CONSTRUCTIONS OF H-ANTIMAGIC GRAPHS USING SMALLER EDGE-ANTIMAGIC GRAPHS

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Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs at least to one subgraph of G isomorphic to a given graph H. An (a, d)-H-*antimagic labeling* of G admitting an H-covering is a bijective function $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$, constitute an arithmetic progression with the initial term a and the common difference d. Such a labeling is called *super* if $f(V) = \{1, 2, \ldots, |V|\}$.

In this paper, we study the existence of super (a, d)-H-antimagic labelings for graph operation G^H, where G is a (super) (b, d^*)-edge-antimagic total graph and H is a connected graph of order at least 3.

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1 Introduction

We consider finite and simple graphs. Let the vertex and edge sets of a graph G be denoted by $V(G)$ and $E(G)$, respectively. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of E

belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. Then it is said that G admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*.

Gutiérrez and Lladó [11] defined an H-magic labeling as follows. The graph G admitting an H-covering is called H-*magic* if there exists a total labeling $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that, for each subgraph H' isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is constant. When $f(V) = \{1, 2, \ldots, |V|\}$, we say that G is H-*supermagic*. The H-(super)magic labelings are an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [14] and Enomoto, Lladó, Nakamigawa and Ringel [10], respectively. In [11], star-(super)magic and path-(super)magic labelings of some connected graphs were considered and proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [16] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati, Salman, Baskoro, Ryan and Miller [19] and also Salman, Ngurah and Izzati [21] proved that certain families of trees are path-supermagic. Ngurah, Salman and Susilowati [20] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [18] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

Simanjuntak, Miller and Bertault [22] introduced an (a, d)-*edge-antimagic total* ((a, d)-EAT) *labeling* of G which is defined as a bijective function $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that the set of edge-weights $\{f(u) + f(uv) + f(v) : uv \in E(G)\}$ is equal to the set $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$ for some positive integers a and d. An (a, d)-EAT labeling f is called super if the vertex labels are the smallest possible labels. Several results related to edge-antimagic total labelings are provided; see for example [8], [17] and [23].

Combining the two previous labelings, Inayah, Salman and Simanjuntak [12] introduced an (a, d)-H-*antimagic labeling* of a graph G admitting an H-covering as a bijective function $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

form an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d)-H-antimagic labeling is called *(super)* (a, d)-H-*antimagic*. In [13], super

(a,d)-H-antimagic labelings for some shackles of a connected graph H are investigated. In [7] was proved that wheels are cycle-antimagic. The existence of super $(a,1)$-tree-antimagic labelings for disconnected graphs are studied in [6].

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [15]. Further information on super d-antimagic labelings can be found in [2, 5].

Let G be an arbitrary graph and H be a connected graph of order at least 2. We define a graph operation G^H in the following way.

1. Denote the edges in G arbitrarily by $e_1, e_2, \dots, e_{|E(G)|}$;
2. Take $|E(G)|$ copies of H, say $H_1, H_2, \dots, H_{|E(G)|}$;
3. In every H_i, $i = 1,2,\dots,|E(G)|$, choose two different vertices, say x_i, y_i;
4. Replace every edge e_i in $E(G)$ by subgraph H_i in such a way that its end vertices and $x_i, y_i \in V(H_i)$ are identified.

The resulting graph G^H is of order $(|V(H)|-2)|E(G)| + |V(G)|$ and size $|E(H)||E(G)|$. Note that the graph G^H is not defined uniquely. It means for graphs G and H there are many non-isomorphic graphs obtained by using this construction.

In this paper we investigate the existence of super (a,d)-H-antimagic labelings for G^H. We show connection between H-antimagic labelings and edge-antimagic total labelings and describe a construction how to obtain the H-antimagic graph from a smaller edge-antimagic total graph G.

2 Partitions with determined differences

For construction H-antimagic labelings of graphs we will use the partitions of a set of integers with determined differences. This concept was introduced in [1].

Let n, k, d and i be positive integers. We will consider the partition $\mathcal{P}^n_{k,d}$ of the set $\{1,2,\dots,kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)$th k-tuple and the sum of the numbers in the ith k-tuple is always equal to the constant d, where $i = 1,2,\dots,n-1$. Thus these sums form an arithmetic sequence with the difference d. By the symbol $\mathcal{P}^n_{k,d}(i)$ we denote the ith k-tuple in the partition with the difference d, where $i = 1,2,\dots,n$.

Let $\sum \mathcal{P}^n_{k,d}(i)$ be the sum of the numbers in $\mathcal{P}^n_{k,d}(i)$. Evidently, from the definition, $\sum \mathcal{P}^n_{k,d}(i+1) - \sum \mathcal{P}^n_{k,d}(i) = d$. It is obvious that if there exists a partition of the set $\{1, 2, \ldots, kn\}$ with the difference d, there also exists a partition with the difference $-d$. By the notation $\mathcal{P}^n_{k,d}(i) \oplus c$ we mean that we add the constant c to every number in $\mathcal{P}^n_{k,d}(i)$.

If $k = 1$ then only the following partition of the set $\{1, 2, \ldots, n\}$ is possible

$$\mathcal{P}^n_{1,1}(i) = \{i\} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

If $k = 2$ then we have several partitions of the set $\{1, 2, \ldots, 2n\}$. Let us define the partitions into 2-tuples in the following way:

$$\begin{aligned} \mathcal{P}^n_{2,0}(i) &= \{i, 2n + 1 - i\}, & \\ \sum \mathcal{P}^n_{2,0}(i) &= 2n + 1, & \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

$$\begin{aligned} \mathcal{P}^n_{2,2}(i) &= \{i, n + i\}, & \\ \sum \mathcal{P}^n_{2,2}(i) &= n + 2i, & \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

$$\begin{aligned} \mathcal{P}^n_{2,4}(i) &= \{2i - 1, 2i\}, & \\ \sum \mathcal{P}^n_{2,4}(i) &= 4i - 1, & \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Moreover, for $3 \le n \equiv 1 \pmod 2$

$$\mathcal{P}^n_{2,1}(i) = \begin{cases} \{\frac{n+1}{2} + \frac{i-1}{2}, n + 1 + \frac{i-1}{2}\} & \text{for} \quad i \equiv 1 \pmod 2, \\ \{\frac{i}{2}, n + \frac{n+1}{2} + \frac{i}{2}\} & \text{for} \quad i \equiv 0 \pmod 2, \end{cases}$$

$$\sum \mathcal{P}^n_{2,1}(i) = n + \tfrac{n+1}{2} + i, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Note that we are able to obtain the partitions into 2-tuples $\mathcal{P}^n_{2,0}(i)$ and $\mathcal{P}^n_{2,2}(i)$ as $\mathcal{P}^n_{1,s}(i) \cup (\mathcal{P}^n_{1,t}(i) \oplus n)$, where $s, t = \pm 1$. We can use this idea to construct the other partitions. More precisely,

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $k = l + m$.

For example, we are able to obtain $\mathcal{P}^n_{3,d}(i)$ from the partitions $\mathcal{P}^n_{1,s}(i)$, $s = \pm 1$ and $\mathcal{P}^n_{2,t}(i)$, $t = 0, \pm 2, \pm 4$ and also $t = \pm 1$ for n odd. It means, $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$. Moreover, we are able to construct $\mathcal{P}^n_{3,9}$ in the following way

$$\begin{aligned} \mathcal{P}^n_{3,9}(i) &= \{3(i-1) + 1, 3(i-1) + 2, 3(i-1) + 3\}, \\ \sum \mathcal{P}^n_{3,9}(i) &= 9i - 3, \quad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Thus $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5, \pm 9$. Note that if $n \equiv 1 \pmod 2$ then also the differences $d = 0, \pm 2$ are realizable.

Summarizing the previous fact we get the following theorem.

k	d for every n	moreover for n odd
1	± 1	
2	$0, \pm 2, \pm 4$	± 1
3	$\pm 1, \pm 3, \pm 5, \pm 9$	$0, \pm 2$
4	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$	$\pm 1, \pm 3, \pm 5$
5	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 25$	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10$
6	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16, \pm 18, \pm 20, \pm 24, \pm 26, \pm 36$	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 15, \pm 17$
7	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 19, \pm 21, \pm 23, \pm 25, \pm 27, \pm 29, \pm 35, \pm 37, \pm 49$	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16, \pm 18, \pm 24, \pm 26$

Table 1: The feasible differences d for partition $\mathcal{P}^n_{k,d}$, $k \leq 7$.

Theorem 1. *Let n, k, d and i be positive integers. There exists a partition $\mathcal{P}^n_{k,d}$ of the set $\{1, 2, \ldots, kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)$th k-tuple and the sum of the numbers in the ith k-tuple is d, $i = 1, 2, \ldots, n-1$ for $d = k^2$ or $d = s + t$, where s and t are realizable differences in partitions $\mathcal{P}^n_{l,s}$ and $\mathcal{P}^n_{m,t}$, $k = l + m$.*

Moreover, the corresponding ith k-tuple in the partition $\mathcal{P}^n_{k,d}$ can be obtained such that

$$\mathcal{P}^n_{k,k^2}(i) = \{k(i-1)+1, k(i-1)+2, \ldots, k(i-1)+k\}$$

or

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $k = l + m$, respectively.

Let us note that each of the defined partition $\mathcal{P}^n_{k,d}$ has the property that

$$\sum \mathcal{P}^n_{k,d}(i) = C^n_{k,d} + di,$$

where $C^n_{k,d}$ is a constant depending on the parameters k and d. Table 1 gives the values of feasible differences for partition $\mathcal{P}^n_{k,d}$ for $k \leq 7$.

It indicates that, for a given k, the number of feasible values of d is quite big. However, for $k \geq 6$ not every number from the set $\pm((k-1)^2+1), \pm((k-1)^2-1), \dots, \pm 1$ for k odd (or $\pm((k-1)^2+1), \pm((k-1)^2-1), \dots, 0$ for k even) can be realizable as a difference d in the partition $\mathcal{P}^n_{k,d}$. However, it is a simple observation that for $k \geq 6$ all numbers from the set

$$\begin{aligned} &\pm 1, \pm 3, \dots, \pm(k+14) && \text{for } k \text{ odd} \\ &0, \pm 2, \dots, \pm(k+14) && \text{for } k \text{ even} \end{aligned} \tag{1}$$

are feasible as a difference d in the partition $\mathcal{P}^n_{k,d}$.

3 Counting the upper bound of the difference d

The next theorem gives the upper bound of the difference d if the graph G^H is super (a,d)-H-antimagic.

Theorem 2. *Let G be a (p_G, q_G)-graph and let H be a connected (p_H, q_H)-graph. If G^H admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in G^H is q_G then*

$$d \leq p_H^2 + q_H^2 - 2p_H + \frac{p_H(p_G-2)}{q_G-1}.$$

Proof. Let G be an arbitrary (p_G, q_G)-graph and let H be a connected (p_H, q_H)-graph. Let G^H contains exactly q_G subgraphs isomorphic to H. Let G^H admits a super (a,d)-H-antimagic labeling f,

$$f : V(G^H) \cup E(G^H) \to \{1, 2, \dots, p+q\},$$

where $p = |V(G^H)| = (p_H - 2)q_G + p_G$ and $q = |E(G^H)| = q_H q_G$.

The smallest possible weight of a subgraph isomorphic to H can be obtained when the smallest possible numbers are used to label its elements. It means, when the numbers $1, 2, \dots, p_H$ are used as the vertex labels and the numbers $p+1, p+2, \dots, p+q_H$ are used as the edge labels. Thus

$$\begin{aligned} a \geq& 1 + 2 + \dots + p_H + (p+1) + (p+2) + \dots + (p+q_H) \\ =& \frac{(p_H+1)p_H}{2} + pq_H + \frac{(q_H+1)q_H}{2}. \end{aligned} \tag{2}$$

The largest possible weight of a subgraph isomorphic to H can be realizable if the largest possible numbers are used to label vertices as well the edges of this subgraph. Thus

$$\begin{aligned} a + (q_G - 1)d \leq& p + (p-1) + \dots + (p - p_H + 1) \\ &+ (p+q) + (p+q-1) + \dots + (p+q-q_H+1) \end{aligned}$$

$$= \frac{(2p - p_H + 1)p_H}{2} + \frac{(2p + 2q - q_H + 1)q_H}{2}. \quad (3)$$

Combining Inequalities (2) and (3) and after some mathematical manipulations we get the upper bound for the difference d in the following form.

$$d \leq p_H^2 + q_H^2 - 2p_H + \frac{p_H(p_G - 2)}{q_G - 1}.$$

□

Remark 1. *Note that if H contains no articulation then trivially G^H contains exactly q_G subgraphs isomorphic to H.*

If G is a tree, i.e., $p_G = q_G + 1$, then from Theorem 2 it follows that $d \leq p_H^2 + q_H^2 - p_H$.

Corollary 1. *Let G be a tree of order p_G and let H be a connected (p_H, q_H)-graph. If G^H admits a super (a,d)-H-antimagic labeling then*

$$d \leq p_H^2 + q_H^2 - p_H.$$

Carlson [9] defines an amalgamation of graphs as follows. Let $G_1, G_2, \ldots, G_k$ be a finite collection of graphs and let each G_i have a fixed vertex v_i called the *terminal*. The *amalgamation* Amal$\{G_i, v_i\}$ is formed by taking all the G_i's and identifying their terminals. By $amal(H,k)$ we denote a graph, where the amalgamation is constructed from k copies of connected graph H.

If the graph G is isomorphic to a star $K_{1,n}$, $n \geq 2$, then the graph $K_{1,n}^H$ is isomorphic to the amalgamation $amal(H,n)$. Using Corollary 1 we immediately obtain the following result.

Corollary 2. *Let H be a connected (p_H, q_H)-graph. If the amalgamation $amal(H,n)$ admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in $amal(H,n)$ is n then*

$$d \leq p_H^2 + q_H^2 - p_H.$$

A *shackle* of $G_1, G_2, \ldots, G_k$, denoted by $shack(G_1, G_2, \ldots, G_k)$, is a graph constructed from non-trivial connected graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \leq i,j \leq k$ with $|i-j| \geq 2$, G_i and G_j have no common vertex, and for every $1 \leq i \leq k-1$, G_i and G_{i+1} share exactly one common vertex, called a *linkage vertex*, where the $k-1$ linkage vertices are all distinct. In the case when all G_i's are isomorphic to a connected graph H, we call the resulting graph as a shackle of H denoted by $shack(H,k)$.

If the graph G is isomorphic to a path P_n, $n \geq 2$, then the graph P_n^H is isomorphic to the shackle $shack(H, n-1)$ and by Corollary 1 the upper bound for the difference d is as follows.

Corollary 3. *Let H be a connected (p_H, q_H)-graph. If $shack(H, n-1)$ admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in $shack(H, n-1)$ is $n-1$ then*

$$d \le p_H^2 + q_H^2 - p_H.$$

Note that this upper bound was proved by Lemma 6 in [13].

On the other hand if the graph H is isomorphic to K_2 then from Theorem 2 it follows.

Corollary 4. *If H is isomorphic to K_2 and G^H admits a super (a,d)-EAT labeling then*

$$d \le 1 + \frac{2p_G - 4}{q_G - 1}.$$

This upper bound for the difference d was proved in [3].

4 Main result

In this section we show connection between H-antimagic labelings and edge-antimagic total labelings. We describe a construction how to obtain the H-antimagic graph from a smaller edge-antimagic total graph G. Note that if $H \cong K_2$ then $G^H \cong G$ and the result trivially holds.

The following theorem gives the main result.

Theorem 3. *Let G be a (b,d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super (a,d)-H-antimagic and $d = d^* + d_v + d_e$, where d_v and d_e are feasible values of differences in the partitions $\mathcal{P}^{q_G}_{p_H-3,d_v}$ and $\mathcal{P}^{q_G}_{q_H,d_e}$, respectively.*

Proof. Let g be a (b,d^*)-EAT labeling of G. The set of all edge-weights of the edges of G under the labeling g is

$$\{wt_g(e) : e \in E(G)\} = \{b, b+d^*, \dots, b+(q_G-1)d^*\}.$$

Denote the edges of G by the symbols $e_1, e_2, \dots, e_{q_G}$ such that

$$wt_g(e_i) = b + (i-1)d^*,$$

where $i = 1, 2, \dots, q_G$.

Let H be a connected (p_H, q_H)-graph, $p_H \ge 3$.

Let G^H contains exactly q_G subgraphs isomorphic to H, say $H_1, H_2, \dots, H_{q_G}$, where the subgraph H_i replaces the edge e_i in G, $i = 1, 2, \dots, q_G$.

Construct a total labeling f, $f : V(G^H) \cup E(G^H) \to \{1, 2, \dots, q_G(p_H + q_H - 2) + p_G\}$ in the following way:

- $f(v) = g(v)$, if there exist integers t, s, $1 \le t < s \le q_G$ such that $v \in V(H_t) \cap V(H_s)$.
- As $p_H \ge 3$ then there exists a vertex x, $x \in V(H_i)$ and $x \ne v$. Then for $i = 1, 2, \dots, q_G$ let
$$f(x) = g(e_i).$$
- For $i = 1, 2, \dots, q_G$ let
$$\{f(y) : y \in V(H_i), y \ne v \text{ and } y \ne x\} = \mathcal{P}^{q_G}_{p_H-3,d_v}(i) \oplus (p_G + q_G).$$
- For $i = 1, 2, \dots, q_G$ let
$$\{f(e) : e \in E(H_i)\} = \mathcal{P}^{q_G}_{q_H,d_e}(i) \oplus \big((p_H - 2)q_G + p_G\big),$$
where d_v depends on p_H and d_e depends on q_H.

It is not difficult to check that the vertices are labeled with the smallest possible numbers $1, 2, \dots, (p_H - 2)q_G + p_G$.

Moreover, for the weight of the subgraph H_i, $i = 1, 2, \dots, q_G$, we obtain

$$\begin{aligned}
wt_f(H_i) &= \sum_{u \in V(H_i)} f(u) + \sum_{e \in E(H_i)} f(e) \\
&= \sum_{\substack{v \sim e_i \\ e_i \in E(G)}} f(v) + f(x) + \sum_{u \in V(H_i) \setminus \{v,x\}} f(u) + \sum_{e \in E(H_i)} f(e) \\
&= \sum_{\substack{v \sim e_i \\ e_i \in E(G)}} g(v) + g(e_i) + \sum \Big(\mathcal{P}^{q_G}_{p_H-3,d_v}(i) \oplus (p_G + q_G)\Big) \\
&\quad + \sum \Big(\mathcal{P}^{q_G}_{q_H,d_e}(i) \oplus ((p_H - 2)q_G + p_G)\Big) \\
&= \Big(b + (i-1)d^*\Big) + \Big(C^{q_G}_{p_H-3,d_v} + d_v i + (p_H - 3)(p_G + q_G)\Big) \\
&\quad + \Big(C^{q_G}_{q_H,d_e} + d_e i + q_H\big((p_H - 2)q_G + p_G\big)\Big) \\
&= C^{q_G}_{p_H-3,d_v} + C^{q_G}_{q_H,d_e} + b - d^* + (p_H - 3)(p_G + q_G) \\
&\quad + q_H\big((p_H - 2)q_G + p_G\big) + (d^* + d_v + d_e)i.
\end{aligned}$$

This concludes the proof.

The largest feasible value of the difference d for a super (a, d)-H-antimagic labeling of G^H is given by the following Corollary.

Corollary 5. *Let G be a (super) (b, d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super $(a, d^* + (p_H - 3)^2 + q_H^2)$-$H$-antimagic graph.*

Proof. From Theorem 1 it follows that the largest possible value of the difference in the partition $\mathcal{P}^{q_G}_{p_H-3,d_v}$ is $(p_H - 3)^2$ and the largest possible value of the difference in the partition $\mathcal{P}^{q_G}_{q_H,d_e}$ is q_H^2. According to Theorem 3 the result follows.

Next corollary gives the formula for another feasible differences of d as a function of p_H and q_H.

Corollary 6. *Let G be a (super) (b, d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super (a, d)-H-antimagic, where*

$$d = d^* + (p_H - 3 - t)^2 + (q_H - s)^2 \pm t \pm s$$

for every $t = 0, 1, \ldots, p_H - 3$ and $s = 0, 1, \ldots, q_H$.

5 Special families of graphs

In this section we consider two special families of graphs, namely amalgamation of graphs and shackle of graphs.

If the graph $G \cong K_{1,n}$, $n \geq 2$, then the graph $K^H_{1,n}$ is known as amalgamation of H. According to Corollary 2, if $K^H_{1,n}$ admits a super (a, d)-H-antimagic labeling and number of subgraphs isomorphic to H in $K^H_{1,n}$ is n then $d \leq p_H^2 + q_H^2 - p_H$.

In [23] is proved the following result.

Theorem 4 ([23]). *The star $K_{1,n}$, $n \geq 2$, admits a super (a, d)-EAT labeling for $d = 0, 1, 2$.*

Theorem 5. *Let H be a connected (p_H, q_H)-graph, $p_H \geq 9$ and let n be an integer, $n \geq 2$. If $K^H_{1,n}$ contains exactly n subgraphs isomorphic to H then $K^H_{1,n}$ admits a super (a, d)-H-antimagic labeling for*

$$0 \leq d \leq p_H + q_H + 27.$$

Proof. It follows from Theorem 3, Theorem 4 and Expression (1) for partition of numbers.

Note that Theorem 3 gives much more feasible values of the difference d for super (a, d)-H-antimagic labeling of $K^H_{1,n}$. Furthermore there exist several feasible differences d which is not possible to obtain from the proof of Theorem 3. For these values of difference d we propose the following.

Open Problem 1. *Determine for which values of differences d, $0 \le d \le p_H^2 + q_H^2 - p_H$, not covered by Theorem 3, there exists a super (a,d)-H-antimagic labeling of $K_{1,n}^H$.*

As we mentioned before, if the graph $G \cong P_n$, $n \ge 2$, then the graph P_n^H is known as shackle of H. According to Corollary 3, if P_n^H admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in P_n^H is $n-1$ then $d \le p_H^2 + q_H^2 - p_H$.

For edge-antimagicness of paths in [4] is proved the following.

Theorem 6 ([4]). *The path P_n, $n \ge 2$, admits a super (a,d)-EAT labeling if and only if $d = 0,1,2,3$.*

Then we get.

Theorem 7. *Let H be a connected (p_H, q_H)-graph, $p_H \ge 9$ and let n be an integer, $n \ge 3$. If P_n^H contains exactly $n-1$ subgraphs isomorphic to H then P_n^H admits a super (a,d)-H-antimagic labeling for*

$$0 \le d \le p_H + q_H + 28.$$

Proof. Using Theorem 3, Theorem 6 and Expression (1) for partition of numbers we immediately obtain that $0 \le d \le p_H + q_H + 28$. ❑

By the same way as for amalgamation we can formulate analogous open problem for shackle of H.

Open Problem 2. *Determine for which values of differences d, $0 \le d \le p_H^2 + q_H^2 - p_H$, not covered by Theorem 3, there exists a super (a,d)-H-antimagic labeling of P_n^H.*

Inayah, Simanjuntak, Salman and Syuhada [13] studied the existence of H-antimagic labeling of shackle of H by using a different method. Their different approach gives different sets of differences obtained by desired constructions.

6 Conclusion

In this paper, we examined the existence of super (a,d)-H-antimagic labelings for graph operation G^H, where G is a (b,d^*)-edge-antimagic total graph and H is a connected graph of order at least 3. We have found super (a,d)-H-antimagic labelings for all differences $d = d^* + d_v + d_e$, where d^* is the feasible value of difference in super edge-antimagic graph G and d_v (respectively, d_e) are feasible values of differences in the partitions $\mathcal{P}_{p_H-3,d_v}^{q_G}$ (respectively, $\mathcal{P}_{q_H,d_e}^{q_G}$). Additionally, we showed that for a connected (p_H,q_H)-graph H the graph $K_{1,n}^H$ (respectively, P_n^H) admits a super (a,d)-H-antimagic labeling for every difference $0 \le d \le p_H + q_H + 27$ (respectively, $0 \le d \le p_H + q_H + 28$).

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