The characterization of graph by positive inertia index *

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Abstract Let G be a graph of order n, the number of positive eigenvalues of G is called the positive inertia index of G and denoted by $p(G)$. The minimum number of complete multipartite subgraphs in any complete multipartite graph edge decomposition of graph G, in which the edge-induced subgraph of each edge subset of the decomposition is a complete multipartite graph, is denoted by $\varepsilon(G)$. In this paper, we prove $\varepsilon(G) \geq p(G)$ for any graph G. Especially, if $\varepsilon(G) = 2$, then $p(G) = 2$. We also characterize the graph G with $p(G) = n - 2$.

Keywords Positive inertia index; Edge decomposition ; Matching number; Covering number

1 Introduction

Throughout the paper, graphs are simple, i.e., without loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G) =$

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$\{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is defined as follows: $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ of A are said to be the eigenvalues of the graph G and to form the spectrum of this graph. The numbers of positive, negative and zero eigenvalues in the spectrum of the graph G are called positive inertia index, negative inertia index and nullity of the graph G, and are denoted by $p(G)$, $n(G)$ and $\eta(G)$, respectively. Obviously $p(G) + n(G) + \eta(G) = n$. There are many studies on nullity of graph (see[1,4-6,8-10,15]). However, the studies on positive inertia index of graph are very few.

Let G be a graph, and $G_1, G_2, \cdots, G_k$ be complete multipartite (bipartite) subgraphs of G, such that

$$E(G_1)\cup E(G_2)\cup\cdots\cup E(G_k) = E(G), \quad E(G_i)\cap E(G_j) = \emptyset \ (i,j = 1,2,\cdots,k, i \neq j). \tag{1}$$

We call $\{G_1, G_2, \cdots, G_k\}$ is a complete multipartite (bipartite) graph edge decomposition of graph G. The minimum number of complete multipartite (bipartite) subgraphs in a complete multipartite (bipartite) graph edge decomposition of graph G is called the complete multipartite (bipartite) graph edge decomposition number of graph G and denoted by $\varepsilon(G)$ ($bd(G)$, respectively). The number $bd(G)$ is also called the biclique decomposition number of graph G.

The well-known theorem of Graham and Pollak [7] (see also [3, 12-14,17]) asserts that biclique decomposition number is at least the maximum of $p(G)$ and

$n(G)$, that is, $bd(G) \geq max\{p(G), n(G)\}$. In this paper, we will generalize this theorem, show that $\varepsilon(G) \geq p(G)$ for any graph G. Especially, if $\varepsilon(G) = 2$, then $p(G) = 2$. We also characterize the graphs G with $p(G) = n - 2$, where $n = |V(G)|$.

Let G be a graph, $U \subseteq V(G)$, the vertex-induced subgraph $G[U]$ is the subgraph of G whose vertex set is U and whose edge set consists of all edges of G which have both ends in U. $G \setminus U$ denotes the graph $G[V(G) \setminus U]$. $F \subseteq E(G)$, the edge-induced subgraph $G[F]$ is the subgraph of G whose edge set is F and whose vertex set consists of all ends of edges of F. Let G and H be two vertex disjoint graphs, $G \cup H$ denotes the union graph of G and H. A matching of G is a collection of independent edges of G. The maximum number of edges in a matching of G is called the matching number of G and denoted by $\mu(G)$. A covering of G is a set of vertices which together meet all edges of the graph. The minimum number of vertices in a covering of G is called the covering number of G and denoted by $\beta(G)$. $u \in V(G)$, $d_G(u)$ denotes the degree of vertex u in G. K_n, C_n and $K_{n_1,n_2,\cdots,n_t}$ denote the complete graph, the cycle and the complete multipartite graph, respectively.

2 Some Lemmas

The following Lemma 2.1 is clear.

Lemma 2.1. $p(G \cup H) = p(G) + p(H)$; $\varepsilon(G \cup H) = \varepsilon(G) + \varepsilon(H)$.

Lemma 2.2([16]). A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph. On the other words, $p(G) = 1$ if and only if $\varepsilon(G) = 1$.

Lemma 2.3([2,11]). (the Courant-Weyl inequalities) Let $\lambda_1(X), \lambda_2(X), \cdots, \lambda_n(X) (\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X))$ be the eigenvalues of a real symmetric matrix X. If A and B are real symmetric matrices of order n, and $C = A + B$. Then

$$\lambda_{i+j+1}(C) \leq \lambda_{i+1}(A) + \lambda_{j+1}(B),$$

$$\lambda_{n-i-j}(C) \geq \lambda_{n-i}(A) + \lambda_{n-j}(B),$$

where $0 \leq i, j, i+j+1 \leq n$.

Lemma 2.4([2,11]). (the Cauchy inequalities) Let A be Hermtian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, B be one of its principal submatrices and B have eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i (i = 1, \cdots, m)$ hold.

Lemma 2.5([2]). Let G be a tree. Then $p(G) = \mu(G)$.

3 The relation between $p(G)$ and $\varepsilon(G)$

Theorem 3.1. Let G be a graph. Then $\varepsilon(G) \geq p(G)$.

Proof. We may ignore isolated vertices. Let $\varepsilon(G) = d$, we use mathematical induction on d. If $d = 1$, then the conclusion holds according to Lemma 2.2. Suppose that there exists a partition

$$\varepsilon_1 \cup \varepsilon_2 \cup \cdots \cup \varepsilon_d = E(G), \quad \varepsilon_i \cap \varepsilon_j = \emptyset \quad (i, j = 1, 2, \cdots, d, i \neq j) \tag{1}$$

of the set $E(G)$ of edges of G and such that the subgraph G_i of G, induced by ε_i is a complete multipartite graph for each $i = 1, 2, \cdots, d$. Denote $G[\varepsilon_2 \cup \cdots \cup \varepsilon_d] = H$. Let $X = V(G_1) \cap V(H), Y = V(G_1) \setminus X, Z = V(H) \setminus X$, so the adjacency matrix of graph G is

$$A(G) = \begin{bmatrix} A & B^T & 0 \\ B & C_1 + C_2 & D^T \\ 0 & D & E \end{bmatrix},$$

where $\begin{bmatrix} A & B^T \\ B & C_1 \end{bmatrix} = A(G_1)$ and $\begin{bmatrix} C_2 & D^T \\ D & E \end{bmatrix} = A(H)$.

As G_1 is a complete multipartite graph, its vertex set naturally has a decomposition, so the vertices of Y also has a decomposition induced by vertex decomposition of G_1. Take one vertex from each part of the decomposition of Y, denote the set of these vertices by Y_1. We can obtain a new vertex-induced subgraph $K = G[Y_1 \cup X \cup Z]$, and the adjacency matrices of K is

$$A(K) = \begin{bmatrix} A_1 & B_1^T & 0 \\ B_1 & C_1 + C_2 & D^T \\ 0 & D & E \end{bmatrix}.$$

Since the vertices of $Y \setminus Y_1$ have the same adjacent relations with some vertex of Y_1, we can obtain $p(G) = p(K)$.

Suppose $x, y \notin V(K)$, we construct a new graph M as follows: $V(M) = V(K) \cup \{x, y\}, E(M) = E(K) \cup \{(x, y)\} \cup \{(x, v_i) | v_i \in Y_1 \cup X\} \cup \{(y, v_i) | v_i \in Y_1 \cup X\}$. Then the adjacency matrix of M is

$$A(M) = \begin{bmatrix} 0 & 1 & j_1^T & j_2^T & 0 \\ 1 & 0 & j_1^T & j_2^T & 0 \\ j_1 & j_1 & A_1 & B_1^T & 0 \\ j_2 & j_2 & B_1 & C_1 + C_2 & D^T \\ 0 & 0 & 0 & D & E \end{bmatrix},$$

where j_1 and j_2 are vectors with dimension $|Y_1|$ and $|X|$, respectively, and the components of j_1 and j_2 are equal to 1. For convenient, denote

$$A_2 = \begin{bmatrix} 0 & 1 & j_1^T \\ 1 & 0 & j_1^T \\ j_1 & j_1 & A_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} j_2 & j_2 & B_1 \end{bmatrix}.$$

Then

$$A(M) = \begin{bmatrix} A_2 & B_2^T & 0 \\ B_2 & C_1 + C_2 & D^T \\ 0 & D & E \end{bmatrix},$$

where $\begin{bmatrix} A_2 & B_2^T \\ B_2 & C_1 \end{bmatrix}$ is a adjacency matrix of a complete multipartite graph. By the matrix $\begin{bmatrix} A_2 & B_2^T \\ B_2 & C_1 \end{bmatrix}$ has only one positive eigenvalue; A_2 is adjacency matrix of $M[Y_1 \cup \{x, y\}]$ (here $M[Y_1 \cup \{x, y\}]$ is a complete graph whose order is not less than 2), whose is invertible and has only one positive eigenvalue; and $A(M)$ is congruent to

$$\begin{bmatrix} A_2 & 0 & 0 \\ 0 & C_1 - B_2 A_2^{-1} B_2^T + C_2 & D^T \\ 0 & D & E \end{bmatrix},$$

we know that $C_1 - B_2 A_2^{-1} B_2^T$ is a negative semi-definite matrix, whose eigenvalues are less than zero. According to the induction hypothesis, we have $p(\begin{bmatrix} C_2 & D^T \\ D & E \end{bmatrix}) = p(H) \leq d-1$. By Lemma 2.3, we can obtain $p(\begin{bmatrix} C_1 - B_2 A_2^{-1} B_2^T + C_2 & D^T \\ D & E \end{bmatrix}) \leq d-1$, so $p(M) \leq d$, and according to Lemma 2.4, we have $p(G) = p(K) \leq p(M) \leq d$. □

Corollary 3.1. Let G be a graph with $\varepsilon(G) = 2$. Then $p(G) = 2$.

proof. According to Theorem 3.1 and Lemma 2.2, the proof is clear. □

Remark 3.1. The inverse of Corollary 3.1 doesn't hold. For the following graph H(see Figure 1), we have $p(H) = 2$, but $\varepsilon(H) = 3$.

(In fact, the edges of H can't be decomposed into union of two complete

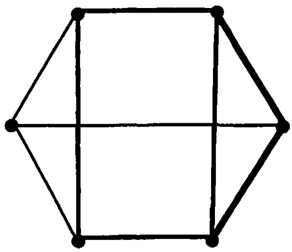

Figure 1: Graph H with $p(H) = 2$, but $\varepsilon(H) = 3$.

multipartite graphs. If not, suppose edges of H can be decomposed into union of two complete multipartite graphs H_1 and H_2, then H_1 and H_2 are not complete $k(\geq 4)$-partite graphs(otherwise, K_4 is a subgraph of H). If H_1 is a complete 3-partite graph and each part contains only one vertex, then $H_1 = K_3$, but the edges in H, not in H_1, can't form a complete multipartite graph; If H_1 is a complete 3-partite graph and there is one part which contains more than two vertices, then $K_{1,1,2}$ is a subgraph of H, but there doesn't exist such subgraph in H; If H_1 is a complete 2-partite graph and there is one part which contains only a vertex, then H_1 is a star graph, but the edges in H, not in H_1, can't form a complete multipartite graph; If H_1 is a complete 2-partite graph and each part contains at least two vertices, if one part contains more than two vertices, then $K_{2,3}$ is a subgraph of H, but there doesn't exist such subgraph in H. Hence $H_1 = K_{2,2}$, $H_2 = K_{2,2}$, but $|E(H_1)| + |E(H_2)| = 8$, a contradiction.

We note that H is the union of three complete multipartite graphs signified by thick, thin and middle-line in Figure 1, respectively. On the other hand, we can easily obtain spectrum of H, $sepc(H) = \{3, 1, 0, 0, -2, -2\}$.)

Remark 3.2. From Table 1 of [2], we can easily prove that the equality in Theorem 3.1 holds for all graphs whose the number of vertices are not greater than 5. Graph H (in Figure 1) is the graph which has the least number of vertex such that equality of Theorems 3.1 does't hold.

Corollary 3.2. Let G be a graph. Then $\beta(G) \geq p(G)$.

proof. As the edges of G can be decomposed into $\beta(G)$ star graphs, and each star graph is a complete multipartite graph, so $\beta(G) \geq \varepsilon(G) \geq p(G)$. □

Corollary 3.3. Let G be a tree. Then $p(G) = \varepsilon(G)$.

proof. For tree G, we have $\varepsilon(G) = \mu(G)$. The conclusion is clear by Lemma 2.5. □

Let $G_1 = (X_1, \mu_1)$ and $G_2 = (X_2, \mu_2)$ be two graphs, the sum $G_1 + G_2 = (X, \mu)$ of G_1 and G_2 is a graph, where $X = X_1 \times X_2$. Let $(x_1, x_2), (y_1, y_2) \in X$, the vertices (x_1, x_2) and (y_1, y_2) are adjacent in the sum $G_1 + G_2$ if and only if either $x_1 = y_1$ and $(x_2, y_2) \in \mu_2$ or $(x_1, y_1) \in \mu_1$ and $x_2 = y_2$. Let n be odd, we know, from [18], $p(C_n + C_{(2t+1)n}) = \varepsilon(C_n + C_{(2t+1)n}) = \frac{(2t+1)(n^2+1)}{2}$. A complete multipartite graph edge decomposition of $C_5 + C_{15}$, which each edge-induced subgraph is a star graph or a 4-cycle, is given by the following Figure 2, where black points denote the center of star graph, cycles denote 4-cycle of decomposition. We can obtain a ring if we curl edges from up to down and from left to right.

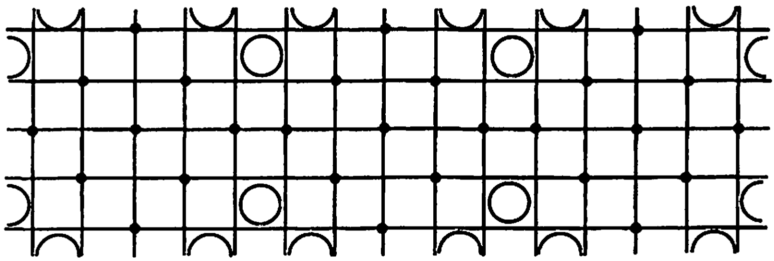

Figure 2: Graph $C_5 + C_{15}$

4 Graph G with $p(G) = n - 2$

In this section, we will characterize the graphs by using positive inertia index. Obviously, if G is a nonempty graph with three vertices, then $p(G) = 1$; If G is a graph with n vertices, then there does not exist graph G with $p(G) = n$.

Lemma 4.1. Let G be a graph of order n. Then $p(G) = n - 1$ if and only if $n = 2$, $G \cong K_2$.

proof. Suppose that $p(G) = n - 1$, by Corollary 3.2, we have $\beta(G) \geq n - 1$. This implies that G is a complete graph. Furthermore, $p(G) = 1$ by Lemma 2.2. Hence $n = 2$ and $G \cong K_2$. □

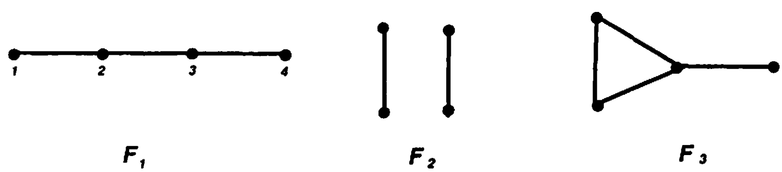

Figure 3: The graph G with four vertices and $p(G) = 2$

Lemma 4.2([2]). (1)Let G be a graph with four vertices. Then $p(G) = 2$ if and only if $G \in \{F_1, F_2, F_3\}$.

(2)Let G be a graph with five vertices. Then $p(G) = 3$ if and only if $G = C_5$.

Lemma 4.3. There doesn't exist graph G of order 6 with $\varepsilon(G) = 4$.

proof. Suppose that there exist a graph G of order 6 with $\varepsilon(G) = 4$, then G has no isolated vertex and isn't a complete graph (if G is a complete graph, then $\varepsilon(G) = 1$, a contradiction. If G has an isolated vertex w, then $\beta(G \setminus \{w\}) \geq \varepsilon(G \setminus \{w\}) = \varepsilon(G) = 4$, this implies that $G \setminus \{w\}$ is a complete graph. so $\varepsilon(G) = \varepsilon(G \setminus \{w\}) = 1$, a contradiction). Let u, v be two non-adjacent vertices of graph G, then $G \setminus \{u, v\}$ is a graph of order 4 with $p(G \setminus \{u, v\}) = 2$ (if $p(G \setminus \{u, v\}) \leq 1$, then $\varepsilon(G \setminus \{u, v\}) \leq 1$ by Remark 3.2, thus $\varepsilon(G) \leq 3$, a contradiction. If $p(G \setminus \{u, v\}) = 3$, then $\varepsilon(G \setminus \{u, v\}) = 3$, thus $G \setminus \{u, v\}$ is a complete graph, so $\varepsilon(G) \leq \varepsilon(G \setminus \{u, v\}) + 2 = 1 + 2 = 3$, a contradiction). By Lemma 4.2 (1), we consider the following two cases.

Case 1. If $G \setminus \{u, v\} \cong F_1, V(F_1) = \{1, 2, 3, 4\}$ (see Figure 4). Without loss of generality to assume that $d_G(u) \geq d_G(v)$. If $d_G(u) = 4$, then the edges of G can be decomposed into a $K_{1,1,2}$ and two star graphs. If $d_G(u) = 3$, then the edges of G can be decomposed into a $K_{1,1,2}$ and two star graphs or a $K_{2,2}$ and two star graphs. If $d_G(u) = 2$ and u is adjacent to vertices 1 and 4 at the same time, (1) if $d_G(v) = 2$, then the edges of G can be decomposed into a K_3 and two star graphs or a $K_{2,2}$ and two star graphs; (2) if $d_G(v) = 1$, then the edges of G can be decomposed into three star graphs. If $d_G(u) = 2$ and u is not adjacent to vertices 1 and 4 at the same time, then the edges of G can be decomposed into a K_3 and two star graphs or a $K_{2,2}$ and two star graphs. If $d_G(u) = 1$,then the edges of G can be decomposed into three star graphs. Therefore, $\varepsilon(G) \leq 3$, this is a contradiction.

Case 2. Similar to the proof of case 1, if $G\backslash\{u,v\} \cong F_2$ or F_3, then $\varepsilon(G) \leq 3$, this is a contradiction. □

Lemma 4.4. There doesn't exist graph G of order 7 with $\varepsilon(G) = 5$.

proof. Suppose that there exist a graph G of order 7 with $\varepsilon(G) = 5$, then G is not a complete graph and has no isolated vertex. Let u, v be two non-adjacent vertices of graph G, then $G \setminus \{u,v\}$ is a graph of order 5 with $p(G \setminus \{u,v\}) = 3$ (if $p(G \setminus \{u,v\}) \leq 2$, then $\varepsilon(G \setminus \{u,v\}) \leq 2$ by Remark 3.2, thus $\varepsilon(G) \leq 4$, a contradiction. If $p(G \setminus \{u,v\}) = 4$, then $\varepsilon(G \setminus \{u,v\}) = 4$, thus $G \setminus \{u,v\}$ is a complete graph, so $\varepsilon(G) \leq \varepsilon(G \setminus \{u,v\}) + 2 = 1 + 2 = 3$, a contradiction). By Lemma 4.2(2), we have $G \setminus \{u,v\} = C_5$. Without loss of generality to assume that $d_G(u) \geq d_G(v)$. If $d_G(u) \geq 4$, then the edges of G can be decomposed into a $K_{1,1,2}$ and three star graphs. If $d_G(u) = 3$, then the edges of G can be decomposed into a $K_{1,1,2}$ and three star graphs or a $K_{2,2}$ and three star graphs. If $d_G(u) = 2$, then the edges of G can be decomposed into a K_3 and three star graphs or a $K_{2,2}$ and three star graphs. If $d_G(u) = 1$, then the edges of G can be decomposed into four star graphs. Therefore, $\varepsilon(G) \leq 4$, this is a contradiction.

□

Theorem 4.1. Let G be a graph of order n. Then $p(G) = n - 2$ if and only

if

1) $n = 2$, $G \cong 2K_1$;

2) $n = 3$, $G \cong K_1 \bigcup K_2$, $K_{1,2}$ or K_3;

3) $n = 4$, $G \cong F_1, F_2$ or F_3;

4) $n = 5$, $G \cong C_5$.

proof. By Lemma 4.1 and Lemma 4.2, we only need to show that there doesn't exist graph G with $p(G) = n - 2$ when $n \geq 6$.

Suppose that there exists a graph G with $p(G) = n - 2$ (here $n \geq 6$). By Theorem 2.1, we have $\varepsilon(G) \geq p(G)$. Hence $\varepsilon(G) = n - 1$ or $n - 2$.

(1) If $\varepsilon(G) = n - 1$, then $\beta(G) = n - 1$ and G is a complete graph. Therefore, $\varepsilon(G) = 1$. However, $n \geq 6$, this is a contradiction.

(2) We will show that there doesn't exist a graph G such that $\varepsilon(G) = n - 2$ when $n \geq 6$. By using mathematical induction on n. The conclusion holds according to Lemma 4.3 and Lemma 4.4 when $n = 6$ or 7. Now, let G be a graph with $n(\geq 8)$ vertices and $\varepsilon(G) = n - 2$, then G is not a complete graph and has no isolated vertex. Furthermore, let u, v are two non-adjacent vertices of graph G, then $\varepsilon(G \setminus \{u, v\}) < n - 3$ (if not, we know that $G \setminus \{u, v\}$ is a complete graph. Therefore, $\varepsilon(G) \leq 3$. However, $\varepsilon(G) = n - 2 \geq 6$, which is a contradiction), and $\varepsilon(G \setminus \{u, v\}) \geq n - 4$ (if not, $\varepsilon(G) < n - 2$). So $\varepsilon(G \setminus \{u, v\}) = n - 4 = |V(G \setminus \{u, v\})| - 2$. According to the induction hypothesis, there doesn't exist such graph $G \setminus \{u, v\}$. □

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