Four-way combinatorial interpretations of some Rogers – Ramanujan type identities

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Abstract

In this paper we provide the 4-way combinatorial interpretations of some Rogers – Ramanujan type identities using partitions with "$n + t$ copies of n", lattice paths, F–partitions and ordinary partitions.

Keywords: Rogers – Ramanujan identities, colored partitions, lattice paths, F – partitions.
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1 Introduction, definitions and notations

1.1 Introduction

The Rogers – Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv \pm 1 (mod 5)}} \frac{1}{(1-q^k)} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv \pm 2 (mod 5)}} \frac{1}{(1-q^k)}, \tag{1.2}$$

where

$$(a;q)_n = \prod_{k=0}^{n-1} (1-aq^k)$$

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were first proved by L.J. Rogers [17] and later independently rediscovered by S. Ramanujan [15]. Many additional q-series=infinite product were found by Ramanujan and were recorded in his lost notebook [15]. A large collection of such identities was produced by L.J. Slater [18]. In literature we find that several q-identities from Slater compendium [18] have been interpreted combinatorially by several authors (for example see Connor [11], Subbarao [20], Subbarao and Agarwal [21], Agarwal [2] and Agarwal and Andrews [4]). In the early nineteen eighties Agarwal and Andrews introduced a new class of partitions called "$(n+t)$-color partitions" or partitions with "$n+t$ copies of n". Using these new partitions many more q-identities have been interpreted combinatorially in [1, 3, 5, 8, 14]. In [8,16] Agarwal and Rana have interpreted a generalised q-series using "$(n+t)$-color partitions", lattice paths and F – partitions. Recently in [13, 14] Goyal and Agarwal and in [19] Sood and Agarwal interpreted five q-identities combinatorially by using above techniques. The purpose of this paper is to extend their work and provide combinatorial interpretations of three more identities of the Rogers – Ramanujan type which appear in Slater's compendium [18] and also derived by Chu and Zhang [10] given below:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q,q^{13};q^{14}]_\infty[q^{16},q^{12};q^{28}]_\infty, \quad (1.3)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4,q^4)_n(q,q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q^3,q^{11};q^{14}]_\infty[q^{20},q^8;q^{28}]_\infty, \quad (1.4)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q^5,q^9;q^{14}]_\infty[q^{24},q^4;q^{28}]_\infty. \quad (1.5)$$

1.2 Definitions and notations

First we recall the definitions of partitions with "$n + t$ copies of n" and their weighted difference.

Definition 1.1. *[9] A partition with "$n + t$ copies of n", $t \geq 0$ is a partition in which a part of size n, $(n \geq 0)$, can come in $n + t$ different colors, denoted by the subscripts, $n_1, n_2, n_3, \cdots, n_{n+t}$.*

Example 1.1. *Partitions of 2 with "$n + 1$ copies of n" are,*

$$2_1, \ 2_1 0_1, \ 1_1 1_1, \ 1_1 1_1 0_1,$$

$$2_2,\ 2_2 0_1,\ 1_2 1_1,\ 1_2 1_1 0_1,$$

$$2_3,\ 2_3 0_1,\ 1_2 1_2,\ 1_2 1_2 0_1.$$

Note that zeros are permitted if and only if $t \geq 1$. Also in no partition are zeros permitted to repeat.

Definition 1.2. *[9] The weighted difference of two parts $m_i, n_j, m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.*

Next we recall the following description of lattice paths from [6] and F – partitions from [7], which we shall be considering in this paper.

Definition 1.3. *[6] All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:*
northeast : from (i, j) to $(i + 1, j + 1)$
southeast : from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$
horizontal: from $(i, 0)$ to $(i + 1, 0)$, only allowed along x-axis
The following terminology will be used in describing lattice paths:

Peak: *Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*
Valley: *A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.*
Mountain: *A section of the path which start on either the x-or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*
Plain: *A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*
The **Height** *of a vertex is its y-coordinate. The* **Weight** *of a vertex is its x-coordinate. The* **Weight** *of a path is the sum of the weights of its peaks.*

Definition 1.4. *[7] A two rowed array of non-negative integers*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ *and* $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$,
is known as a generalized F *– partition(Frobenius partition) or more simply an* F *– partition of* n *if* $n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$.

Example 1.2. *n=28=4+(6+5+2+0)+(5+3+2+1) and the corresponding Frobenius notation is* $\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}$.

In Section 2, we provide the combinatorial interpretations of $(1.3) - (1.5)$ using "n- color partitions" and ordinary partitions. In Section 3, we then provide the combinatorial interpretations of $(1.3) - (1.5)$ using lattice paths and also establish a bijection between "n- color partitions" and lattice paths. Further the results are extended in Section 4 using F – partitions.

2 Combinatorial interpretations using "n- color partitions"

Theorem 2.1. *For* $\nu \geq 0$, *let* $A_1(\nu)$ *denote the number of partitions of* ν *with 'n copies of* n*' into parts greater than or equal to* 3 *such that if* m_i *is the least or the only part in the partition then* $m - i \equiv 2(mod4)$ *and weighted difference between consecutive parts is non negative and* $\equiv 0(mod4)$ *and let*

$$B_1(\nu) = \sum_{k=0}^{\nu} C_1(\nu - k) D_1(k),$$

where $C_1(\nu)$ *denote the number of partitions of* ν *into parts* $\equiv \pm 4,\ \pm 6,\ \pm 8,\ \pm 10\ (mod28)$ *and* $D_1(\nu)$ *denote the number of partitions of* ν *into distinct parts* $\equiv \pm 3, \pm 5, 7(mod14)$. *Then*

$$A_1(\nu) = B_1(\nu),$$

for all ν.

Example 2.1. *We demonstrate Theorem* 2.1 *by showing that*

$$A_1(8) = B_1(8) = 3.$$

The relevant n-color partitions corresponding to $A_1(8)$ *are*

$$8_2, \quad 8_6, \quad 5_1 3_1$$

and $B_1(8) = 3$, *since*

$$B_1(8) = \sum_{k=0}^{8} C_1(8-k)D_1(k),$$

where the relevant partitions corresponding to $C_1(\nu)$ and $D_1(\nu)$ are given in the table below;

ν	$C_1(\nu)$	*partitions enumerated by $C_1(\nu)$*	$D_1(\nu)$	*partitions enumerated by $D_1(\nu)$*
0	1	*empty partition*	1	*empty partition*
1	0	-	1	1
2	0	-	0	-
3	0	-	1	3
4	1	4	0	-
5	0	-	1	5
6	1	6	0	-
7	0	-	1	7
8	2	$8, 4+4$	1	$5+3$

hence,

$$\begin{aligned} B_1(8) &= C_1(8)D_1(0) + C_1(7)D_1(1) + \cdots + C_1(0)D_1(8) \\ &= 3. \end{aligned}$$

Theorem 2.2. *For $\nu \geq 0$, let $A_2(\nu)$ denote the number of partitions of ν with 'n copies of n' into parts such that if m_i is the least or the only part in the partition then $m \equiv i(mod4)$ and weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$ and let*

$$B_2(\nu) = \sum_{k=0}^{\nu} C_2(\nu-k)D_2(k),$$

where $C_2(\nu)$ denote the number of partitions of ν into parts $\equiv \pm2, \pm4, \pm10, \pm12(mod28)$ and $D_2(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm1, \pm5, 7(mod14)$. Then

$$A_2(\nu) = B_2(\nu),$$

for all ν.

Theorem 2.3. *For $\nu \geq 0$, let $A_3(\nu)$ denote the number of partitions of ν with '$n+2$ copies of n' into parts such that for some i, i_{i+2} must be a part and weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$ and let*

$$B_3(\nu) = \sum_{k=0}^{\nu} C_3(\nu-k)D_3(k),$$

where $C_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 6, \pm 8, \pm 12 (mod 28)$ and $D_3(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 1, \pm 3, 7 (mod 14)$. Then

$$A_3(\nu) = B_3(\nu),$$

for all ν.

Proof of Theorem 2.1
Let $A_1(\nu, m)$ denote the number of partitions of ν enumerated by $A_1(\nu)$ into m parts.
We split the partitions enumerated by $A_1(\nu, m)$ into three classes:
(i) those that do not contain k_{k-2} as a part,
(ii) those that contain 3_1 as a part,
(iii) those that contain $k_{k-2} (k > 3)$ as a part.
We now transform the partitions into class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and transformed partition will be of the type enumerated by $A_1(\nu - 4m, m)$.
Next transform the partitions in class (ii) by deleting the least part 3_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1(\nu - 2m - 1, m - 1)$.
Finally we transform the partitions in class (iii) by replacing the part k_{k-2} by $(k + 1)_{k-3}$ and then subtracting 2 from all the other parts. This will produce the partitions of $(\nu - 2m + 1)$ into m parts. Note here that, by this transformation we will get only those partitions of $(\nu - 2m + 1)$ into m parts which contain a part of the form k_{k-2}. Therefore the actual number of partitions which belong to class (iii) is $A_1(\nu - 2m + 1, m) - A_1(\nu - 6m + 1, m)$ where $A_1(\nu - 6m + 1, m)$ is the number of partitions of $\nu - 2m + 1$ into m parts which are free from parts like k_{k-2}.

The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $A_1(\nu, m)$ and those enumerated by

$$A_1(\nu - 4m, m) + A_1(\nu - 2m - 1, m - 1) + A_1(\nu - 2m + 1, m) \\ - A_1(\nu - 6m + 1, m).$$

This leads to the identity

$$\begin{aligned} A_1(\nu, m) =& A_1(\nu - 4m, m) + A_1(\nu - 2m - 1, m - 1) \\ &+ A_1(\nu - 2m + 1, m) - A_1(\nu - 6m + 1, m). \end{aligned} \tag{2.1}$$

Now let

$$f_1(z; q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_1(\nu, m) z^m q^\nu, \tag{2.2}$$

substitute $A_1(\nu, m)$ from (2.1) into (2.2), we get

$$f_1(z;q) = f_1(zq^4;q) + zq^3 f_1(zq^2;q) + q^{-1} f_1(zq^2;q) - q^{-1} f_1(zq^6;q). \quad (2.3)$$

Consider

$$f_1(z;q) = \sum_{n=0}^{\infty} a_n(q) z^n \quad (2.4)$$

since $f_1(0;q) = 1$, using (2.4) in (2.3) and then comparing the coefficients of z^n, we get

$$a_n(q) = \frac{q^{n(n+2)}}{(q^4, q^4)_n (q, q^2)_n},$$

$$f_1(z;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)} z^n}{(q^4, q^4)_n (q, q^2)_n},$$

$$f_1(1;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4, q^4)_n (q, q^2)_n},$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_1(\nu) q^{\nu} &= \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} A_1(\nu, m) q^{\nu} \\ &= f_1(1;q) \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4, q^4)_n (q, q^2)_n} \\ &= \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} (q^{14}, q, q^{13}; q^{14})_{\infty} (q^{16}, q^{12}; q^{28})_{\infty} \\ &= \sum_{\nu=0}^{\infty} C_1(\nu) q^{\nu}\end{aligned}$$

which completes the proof of Theorem (2.1).

Sketch proofs of Theorems 2.2 and 2.3

The proofs are similar to that of Theorem 2.1, hence we omit the details and give only the main steps.
The following are the recurrence relations corresponding to Theorem 2.2 and 2.3 respectively:

$$A_2(\nu,m) = A_2(\nu-4m,m) + A_2(\nu-2m+1,m-1) + A_2(\nu-2m+1,m) - A_2(\nu-6m+1,m), \tag{2.5}$$

$$A_3(\nu-1,m-1) = A_2(\nu,m) - A_2(\nu-4m,m). \tag{2.6}$$

and the respective q-functional equations are

$$f_2(z;q) = f_2(zq^4;q) + zqf_2(zq^2;q) + q^{-1}f_2(zq^2;q) - q^{-1}f_2(zq^6;q), \tag{2.7}$$

$$zqf_3(z;q) = f_2(z;q) - f_2(zq^4;q). \tag{2.8}$$

3 Combinatorial interpretations using lattice paths

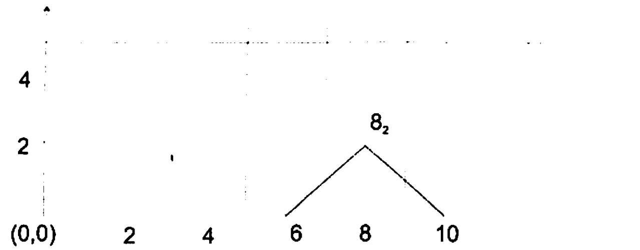
Theorem 3.1. *For $\nu \geq 0$, let $E_1(\nu)$ denote the number of lattice paths of weight ν which start at $(0,0)$, have no valley above height 0, there is a plain of length $\equiv 2(mod4)$ in the beginning of the path, the length of the other plains, if any, are $\equiv 0(mod4)$. Then*

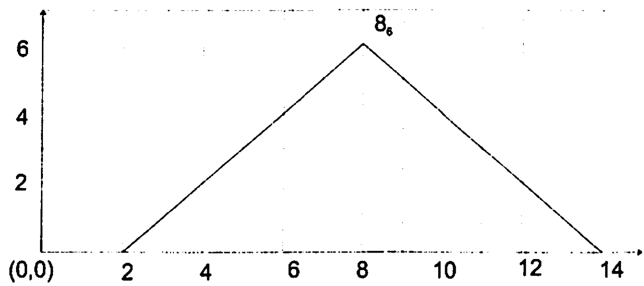
$$E_1(\nu) = B_1(\nu) = A_1(\nu), \textit{ for all } \nu.$$

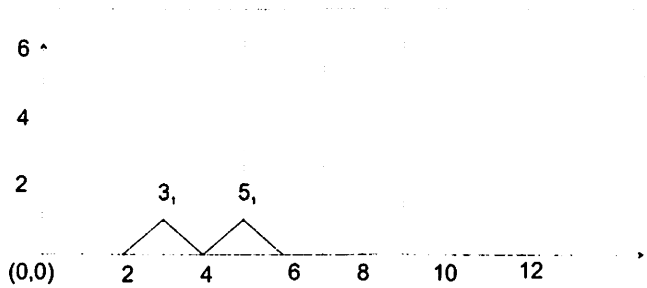
Example 3.1. *We demonstrate Theorem 3.1 by showing that*

$$E_1(8) = B_1(8) = A_1(8) = 3.$$

The relevant lattice paths corresponding to $E_1(8)$ are as follows,







and by Example 2.1, $A_1(8) = B_1(8) = 3$.

Theorem 3.2. *For* $\nu \geq 0$, *Let* $E_2(\nu)$ *denote the number of lattice paths of weight* ν *which start at* $(0,0)$, *have no valley above height* 0, *the length of the plains, if any, are* $\equiv 0(mod 4)$. *Then*

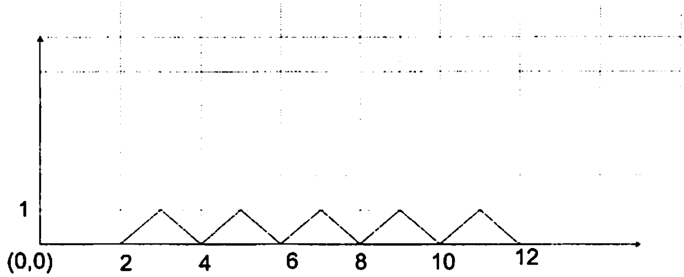
$$E_2(\nu) = B_2(\nu) = A_1(\nu), \textit{ for all } \nu.$$

Theorem 3.3. *For* $\nu \geq 0$, *let* $E_3(\nu)$ *denote the number of lattice paths of weight* ν *which start at* $(0,2)$, *have no valley above height* 0, *the length of the plains, if any, are* $\equiv 0(mod 4)$. *Then*

$$E_3(\nu) = B_3(\nu) = A_3(\nu), \textit{ for all } \nu.$$

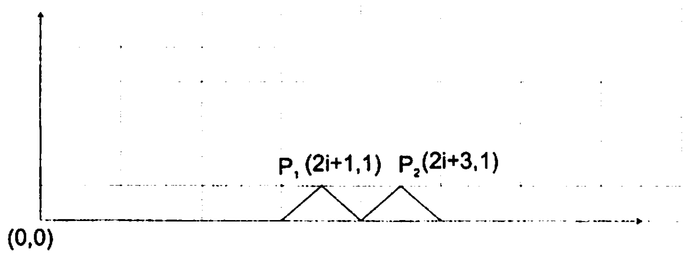
Proof of Theorem 3.1

In $\frac{q^{n(n+2)}}{(q^4;q^4)_n(q;q^2)_n}$ the factor $q^{n(n+2)}$ generates a lattice path from $(0,0)$ to $(2n+2,0)$ having n peaks each of height 1 and plain of length 2 in the beginning of the path.
For n=5, the path begins as



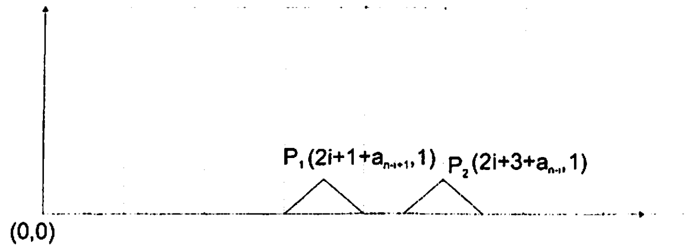
Graph A

In the above graph we consider two successive peaks say, i^{th} and $(i+1)^{th}$ and denote them by P_1 and P_2, respectively.



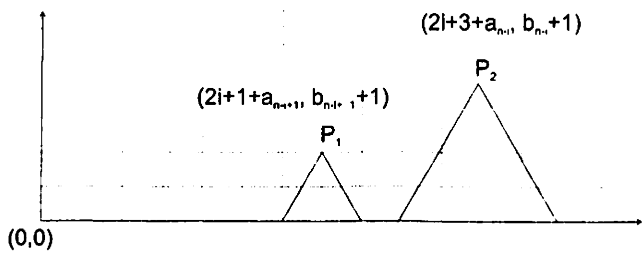
Graph B

The factor $1/(q^4;q^4)_n$ generates n nonnegative parts $\equiv 0(mod 4)$, say $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, which are encoded by inserting a_n horizontal steps in front of the first mountain, and $a_i - a_{i+1}$ horizontal steps in front of the $(n-i+1)st$ mountain, $1 \leq i \leq n-1$. Thus the x-coordinate of the i^{th} peak is increased by $a_n + (a_{n-1} - a_n) + (a_{n-2} - a_{n-1}) + \cdots + (a_{n-i+1} - a_{n-i+2}) = a_{n-i+1}$ and the x-coordinate of the $(i+1)^{th}$ peak is increased by a_{n-i}.
Graph B now becomes Graph C.



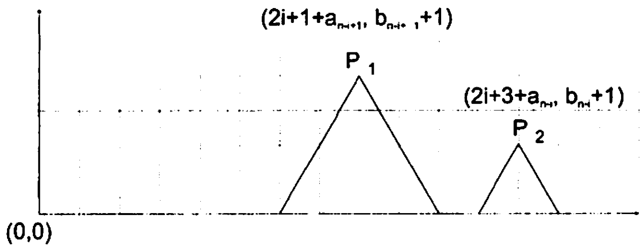
Graph C

$P_1 \equiv (1 + 2i + a_{n-i+1}, 1)$,
$P_2 \equiv (3 + 2i + a_{n-i}, 1)$.
The factor $1/(q;q^2)_n$ generates non negative multiples of $(2i-1)$, $1 \le i \le n$, say, $b_1 \times 1, b_2 \times 3, \cdots, b_n \times (2n-1)$.
This is encoded by having the i^{th} peak grow to height $b_{n-i+1} + 1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. Graph C now changes to Graph D or Graph E depending on whether $b_{n-i} > b_{n-i+1}$ or $< b_{n-i+1}$. In the case when $b_{n-i} = b_{n-i+1}$, the new graph looks like Graph D.



Graph D

The Graph E looks like



Graph E

Every lattice path enumerated by $E_1(\nu)$ is uniquely generated in this manner. This proves Theorem 3.1.

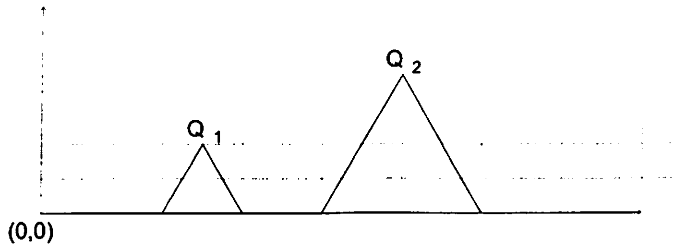
Bijection between n-color partitions and lattice paths

We now establish a $1-1$ correspondence between the lattice paths enumerated by $B_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$. We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Thus if we denote the two peaks in Graph D (or Graph E) by A_x and B_y, $(B \geq A)$ respectively, then

$$A = 1 + 2i + a_{n-i+1} + 2(b_n + b_{n-1} + \cdots + b_{n-i+2}) + b_{n-i+1}$$
$$x = b_{n-i+1} + 1$$
$$B = 3 + 2i + a_{n-i} + 2(b_n + b_{n-1} + \cdots + b_{n-i+1}) + b_{n-i}$$
$$y = b_{n-i} + 1.$$

Clearly, the weighted difference of these two parts is $((B_y - A_x)) = B - A - x - y = a_{n-i} - a_{n-i+1}$ which is non negative and $\equiv 0(mod 4)$.

To see the reverse implication we consider two n-color parts of a partition enumerated by $A_1(\nu)$, say C_u and D_v with $D \geq C \geq 3$. Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.

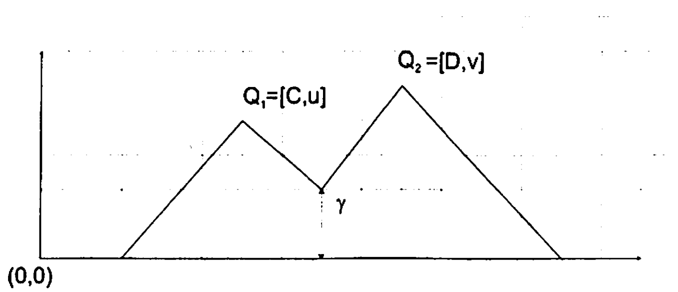


Graph F

The length of the plain between the two peaks is $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is therefore nonnegative and $\equiv 0(mod 4)$.

If C_u were the smallest part of the partition, the corresponding peak in the associated path would be the first peak preceded by a plain of length $2 + a$. where $a \equiv 0(mod 4)$.

Finally, we show that there can not be a valley above height 0. This can be proved by contradiction.
Suppose there is a valley V of height γ ($\gamma > 0$) between the peaks Q_1 and Q_2.



Graph G

In this case there is a descent of $u - \gamma$ from Q_1 to V and an ascent of $v - \gamma$ from V to Q_2. This implies that $D = C + (u - \gamma) + (v - \gamma)$, or $D - C - u - v = -2\gamma$. But since the weighted difference is non negative, therefore γ=0. This completes the bijection between the lattice paths enumerated by $B_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$.

Sketch Proofs of Theorems 3.2 and 3.3

Theorem 3.2 is treated in exactly the same manner as the Theorem 3.1 except that now the path begins from $(0, 0)$.
For Theorem 3.3, comparing it with Theorem 3.2, we see that in this case there

are two extra factors, viz., q^{2n} and $(1-q^{2n+1})^{-1}$. The extra factor q^{2n} puts two south east steps: $(0,2)$ to $(1,1)$ and $(1,1)$ to $(2,0)$. Thus there are now $n+1$ peaks starting from $(0,2)$ and the extra factor $(1-q^{2n+1})^{-1}$ introduces a non negative multiple of $2n+1$, say $b_{n+1}\times(2n+1)$. This is encoded by having first peak grow to height $b_{n+1}+2$. Clearly, $(b_{n+1})_{b_{n+1}+2}$ which is of the form i_{i+2} will be the colored part corresponding to the first peak.

4 Combinatorial interpretations using F – partition

Theorem 4.1. *For $\nu \geq 0$, let $F_1(\nu)$ denote the number of F – partitions of ν such that*
(1.a) $a_r \equiv 1(mod2)$,
(1.b) $a_i \leq b_i$ and
(1.c) $a_i > b_{i+1}$ and are of opposite parity.

$A_1(\nu)$ denote the number of partitions of ν with 'n copies of n' such that (1.d) each part ≥ 3,
(1.e) the weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$, and
(1.f) if m_i is the least or only part in the partition then $m-i \equiv 2(mod4)$.
Then

$$A_1(\nu) = F_1(\nu) \ \ for \ \ all \ \ \nu.$$

Theorem 4.2. *For $\nu \geq 0$, let $F_2(\nu)$ denote the number of F – partitions of ν such that*
(1.a) $a_r \equiv 0(mod2)$,
(1.b) $a_i \leq b_i$ and
(1.c) $a_i > b_{i+1}$ and are of opposite parity.

$A_2(\nu)$ denote the number of partitions of ν with 'n copies of n' such that
(1.d) the weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$, and
(1.e) if m_i is the least or only part in the partition then $m \equiv i(mod4)$.
Then

$$A_2(\nu) = F_2(\nu) \ \ for \ \ all \ \ \nu.$$

Theorem 4.3. *For $\nu \geq 0$, let $F_3(\nu)$ denote the number of F – partitions of ν such that*
(1.a) a_r is 0,
(1.b) $a_i \leq b_i + 2$ and
(1.c) $a_i > b_{i+1} + 2$ and are of opposite parity.

$A_3(\nu)$ denote the number of partitions of ν with '$n+2$ copies of n' such that

(1.d) the weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$, and
(1.e) for some i, i_{i+2} must be a part.
Then

$$A_3(\nu) = F_3(\nu) \ for \ all \ \nu.$$

Note. $A_k(\nu)$ (for $1 \leq k \leq 3$) of Theorem 4.1 – 4.3 are same as Theorem 2.1 – 2.3. respectively.

Proof of Theorem 4.1
We establish a 1-1 correspondence between the F – partitions enumerated by $F_1(\nu)$ and the n - color partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\begin{pmatrix} a \\ b \end{pmatrix}$ of the F – partition to a single part m_i of an n - color partition enumerated by $A_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a+b+1)_{b-a+1}, \tag{4.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i = \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix}. \tag{4.2}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in an F – partition enumerated by $F_1(\nu)$ with

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi : \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a+b+1)_{b-a+1} = m_i$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix} \rightarrow (c+d+1)_{d-c+1} = n_j,$$

we have

$$\begin{aligned} ((m_i - n_j)) &= m - n - i - j \\ &= (a+b+1)-(c+d+1)-(b-a+1)-(d-c+1) \end{aligned}$$

$$= 2(a - d - 1). \tag{4.3}$$

Clearly (4.3) and (1.c) imply (1.e).
Also (4.2), (1.a) and (1.b) imply (1.d).

Now if $a_r \equiv 1(mod 2)$, then

$$m - i = (a_r + b_r + 1) - (b_r - a_r + 1) = 2a_r$$

which imply $(m - i) \equiv 2(mod 4)$, hence $(1.f)$ holds.

To see the reverse implication, we consider the inverse images of two consecutive parts m_i, n_j of an n - color partition enumerated by $A_1(\nu)$

$$\phi^{-1} : m_i = \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\phi^{-1} : n_j = \begin{pmatrix} (n-j)/2 \\ (n+j-2)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

that is,

$$a = (m - i)/2 \tag{4.4}$$

$$b = (m + i - 2)/2 \tag{4.5}$$

$$c = (n - j)/2 \tag{4.6}$$

$$d = (n + j - 2)/2 \tag{4.7}$$

and so

$$b - a = i - 1 \tag{4.8}$$

$$d - c = j - 1 \tag{4.9}$$

$$a - d = \frac{1}{2}((m_i - n_j)) + 1 \tag{4.10}$$

(4.10) and (1.e) imply (1.c).
(4.8) and (4.9) imply (1.b).
(1.f) implies that there is a column of the form $\begin{pmatrix} a_r \\ b_r \end{pmatrix}$ such that a_r is odd. Also such a column has to be the last in the F – partition. This completes the proof of the Theorem 4.1.

To illustrate the bijection we have constructed the example for $\nu = 8$ shown in the following table:

Table 1

F – partitions enumerated by $A(8)$	Image under ϕ
$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$	8_2
$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$	8_6
$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$	$5_1 + 3_1$

Sketch Proof of Theorems 4.2 and 4.3

For Theorem 4.2, proceed in the same manner as Theorem 4.1. Here since $a_r \equiv 0(mod 2)$, then

$$m - i = (a_r + b_r + 1) - (b_r - a_r + 1) = 2a_r$$

which imply $(m - i) \equiv 0(mod 4)$.

For Theorem 4.3, the mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a + b + 1)_{b-a+3},$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i = \begin{pmatrix} (m - i + 2)/2 \\ (m + i - 4)/2 \end{pmatrix} .$$

Also here the part 0_2 corresponds to a "phantom" column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, which is dropped from the corresponding F – partition.

5 Conclusion

The most obvious question which arises from this work is: Is it possible to generalize Theorems 2.1 – 2.3 analogous to Gordon's generalisation [12] of Rogers – Ramanujan Identities?

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