximum degree $\Delta(G)$ (in short, Δ). For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v. A vertex of degree k is called a k-*vertex*. The *distance* between two vertices u and v is the length of a shortest path connecting them in G. The *square* G^2 of a graph G is the graph defined on

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the vertex set $V(G)$ such that two vertices are adjacent in G^2 if and only if their distance is 1 or 2 in G. A *k-coloring* of a graph G is a mapping f from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for every edge xy of G. The *chromatic number* $\chi(G)$ of G is the smallest k such that G has a k-coloring.

Wegner [14] proved that $\chi(G^2) \leq 8$ for a planar graph G with $\Delta = 3$ and conjectured that 8 can be reduced to 7. Moreover, he also proposed the following conjecture.

Conjecture 1 *For a planar graph G,*

$$\chi(G^2) \leq \begin{cases} \Delta + 5, & \text{if } 4 \leq \Delta \leq 7; \\ \lfloor 3\Delta/2 \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

This conjecture remains open. Van den Heuvel and McGuinness [3] proved that $\chi(G^2) \leq 2\Delta + 25$ for any planar graph G. The best known result so far is $\chi(G^2) \leq \lceil 5\Delta/3 \rceil + 78$, due to Molloy and Salavatipour [8]. Lih, Wang and Zhu [7] established the conjecture for a K_4-minor free graph. It is shown [10, 11] that every outerplanar graph G with $\Delta \geq 3$ has $\chi(G^2) \leq \Delta + 2$, and $\chi(G^2) = \Delta + 1$ if $\Delta \geq 6$. Wang and Lih [12] proved that if G is planar graph without 3-cycles and 4-cycles, then $\chi(G^2) \leq \Delta + 16$. Zhu et al. [15] extended this result by showing that if a planar graph G contains no 4-cycles or no 5-cycles, then $\chi(G^2) \leq \Delta + 7$.

Let T be a tree with $\Delta \geq 3$ and without 2-vertices. A 1-vertex of T is called a *leaf.* A *Halin graph* is a plane graph $G = T \cup C$, where C is a cycle connecting the leaves of T in the cyclic order determined by the planar drawing of T. Vertices of C are called *outer vertices* of G and vertices in $V(G) \setminus V(C)$ are called *inner vertices* of G. Let $V_{\text{in}}(G)$ denote the set of inner vertices in G. A Halin graph G is called a *wheel* if $|V_{\text{in}}(G)| = 1$. An inner vertex is called *a handle* if it is adjacent to only one inner vertex. A *k-handle* is a handle of degree k.

It is straightforward to see that Halin graphs are 3-connected plane graphs. Some properties and parameters on Halin graphs have been extensively investigated in [1, 2, 4–6, 9].

Wang [13] showed that every Halin graph G with $\Delta \geq 6$ has $\chi(G^2) = \Delta + 1$, and proposed the following conjecture:

Conjecture 2 *If G is a Halin graph with $\Delta = 5$, then $\chi(G^2) = 6$.*

This paper gives a positive solution to Conjecture 2.

2 Structural lemma

Suppose that G is a Halin graph. Since G is 3-connected, every edge e of G is incident to exactly two faces f_1 and f_2. Let $m^*(e) = \max\{d_G(f_1), d_G(f_2)\}$.

Lemma 1 *Let $G = T \cup C$ be a Halin graph with $\Delta \leq 5$ that is not a wheel. Then C contains a path $P_k = x_1x_2\cdots x_k$ such that one of the following holds (see Fig. 1):*

(B1) *There exist a k-handle u and a vertex v with $N_G(u) = \{v, x_1, \dots, x_{k-1}\}$ and $vx_k \in E(G)$ such that either $k = 5$, or $3 \leq k \leq 4$ and $d_G(v) = 3$.*

(B2) *$k \geq 4$ and there exist two handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_1, \dots, x_p\}$ and $N_G(u_2) = \{v, x_{p+1}, \dots, x_k\}$, where $2 \leq p \leq k-2$.*

(B3) *There exist two handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_1, \dots, x_p\}$, $N_G(u_2) = \{v, x_{p+2}, \dots, x_k\}$, $vx_{p+1} \in E(G)$, where $2 \leq p \leq k-3$, such that either $k \geq 6$, or $k = 5$ and $d_G(v) = 4$.*

(B4) *$4 \leq k \leq 5$ and there exist a handle u and a vertex v with $N_G(u) = \{v, x_1, \dots, x_{k-2}\}$ and $vx_{k-1}, vx_k \in E(G)$.*

(B5) *$k = 6$ and there exist two 3-handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_2, x_3\}$, $N_G(u_2) = \{v, x_5, x_6\}$ and $vx_1, vx_4 \in E(G)$.*

(B6) *$k = 5$ and there exist a 4-handle u and a 4-vertex v with $N_G(u) = \{v, x_2, x_3, x_4\}$ and $vx_1, vx_5 \in E(G)$.*

(B7) *$k = 5$ and there exist a 3-handle u and a 4-vertex v with $N_G(u) = \{v, x_2, x_3\}$ and $N_G(v) = \{u, w, x_1, x_4\}$ such that $d_G(w, x_5) \leq 2$.*

Proof. Since G is not a wheel, G contains at least two inner vertices. If G has exactly two inner vertices, then (B1) or (B6) holds obviously. So assume that G contains at least three inner vertices. Among all the longest paths in the subgraph $G - V(C)$, we choose a path $Q = y_1y_2\cdots y_n$ such that $m^*(y_1y_2)$ is as large as possible. Then $n \geq 3$, and both y_1 and y_n are handles. Let $y_3, z_1, z_2, \dots, z_m$ denote the neighbors of y_2 in T in clockwise direction, where $2 \leq m \leq 4$, and $y_1 = z_l$ for some $1 \leq l \leq m$. Thus each z_i is either a handle or a leaf of T by the definition of Q. If there are two consecutive vertices in $\{y_3, z_1, z_2, \dots, z_m\}$ that are handles, then (B2) holds. Otherwise, suppose that no two consecutive vertices in $\{y_3, z_1, z_2, \dots, z_m\}$ are handles. If z_i is a 5-handle for some $1 \leq i \leq m$, then (B1) holds since both z_{i-1} and z_{i+1} are leaves.

Assume that some z_i is a 4-handle. Then z_{i-1} and z_{i+1} are leaves of T (if they exist). If either z_{i-2} or z_{i+2} is a handle, then (B3) holds. Otherwise, both z_{i-2} and z_{i+2} are leaves of T (if they exist). If $d_T(y_2) = 3$, then (B1) holds. If $d_T(y_2) = 5$, then (B4) holds. If $d_T(y_2) = 4$, that is, $m = 3$, we have two possibilities: when $i = 2$, (B6) holds; when $i = 1$ or $i = 3$, (B4) holds.

Now assume that each z_i, for $1 \leq i \leq m$, is either a 3-handle or a leaf. Note that y_1 is a 3-handle. If $d_T(y_2) = 3$, then (B1) holds. If $d_G(y_2) = 5$, then (B4) or (B5) holds. Hence assume that $d_G(y_2) = 4$, that is, $m = 3$. If both z_1 and z_3 are 3-handles, then (B3) holds. If exactly one of z_1 and z_3 is a 3-handle, then (B4) holds. This leaves to the only possibility that z_2 is a 3-handle and z_1, z_3 are leaves, where $y_1 = z_2$. Let $f_1 = [y_3 y_2 z_3 s_1 s_2 \cdots s_p]$ and $f_2 = [z_1 y_2 y_3 t_q t_{q-1} \cdots t_1]$ denote the incident faces of $y_2 y_3$ in G. It is easy to see that $s_1, t_1 \in V(C)$, $p, q \geq 1$, and all s_i's and t_j's, for $i, j \geq 2$, are inner vertices of G. This is because every inner face has exactly one common edge with outer face. If $\min\{p, q\} \leq 2$, then $d_G(y_3, s_1) \leq 2$, or $d_G(y_3, t_1) \leq 2$, and hence (B7) holds. Otherwise, assume that $p, q \geq 3$. If $p, q \geq 4$, then there is a path $Q_1 = s_2 s_3 \cdots s_p y_3 \cdots y_n$, or $Q_2 = t_2 t_3 \cdots t_q y_3 \cdots y_n$ with length more than the length of Q, a contradiction. Thus, we may assume, without loss of generality, that $p = 3$. It follows that $Q' = s_2 s_3 y_3 \cdots y_n$ is also a longest path in $G - V(C)$. However, $m^*(s_2 s_3) \geq 6 > 4 = m^*(y_1 y_2)$, contradicting the choice of Q. This completes the proof of the lemma. □

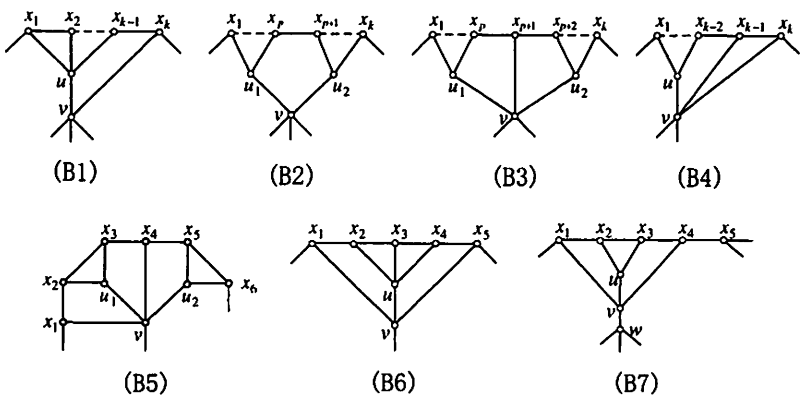


Fig. 1: Reducible configurations in Lemma 1.

3 Main result

Note that the graph H_0, depicted in Fig. 2, is a Halin graph with $\Delta = 4$ and $\chi(H_0^2) = 7 = |H_0|$. In what follows, a k-coloring of G^2 is called a *square-k-coloring* of G.

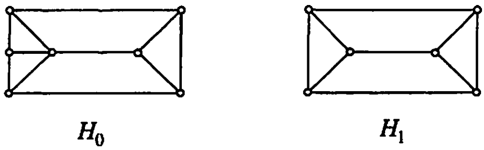


Fig. 2: Two Halin graphs H_0 and H_1.

Theorem 2 *If G is a Halin graph with $\Delta \leq 5$ and $G \ncong H_0$, then $\chi(G^2) \leq 6$.*

Proof. We prove the theorem by induction on the vertex number $|G|$. If $|G| \leq 6$, then the result is trivial, since we may assign different colors to the vertices of G. If $|G| = 7$, it is easy to confirm that G is isomorphic to H_0. Let $G = T \cup C$ be a Halin graph with $|G| \geq 8$. The fact that $\Delta \leq 5$ implies that $|V_{\text{in}}(G)| \geq 2$. Thus, G is not a wheel. If $|V_{\text{in}}(G)| \leq 3$, all Halin graphs (up to isomorphism) and their square-6-coloring are collected in Appendix I. So assume that $|V_{\text{in}}(G)| \geq 4$. By Lemma 1, there exists a path $P_k = x_1x_2\cdots x_k$ in C such that at least one of (B1) to (B7) holds. In each case, we first construct a Halin graph H from G such that $|H| < |G|$ and $\Delta(H) \leq 5$. If $H \ncong H_0$, then by the induction hypothesis, H has a square-6-coloring f. Afterwards, we extend f into a square-6-coloring of G. If $H \cong H_0$, then G will be a graph with a few vertices, and we can give it a square-6-coloring.

In the sequel, let $y \in N_C(x_1) \setminus \{x_2\}$, $z \in N_C(x_k) \setminus \{x_{k-1}\}$, $N_G(y) = \{x_1, y_1, y_2\}$, and $N_G(z) = \{x_k, z_1, z_2\}$. Moreover, let $S = \{1, 2, \ldots, 6\}$ denote a set of six colors. Set $Y = \{f(y_1), f(y_2)\}$ and $Z = \{f(z_1), f(z_2)\}$. We reduce seven configurations (B1)-(B7) as follows:

Case (B1) There exist a k-handle u and a vertex v with $N_G(u) = \{v, x_1, \ldots, x_{k-1}\}$ and $vx_k \in E(G)$ such that either $k = 5$, or $3 \leq k \leq 4$ and $d_G(v) = 3$.

Subcase (B1.1) $k = 5$.

Let $H = G - \{x_1, x_2, x_3, x_4\} + \{uy, ux_5\}$. Then $|H| \geq |G| - 4 \geq 4$, and $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 1 \geq 4 - 1 = 3$ since the inner vertex, u, of G becomes an outer vertex of H. Thus, $H \not\cong H_0$. By the induction hypothesis, H has a square-6-coloring f with the color set S. Without loss of generality, assume that $f(u) = 1$, $f(v) = 2$, $f(y) = 3$, and $f(x_5) = 4$. To extend f to the whole graph G, we consider two possibilities as follows: If $Y \neq \{5, 6\}$, we color x_1 with $a \in \{5, 6\} \setminus Y$, x_2 with 4, x_4 with $b \in \{3, 5, 6\} \setminus \{f(z), a\}$, and x_3 with a color in $\{3, 5, 6\} \setminus \{a, b\}$. If $Y = \{5, 6\}$, we color x_1 with 4, x_2 with 5, x_4 with $a \in \{3, 6\} \setminus \{f(z)\}$, and x_3 with a color in $\{3, 6\} \setminus \{a\}$.

Subcase (B1.2) $3 \leq k \leq 4$ and $d_G(v) = 3$.

Let w denote the neighbor of v in G other than u and x_k. It is easy to check that $w \neq y$ and $w \neq z$ since $|V_{\text{in}}(G)| \geq 4$. Let $H = G - \{x_1, x_2, \ldots, x_k, u\} + \{vy, vz\}$. If $H \not\cong H_0$, then the induction hypothesis asserts that H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$, and $f(z) = 4$. To extend f to the whole graph G, we consider two cases as follows:

- $k = 4$. If $Y \neq \{5, 6\}$, we color x_2 with 2, u with 4, x_1 with $a \in \{5, 6\} \setminus Y$, x_4 with $b \in \{3, 5, 6\} \setminus Z$, and x_3 with a color in $\{3, 5, 6\} \setminus \{a, b\}$. If $Y = \{5, 6\}$, we color x_1 with 2, u with 4, x_4 with $a \in \{3, 5, 6\} \setminus Z$, x_2 with $b \in \{5, 6\} \setminus \{a\}$, and x_3 with a color in $\{3, 5, 6\} \setminus \{a, b\}$.
- $k = 3$. If $2 \notin Y$, we color x_1 with 2, u with 4, x_3 with $a \in \{3, 5, 6\} \setminus Z$, and x_2 with $b \in \{5, 6\} \setminus \{a\}$. If $4 \notin Y$, we color x_1 with 4, x_2 with 2, x_3 with $a \in \{3, 5, 6\} \setminus Z$, and u with a color in $\{5, 6\} \setminus \{a\}$. If $Y = \{2, 4\}$, we color x_2 with 2, u with 4, x_3 with $a \in \{3, 5, 6\} \setminus Z$, and x_1 with $b \in \{5, 6\} \setminus \{a\}$.

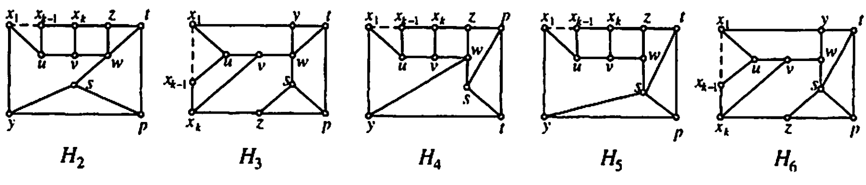


Fig. 3: $H \cong H_0$ in Subcase (B1.2).

Now assume that $H \cong H_0$. It is not difficult to inspect that G is one of the graphs H_2-H_6, as shown in Fig. 3. We construct a square-6-coloring f for each H_i for $i = 2, 3, \ldots, 6$.

Let $G = H_2$. If $k = 3$, we color $\{v,p\}$ with 1, $\{t,x_1\}$ with 2, $\{s,x_3\}$ with 3, $\{w,x_2\}$ with 4, $\{y,z\}$ with 5, and u with 6. If $k = 4$, we color $\{v,p\}$ with 1, $\{t,x_1\}$ with 2, $\{s,x_4\}$ with 3, $\{w,x_2\}$ with 4, $\{y,x_3\}$ with 5, and $\{u,z\}$ with 6.

Let $G = H_3$. If $k = 3$, we color $\{v,p\}$ with 1, $\{s,x_2\}$ with 2, $\{z,x_1\}$ with 3, $\{y,x_3\}$ with 4, $\{t,u\}$ with 5, and w with 6. If $k = 4$, we color $\{v,p\}$ with 1, $\{s,x_3\}$ with 2, $\{z,x_1\}$ with 3, $\{y,x_4\}$ with 4, $\{t,u\}$ with 5, and $\{w,x_2\}$ with 6.

Let $G = H_4$. If $k = 3$, we color $\{v,p\}$ with 1, $\{t,x_3\}$ with 2, $\{z,x_1\}$ with 3, $\{w,x_2\}$ with 4, $\{s,u\}$ with 5, and y with 6. If $k = 4$, we color $\{v,p\}$ with 1, $\{t,x_4\}$ with 2, $\{z,x_1\}$ with 3, $\{w,x_2\}$ with 4, $\{s,u\}$ with 5, and $\{y,x_3\}$ with 6.

Let $G = H_5$. If $k = 3$, we color $\{v,p\}$ with 1, $\{t,u\}$ with 2, $\{y,x_3\}$ with 3, $\{w,x_2\}$ with 4, $\{z,x_1\}$ with 5, and s with 6. If $k = 4$, we color $\{v,p\}$ with 1, $\{t,u\}$ with 2, $\{y,x_4\}$ with 3, $\{w,x_2\}$ with 4, $\{z,x_1\}$ with 5, and $\{s,x_3\}$ with 6.

Let $G = H_6$. If $k = 3$, we color $\{v,p\}$ with 1, $\{t,x_3\}$ with 2, $\{y,z\}$ with 3, $\{s,x_1\}$ with 4, $\{w,x_2\}$ with 5, and u with 6. If $k = 4$, we color $\{v,p\}$ with 1, $\{t,x_4\}$ with 2, $\{y,x_3\}$ with 3, $\{s,x_1\}$ with 4, $\{w,x_2\}$ with 5, and $\{u,z\}$ with 6.

Case (B2) $k \geq 4$ and there exist two handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_1, \ldots, x_p\}$ and $N_G(u_2) = \{v, x_{p+1}, \ldots, x_k\}$, where $2 \leq p \leq k-2$.

Subcase (B2.1) $k \geq 5$.

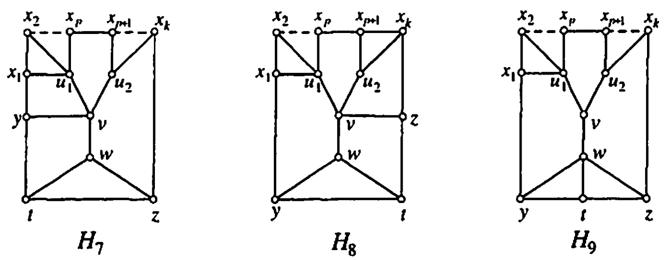


Fig. 4: $H \cong H_0$ in Subcase (B2.1).

By symmetry, we assume that $p \geq \lceil k/2 \rceil$. Note that $k \leq 8$ and u_1, u_2, v are inner vertices. Let $H = G - \{x_1, x_2, \ldots, x_k\} + \{yu_1, u_1u_2, u_2z\}$. If

$H \not\cong H_0$, then the induction hypothesis asserts that H has a square-6-coloring f using S such that $f(v) = 1$, $f(y) = 2$, $f(u_1) = 3$, $f(u_2) = 4$, and $f(z) \in \{2,5\}$. To extend f to the whole graph G, we first recolor u_1 with 4, u_2 with 3, then color x_1 with 3 and x_k with 4. If $k = 8$, we color $\{x_2, x_5\}$ with 5, $\{x_3, x_6\}$ with 2, and $\{x_4, x_7\}$ with 6. If $k = 7$, we color $\{x_2, x_5\}$ with 5, $\{x_3, x_6\}$ with 6, and x_4 with 2. If $k = 6$, we color $\{x_2, x_5\}$ with 6, x_3 with 2, and x_4 with 5. If $k = 5$, we color x_2 with 5, x_3 with 2, and x_4 with 6.

If $H \cong H_0$, then it is easy to check that G is H_7, H_8 or H_9, as shown in Fig. 4.

Let $G = H_7$. We first color $\{t, u_2\}$ with 1, w with 2, $\{z, u_1\}$ with 3, y with 4, and v with 5. If $k = 8$, we color $x_1, x_2, \ldots, x_8$ with $2,6,1,4,6,3,2,4$, respectively. If $k = 7$, we color $x_1, x_2, \ldots, x_7$ with $6,1,4,2,3,6,4$, respectively. If $k = 6$, then $3 \le p \le 4$, we color $x_1, x_2, \ldots, x_6$ with $6,1,4,2,6,4$, respectively. If $k = 5$, then $p = 3$, and we color $x_1, x_2, \ldots, x_5$ with $6,1,2,4,6$, respectively. If $k = 4$, we color x_1, x_2, x_3, x_4 with $6,2,4,6$, respectively.

Let $G = H_8$. We first color $\{y, u_1\}$ with 1, w with 2, $\{t, u_2\}$ with 3, z with 4, and v with 5. If $k = 8$, we color $x_1, x_2, \ldots, x_8$ with $2,6,3,2,6,4,2,1$, respectively. If $k = 7$, we color $x_1, x_2, \ldots, x_7$ with $2,6,3,2,4,6,2$, respectively. If $k = 6$, then $3 \le p \le 4$, we color $x_1, x_2, \ldots, x_6$ with $6,3,2,4,6,1$, respectively. If $k = 5$, then $p = 3$, and we color $x_1, x_2, \ldots, x_5$ with $6,3,2,6,1$, respectively. If $k = 4$, we color x_1, x_2, x_3, x_4 with $4,6,2,1$, respectively.

Let $G = H_9$. We first color $\{y, u_2\}$ with 1, w with 2, v with 3, $\{z, u_1\}$ with 4, $\{x_1, x_k\}$ with 5, and t with 6. If $k = 8$, we color $x_2, x_3, \ldots, x_7$ with $6,1,2,6,4,2$, respectively. If $k = 7$, we color $x_2, x_3, \ldots, x_6$ with $6,1,2,4,6$, respectively. If $k = 6$, we color x_2, x_3, x_4, x_5 with $6,1,2,6$, respectively. If $k = 5$, we color x_2, x_3, x_4 with $6,1,2$, respectively. If $k = 4$, we color x_2 with 6 and x_3 with 2.

Subcase (B2.2) $k = 4$, implying $p = 2$.

(B2.2.1) Assume that $d_G(v) = 3$. Let w denote the neighbor of v other than u_1 and u_2. Since $|V_{\text{in}}(G)| \ge 4$, we see that w is an inner vertex. Let $H = G - \{x_1, x_2, x_3, x_4, u_1, u_2\} + \{vy, vz\}$. If $H \not\cong H_0$, then by the induction hypothesis, H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$, and $f(z) = 4$. If $Y \ne \{5,6\}$, we color x_1, u_2 with $a \in \{5,6\} \setminus Y$, x_2 with 4, u_1 with $b \in \{5,6\} \setminus \{a\}$, x_4 with $c \in \{2,3,5,6\} \setminus (Z \cup \{a\})$, and x_3 with a color in $\{2,3\} \setminus \{c\}$. If $Y = \{5,6\}$, we color x_1 with 2, x_2 with 4, u_2 with 3, x_4 with $a \in \{2,5,6\} \setminus Z$, x_3 with $b \in \{5,6\} \setminus \{a\}$, and u_1 with a color in $\{5,6\} \setminus \{b\}$.

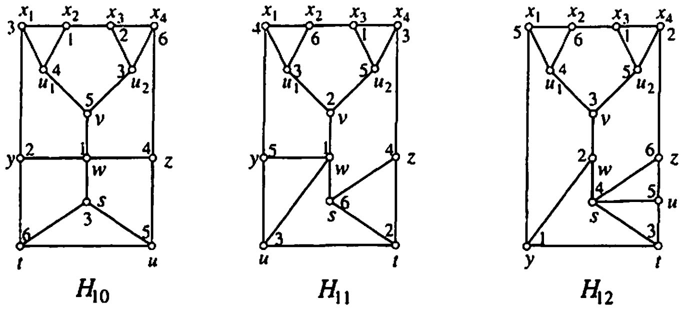


Fig. 5: $H \cong H_0$ in Subcase (B2.2.1).

If $H \cong H_0$, then G is one of H_{10}, H_{11}, H_{12}, and the corresponding square-6-colorings are given, as shown in Fig. 5. Note that the number lying aside a vertex represents its color.

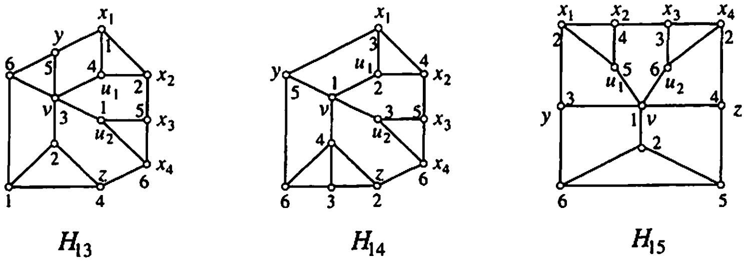


Fig. 6: $H \cong H_0$ in Subcase (B2.2.2).

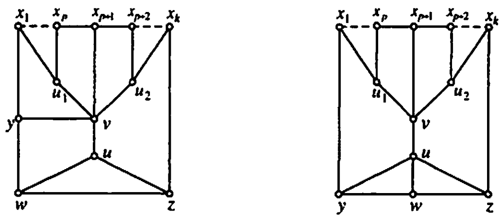
(B2.2.2) Suppose that $d_G(v) \geq 4$. Let $H = G - \{x_1, x_2, x_3, x_4, u_1\} + \{yu_2, zu_2\}$. If $H \not\cong H_0$, then H has a square-6-coloring f using S such that y, z, v, u_2 have distinct colors. We color u_1 with a color $a \in S$ that differs from the colors of vertices in $N_H(v) \cup \{v\}$. Moreover, if $a = f(y)$, then $a \neq f(z)$, we switch the colors of u_1 and u_2. Thus, we can assume that $f(u_1) = 6$, $f(u_2) = 1$, and $f(v) = 2$. Since $1 \notin Y$, we color x_1 with 1. If $f(z) \neq 6$, we color x_2 with $f(z)$, x_4 with $b \in \{3,4,5,6\} \setminus (Z \cup \{f(z)\})$, and x_3 with a color in $\{3,4,5\} \setminus \{b, f(z)\}$. If $f(z) = 6$, we color x_4 with $c \in \{3,4,5\} \setminus Z$, x_2 with $d \in \{3,4,5\} \setminus \{c, f(y)\}$, and x_3 with a color in $\{3,4,5\} \setminus \{c, d\}$.

If $H \cong H_0$, then G is one of H_{13}, H_{14}, H_{15} whose square-6-colorings are given in Fig. 6.

Case (B3) There exist two handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_1, \ldots, x_p\}$, $N_G(u_2) = \{v, x_{p+2}, \ldots, x_k\}$, $vx_{p+1} \in E(G)$, where $2 \leq p \leq$

$k-3$, such that either $k \geq 6$, or $k=5$ and $d_G(v)=4$.

By Case (B1), we may assume that $2 \leq \lceil (k-1)/2 \rceil \leq p \leq 3$. This implies that $5 \leq k \leq 7$. Let $H = G - \{x_1, x_2, \ldots, x_k\} + \{yu_1, u_1u_2, u_2z\}$. If $H \not\cong H_0$, then H has a square-6-coloring f using S such that $f(v)=1$, $f(u_1)=2$, $f(u_2)=3$, and $f(y)=4$. In the following, let $M_f(v)$ denote the set of colors assigned to the neighbors of v other than u_1 and u_2 in H. In G, we recolor u_1 with 3 and u_2 with 2, and color x_1 with 2, x_k with 3, and x_{p+1} with a color $\beta \in \{4,5,6\} \setminus M_f(v)$. Then we consider the following two cases by symmetry.



H_{16} H_{17}

Fig. 7: $H \cong H_0$ in Case (B3).

- $f(z)=4$. Let $\beta \in \{4,5\}$. If $k=7$ and $\beta=4$, we color $\{x_2, x_5\}$ with 5, and $\{x_3, x_6\}$ with 6. If $k=7$ and $\beta=5$, we switch the colors of u_1 and x_4, then we color $\{x_2, x_6\}$ with 6, x_3 with 4, and x_5 with 5. If $k=6$, we color $\{x_2, x_5\}$ with $a \in \{5,6\} \setminus \{\beta\}$ and x_3 with a color in $\{4,5,6\} \setminus \{a, \beta\}$. If $k=5$, then $d_G(v)=4$ by the definition of (B3). Let $w \in N_G(v) \setminus \{u_1, u_2, x_3\}$. If $\beta=4$, we color x_2 with 5 and x_4 with 6. Thus assume that $\beta=5$. If $f(w)=4$, we recolor u_1 with 6, and color x_2 with 3 and x_4 with 6. If $f(w) \neq 4$, we recolor x_3 with 4 and then reduce the proof to the previous case.
- $f(z)=5$. Let $\beta \in \{4,5,6\}$. If $k=7$ and $\beta=4$, we color $\{x_2, x_5\}$ with 5, and $\{x_3, x_6\}$ with 6. If $k=7$ and $\beta=5$, we color $\{x_2, x_6\}$ with 6, and $\{x_3, x_5\}$ with 4. If $k=7$ and $\beta=6$, we color $\{x_2, x_5\}$ with 5, and x_3, x_6 with 4. If $k=6$ and $\beta \neq 6$, we color $\{x_2, x_5\}$ with 6 and x_3 with a color in $\{4,5\} \setminus \{\beta\}$. If $k=6$ and $\beta=6$, we switch the colors of u_2 and x_4, and color x_2 with 6, x_3 with 5, and x_5 with 4. If $k=5$ and $\beta=6$, we color x_2 with 5 and x_4 with 4. If $k=5$ and $\beta=5$, we color x_2 with 6 and x_4 with 4. If $k=5$ and $\beta=4$, we color x_2 with 5 and x_4 with 6.

If $H \cong H_0$, then G is H_{16} or H_{17}, as shown in Fig. 7.

Let $G = H_{16}$. We first color $\{z, u_1\}$ with 1, $\{w, u_2\}$ with 2, u with 3, v with 4, y with 5, and x_{p+1} with 6. If $k = 5$, then $p = 2$ and we color x_1, x_2, x_4, x_5 with $3, 2, 3, 5$, respectively. Assume that $k = 6$. If $p = 3$, we color x_1, x_2, x_3, x_5, x_6 with $6, 3, 2, 1, 5$, respectively. If $p = 2$, we color x_1, x_2, x_4, x_5, x_6 with $3, 2, 1, 5, 6$, respectively. Assume that $k = 7$. If $p = 3$, then we color $x_1, x_2, x_3, x_5, x_6, x_7$ with $6, 3, 2, 1, 3, 6$, respectively. If $p = 2$, then we color $x_1, x_2, x_4, x_5, x_6, x_7$ with $3, 2, 1, 5, 3, 6$, respectively.

Let $G = H_{17}$. We first color u with 1, $\{y, u_2\}$ with 2, $\{z, u_1\}$ with 3, w with 4, v with 5, and x_{p+1} with 6. If $k = 5$, then $p = 3$, we recolor w with 6, and color color x_1, x_2, x_4, x_5 with $4, 2, 3, 4$, respectively. If $k = 6$, then $p = 3$, we recolor x_4 with 4 and y with 6, and then color x_1, x_2, x_3, x_5, x_6 with $2, 1, 6, 1, 6$, respectively. If $k = 7$, then $p = 3$, we color $x_1, x_2, x_3, x_5, x_6, x_7$ with $6, 4, 2, 3, 4, 6$, respectively.

Case (B4) $4 \le k \le 5$ and there exist a handle u and a vertex v with $N_G(u) = \{v, x_1, \ldots, x_{k-2}\}$ and $vx_{k-1}, vx_k \in E(G)$.

Since $|V_{\text{in}}(G)| \ge 4$, it is easy to see that $d_G(v) \ge 4$. We discuss two possibilities below.

Subcase (B4.1) $d_G(v) = 4$, and let $w \in N_G(v) \setminus \{u, x_{k-1}, x_k\}$.

(4.1.1) Assume that $k = 5$. Let $H = G - \{x_1, x_2, x_3, u\} + yx_4$. Then H is a Halin graph with $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 1 \ge 3$ and $|H| < |G|$. Thus, $H \ncong H_0$. Let f be a square-6-coloring f of H such that $f(x_4) = 1$, $f(v) = 2$, $f(y) = 3$, and $f(x_5) = 4$. In G, we color x_1 with 1, x_2 with 4, x_3 with 3, and u with a color in $\{5, 6\} \setminus \{f(w)\}$.

(4.1.2) Assume that $k = 4$. Let $H = G - \{x_1, x_2, x_3, x_4, u\} + \{yv, zv\}$. Then H is a Halin graph with $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 2 \ge 2$ and $|H| < |G|$. If $H \ncong H_0$, then H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$, and $f(z) = 4$. In G, we first color x_4 with $a \in \{3, 5, 6\} \setminus Z$. If $Y = \{5, 6\}$, then we color x_1 with 4, x_2 with 2, u with $b \in \{5, 6\} \setminus \{a\}$, and x_3 with a color in $\{3, 5, 6\} \setminus \{a, b\}$. Otherwise, $Y \neq \{5, 6\}$, we color x_1 with $c \in \{5, 6\} \setminus Y$, say $c = 5$, then color x_2 with 2, u with 4, x_3 with $\{3, 6\} \setminus \{a\}$.

If $H \cong H_0$, then G is one of H_{18}-H_{22} whose square-6-colorings are established in Fig. 8.

Subcase (B4.2) $d_G(v) = 5$.

Let $H = G - \{x_1, x_2, \ldots, x_k\} + \{yu, uz\}$. Then H is a Halin graph with $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 1 \ge 3$ and $|H| < |G|$. Thus, $H \ncong H_0$. By the induction hypothesis, H has a square-6-coloring f such that $f(v) = 1$,

$f(u) = 2$, $f(y) = 3$, and $f(z) = 4$. After erasing the color of u, we color $\{x_1, x_k\}$ with 2. Let $M_f(v)$ denote the subset of colors assigned to the neighbors of v in H other than u. Then $|M_f(v)| = 2$. If $3 \notin M_f(v)$, we color x_{k-1} with 3, u with a color $a \in \{4,5,6\} \setminus M_f(v)$, and properly color $x_2, \ldots, x_{k-2}$ with the colors in $\{4,5,6\} \setminus \{a\}$. So assume that $3 \in M_f(v)$. If $4 \notin M_f(v)$, we color u with 4, x_{k-1} with a color $a \in \{5,6\} \setminus M_f(v)$, and then properly color $x_2, \ldots, x_{k-2}$ with the colors in $\{3,5,6\} \setminus \{a\}$. If $4 \in M_f(v)$, then we color u with 5, x_2 with 4, x_{k-1} with 6, and moreover x_3 with 3 if $k = 5$.

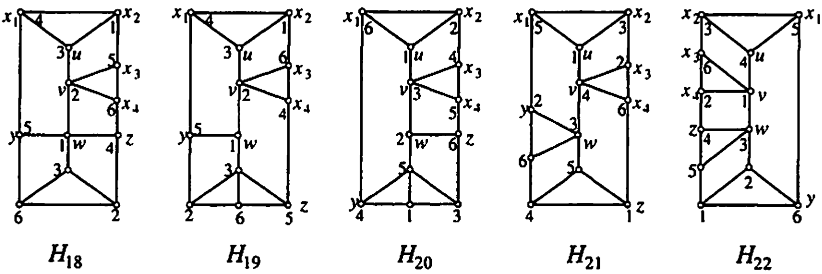


Fig. 8: $H \cong H_0$ in Subcase (B4.1.2).

Case (B5) $k = 6$ and there exist two 3-handles u_1, u_2 and a vertex v with $N_G(u_1) = \{v, x_2, x_3\}$, $N_G(u_2) = \{v, x_5, x_6\}$ and $vx_1, vx_4 \in E(G)$.

Since $|V_{\text{in}}(G_| \geq 4$, $d_G(v) = 5$. Let $w \in N_G(v) \setminus \{x_1, x_4, u_1, u_2\}$. Let $H = G - \{x_1, x_2, \ldots, x_6, u_1, u_2\} + \{yv, vz\}$. Then H is a Halin graph with $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 2 \geq 2$ and $|H| < |G|$. If $H \ncong H_0$, then H has a square-6-coloring f such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$, and $f(z) = 4$. In G, we first color $\{x_2, x_5\}$ with 2. If $Y \neq \{5,6\}$, then we further color x_1 with $a \in \{5,6\} \setminus Y$, x_6 with $b \in \{3,5,6\} \setminus Z$, $\{x_3, u_2\}$ with $c \in \{3,5,6\} \setminus \{a,b\}$, and u_1 with a color in $\{3,5,6\} \setminus \{a,c\}$, x_4 with 4. If $Y = \{5,6\}$, we color x_1 with 4, x_6 with $a \in \{3,5,6\} \setminus Z$, $\{x_3, u_2\}$ with $b \in \{3,5,6\} \setminus \{a\}$, x_4 with $c \in \{3,5,6\} \setminus \{a,b\}$, and u_1 with a color in $\{3,5,6\} \setminus \{b,c\}$.

If $H \cong H_0$, then G is one of H_{23}-H_{27} whose square-6-colorings are given in Fig. 9.

Case (B6) $k = 5$ and there exist a 4-handle u and a 4-vertex v with $N_G(u) = \{v, x_2, x_3, x_4\}$ and $vx_1, vx_5 \in E(G)$.

Let $w \in N_G(v) \setminus \{u, x_1, x_5\}$. Let $H = G - \{x_2, x_3, x_4, u\} + x_1x_5$. Then H is a Halin graph with $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 1 \geq 3$ and $|H| < |G|$. Thus, $H \ncong H_0$. Let f be a square-6-coloring of H with $f(v) = 1$, $f(x_1) = 2$,

$f(x_5) = 3$, and $f(w) = 4$. In G, we color x_2 with 3, x_4 with 2, x_3 with 4, and u with 5.

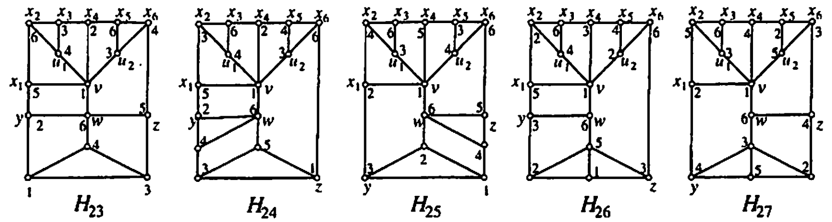


Fig. 9: $H \cong H_0$ in Case (B5).

Case (B7) $k = 5$ and there exist a 3-handle u and a 4-vertex v with $N_G(u) = \{v, x_2, x_3\}$ and $N_G(v) = \{u, w, x_1, x_4\}$ such that $d_G(w, x_5) \leq 2$.

Let $H = G - \{u, x_2, x_3\} + x_1x_4$. Since $|V_{\text{in}}(H)| = |V_{\text{in}}(G)| - 1 \geq 3$ and $|H| < |G|$, we see that $H \not\cong H_0$. Hence H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(x_1) = 3$, and $f(x_4) = 4$. Since $d_G(w, x_5) \leq 2$, we derive that $f(x_5) \neq 2$. In G, we color x_3 with 2, x_2 with $a \in \{5, 6\} \setminus \{f(y)\}$, and u with a color in $\{5, 6\} \setminus \{a\}$. □

Corollary 3 *If G is a Halin graph with $\Delta = 5$, then $\chi(G^2) = 6$.*

We see from Theorem 2 that a Halin graph G with $\Delta = 3$ has $4 \leq \chi(G^2) \leq 6$. If $|G| \geq 6$ and $\Delta = 3$, then $\chi(G^2) \geq 5$ as G contains a 5-cycle. This implies that $\chi(G^2) = 4$ if and only if $G \cong K_4$. Note that there exist 3-regular Halin graphs G such that $\chi(G^2) = 6$. Such an example is the graph H_1, depicted in Fig. 2. Theorem 2 also implies that a Halin graph G with $\Delta = 4$ has $5 \leq \chi(G^2) \leq 7$; and $\chi(G^2) = 7$ if and only if $G \cong H_0$. Therefore, it is interesting to characterize Halin graphs with $3 \leq \Delta \leq 4$ according to the chromatic number of their squares being 5 or 6.

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Appendix I

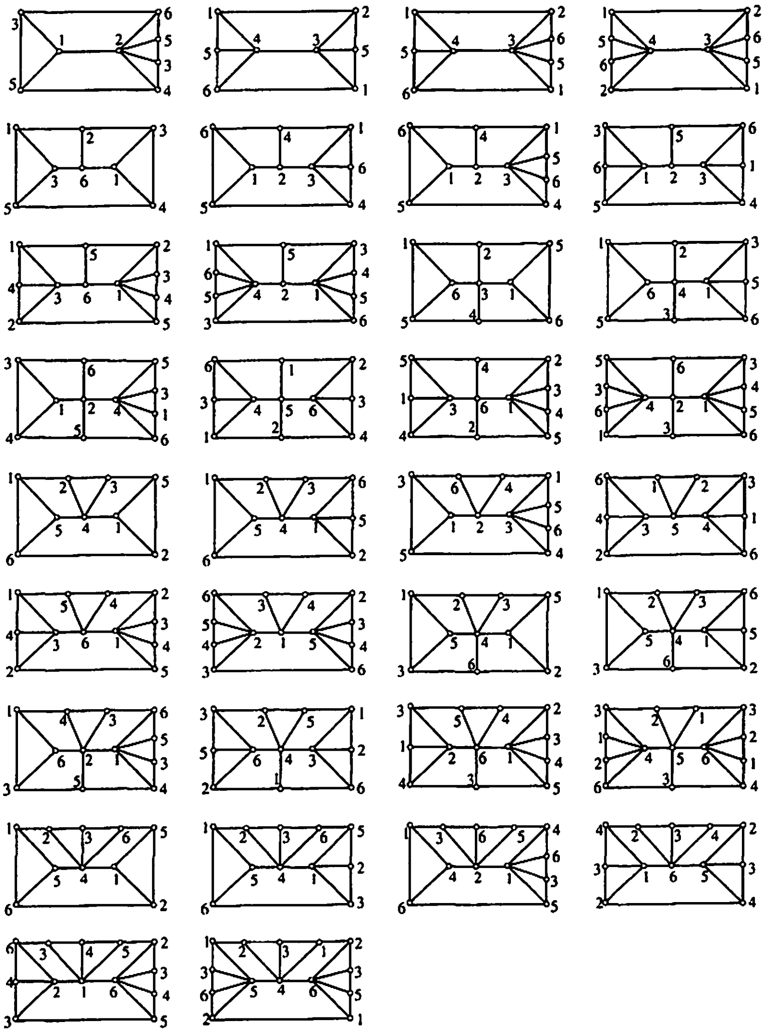

Fig. 10: All Halin graphs G with $|G| \geq 8$, $\Delta \leq 5$ and $2 \leq |V_{\text{in}}(G)| \leq 3$ and their square-6-colorings.