# The number of spanning trees of power graphs associated with specific groups and some applications

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*In memory of Professor Michael Neumann.*

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**Abstract**

Given a group $G$, we define the power graph $\mathcal{P}(G)$ as follows: the vertices are the elements of $G$ and two vertices $x$ and $y$ are joined by an edge if $\langle x\rangle \subseteq \langle y\rangle$ or $\langle y\rangle \subseteq \langle x\rangle$. Obviously the power graph of any group is always connected, because the identity element of the group is adjacent to all other vertices. In the present paper, among other results, we will find the number of spanning trees of the power graph associated with specific finite groups. We also determine, up to isomorphism, the structure of a finite group $G$ whose power graph has exactly $n$ spanning trees, for $n < 5^3$. Finally, we show that the alternating group $\mathbb{A}_5$ is uniquely determined by tree-number of its power graph among all finite simple groups.

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# 1 Problem Statement and Motivation

Throughout this paper, only finite groups will be considered. Moreover, all the graphs under consideration are finite, simple (with no loops or multiple edges) and undirected. Given a connected graph $\Gamma$ with $n$ vertices, a *spanning tree* of $\Gamma$ is a connected subgraph $T$ of $\Gamma$ which has $n-1$ edges. Spanning trees of connected graphs have been the focus of considerable research. Actually, one of the interesting problems in Graph Theory is the problem of finding the number of spanning trees of a connected graph (also called the *complexity* of $\Gamma$, see [3]), which arises in a variety of applications. Especially, it is of interest in the analysis of electrical networks.

There are scattered results in the literature finding an explicit simple formula for the number of spanning trees of special graphs. Nevertheless, the problem becomes more interesting when we are dealing with a graph which is mainly associated with an algebraic structure, for instance, a group. Given a finite group $G$, there are many different ways to associate a simple graph $\Gamma_G$ to $G$ by choosing families of its elements or subgroups as vertices and letting two vertices be joined by an edge if and only if they satisfy a certain relation. Now, one of the interesting questions is to ask about characterizing certain properties of the group by means of the properties of the associated graph, in other words, to study the influence of a property of the graph on the structure of the group. This line of research has attracted considerable attention in recent years (see, for instance, [2, 7, 11, 15]).

A graph which has recently deserved a lot of attention is the *power graph* associated with a group $G$ (see [1, 5, 8, 9]). Note that, the term "power graph" was introduced and first considered in [8]. In this graph, the vertices are all elements of the group $G$ and two different vertices $x$ and $y$ are adjacent, and we write $x \sim y$, when $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. It is evident from the definition that the power graph of any group is always *connected*, because the identity element of the group is adjacent to all other vertices. We denote by $\kappa(G)$ the number of spanning trees of the power graph $\mathcal{P}(G)$ of a group $G$ and call this number the tree-number of $\mathcal{P}(G)$, which will be investigated for certain finite groups in this paper. More precisely, we will obtain the explicit formulas for the number of spanning trees of power graphs associated with the *cyclic group* $\mathbb{Z}_n$, *dihedral group* $D_{2n}$ and the *generalized quaternion group* $Q_{4n}$.

*Remark* 1. These groups are considered as they play an important role in some of the deeper parts of Finite Group Theory and in most cases they appear as subgroups of a given group. For example, the Cauchy's theorem [14, Theorem 5.11] states that if $G$ is a finite group and $p$ is a prime divisor of $|G|$, then $G$ has at least one cyclic subgroup of order $p$.

*Remark* 2. The main tool for computing these explicit formulas is a well-known theorem due to Temperley (Theorem 3.1), which deals with the explicit computation of a determinant. Note that, many useful and efficient tools for evaluating determinants are provided in [10]. It is worth stating at this point that although we are mainly trying to obtain these explicit formulas for the number of spanning trees of specific graphs, meanwhile we will be faced with some *interesting integer matrices* whose determinants are needed.

Evidently for two isomorphic groups $G$ and $H$ we have $\kappa(G) = \kappa(H)$. Dose the converse hold? The answer to this question is negative in the general case. Actually, $\kappa(G)$ can not determine the structure of a group $G$ uniquely. There are many examples which justify this matter. For instance, for all finite elementary abelian 2-groups $G$ we have $\kappa(G) = 1$, or as another example $\kappa(\mathbb{Z}_3) = \kappa(\mathbb{S}_3) = 3$, where $\mathbb{S}_3$ denotes the symmetric group on 3 letters.

A group $G$ from a class $\mathcal{C}$ is said to be recognizable in $\mathcal{C}$ by $\kappa(G)$ (shortly, $\kappa$-recognizable in $\mathcal{C}$) if every group $H \in \mathcal{C}$ with $\kappa(H) = \kappa(G)$ is isomorphic to $G$. In other words, $G$ is $\kappa$-recognizable in $\mathcal{C}$ if $h_{\mathcal{C}}(G) = 1$, where $h_{\mathcal{C}}(G)$ is the (possibly infinite) number of pairwise non-isomorphic groups $H \in \mathcal{C}$ with $\kappa(H) = \kappa(G)$. We denote by $\mathcal{F}$ and $\mathcal{S}$ the classes of all finite groups and all finite simple groups, respectively. There are some examples of groups with $1 < h_{\mathcal{F}}(G) < \infty$. For instance, $h_{\mathcal{F}}(\mathbb{Z}_3) = 2$ and $\kappa(\mathbb{Z}_3) = \kappa(\mathbb{S}_3) = 3$ (see Corollary 6.1). In the present paper, we find the first example of $\kappa$-recognizable group in class $\mathcal{S}$. It turns out that the following is true:

**Theorem 1.1** *The alternating group $\mathbb{A}_5$ is $\kappa$-recognizable in the class of all finite simple groups, that is, $h_{\mathcal{S}}(\mathbb{A}_5) = 1$.*

We will also continue to ask the following two questions:

**Question 1.1** *Is a group $G$ isomorphic to $\mathbb{A}_n$ ($n \geqslant 4$) if and only if $\kappa(G) = \kappa(\mathbb{A}_n)$?*

**Question 1.2** *Given a natural number $n$, determine all groups $G$ whose power graph has exactly $n$ spanning trees, that is $\kappa(G) = n$.*

*Remark* 3. Note that $\kappa(\mathbb{A}_4) = \kappa(\mathbb{Z}_3 \times \mathbb{Z}_3) = 3^4$ (see Table 1). Moreover, there are some natural numbers $n$ for which a group $G$ does not exist with $\kappa(G) = n$. For example, there does not exist a group $G$ with $\kappa(G) = 2$.

After this introduction, the structure of this paper is organized as follows: basic definitions and notation are summarized in Section 2. In Section 3, we derive some preparatory results. In Sections 4 and 5, we obtain explicit formulas for the tree-number of power graphs associated with a cyclic group $\mathbb{Z}_n$, a dihedral group $D_{2n}$ and a generalized quaternion group $Q_{4n}$. Finally, a few applications of obtained results are presented in Section 6:

(1) a classification of groups $G$ for which $\kappa(G) < 5^3$, (2) a new chracterization of the alternating group $\mathbb{A}_5$ by $\kappa(\mathbb{A}_5)$, and (3) a list of $\kappa(G)$ for all groups $G$ with $|G| \leqslant 15$.

## 2 Basic Definitions and Notation

The notation and definitions used in this paper are standard and taken mainly from [3, 14, 17]. We will cite only a few. Let $\Gamma = (V, E)$ be a simple graph. We denote by $\mathbf{A} = \mathbf{A}(\Gamma)$ the adjacency matrix of $\Gamma$. The complement $\overline{\Gamma}$ of $\Gamma$ is the simple graph whose vertex set is $V$ and whose edges are the pairs of non-adjacent vertices of $\Gamma$. When $U \subseteq V$, the induced subgraph $\Gamma[U]$ is the subgraph of $\Gamma$ whose vertex set is $U$ and whose edges are precisely the edges of $\Gamma$ which have both ends in $U$. Two graphs are disjoint if they have no vertex in common, and edge-disjoint if they have no edge in common. If $\Gamma_1$ and $\Gamma_2$ are disjoint, we refer to their union as a disjoint union, and generally denote it by $\Gamma_1 \oplus \Gamma_2$. By starting with a disjoint union of two graphs $\Gamma_1$ and $\Gamma_2$ and adding edges joining every vertex of $\Gamma_1$ to every vertex of $\Gamma_2$, one obtains the join of $\Gamma_1$ and $\Gamma_2$, denoted $\Gamma_1 \vee \Gamma_2$. A clique in a graph is a set of pairwise adjacent vertices. The clique number of a graph $\Gamma$, written $\omega(\Gamma)$, is the number of vertices in a maximum clique of $\Gamma$. Given a group $G$, we denote by $\omega(G)$ the set of orders of all elements in a group $G$ and call this set the spectrum of $G$. The spectrum $\omega(G)$ of $G$ is closed under divisibility and determined uniquely from the set $\mu(G)$ of those elements in $\omega(G)$ that are maximal under the divisibility relation. In the case when $\mu(G)$ is a one-element set $\{n\}$, we write $\mu(G) = n$. For a natural number $m$, the alternating and symmetric group of degree $m$ denoted by $\mathbb{A}_m$ and $\mathbb{S}_m$, respectively.

## 3 Auxiliary Results

In this section we give several auxiliary results to be used later. The first of them is the following lemma (See Lemma 3.4 and Corollary 3.1 in [12]).

**Lemma 3.1** *Let $G = \langle x \rangle$ be a cyclic group of order $n$ and $\Gamma = \mathcal{P}(G)$. Then the degree of $x^m \in G$ in the power graph $\Gamma$ is given by*

$$d_\Gamma(x^m) = \frac{n}{(m,n)} - 1 + \sum_{\substack{d \mid (m,n) \\ d \neq (m,n)}} \phi\left(\frac{n}{d}\right),$$

*where $\phi(k)$ signifies the Euler function of a natural number $k$. In particular, all non-trivial elements of a cyclic group with the same orders have the same degrees in its power graph.*

A complete graph is a simple graph in which any two vertices are adjacent. The complete graph on $n$ vertices is denoted by $K_n$. Next lemma is taken from [5].

**Lemma 3.2** [5, Theorem 2.12] *Let $G$ be a finite group. Then $\mathcal{P}(G)$ is complete if and only if $G$ is a cyclic group of order 1 or $p^m$ for some prime number $p$ and for some natural number $m$.*

As already mentioned in the Introduction, the number of spanning trees of a graph is one of the most important graph-theoretical parameters and appears in a number of applications. Given a graph $\Gamma$, we denote by $\kappa(\Gamma)$, the number of spanning trees of a graph $\Gamma$. In [3], Biggs refers to $\kappa(\Gamma)$ as the *tree-number* of $\Gamma$. Clearly $\kappa(\Gamma) = 0$ if and only if $\Gamma$ is disconnected. In [4], Cayley devised the well-known formula $\kappa(K_n) = n^{n-2}$. The Laplacian matrix $\mathbf{Q}$ of a graph $\Gamma$ is $\Delta - \mathbf{A}$, where $\Delta$ is the diagonal matrix whose $i$-th diagonal entry is the degree $v_i$ in $\Gamma$ and $\mathbf{A}$ is the adjacency matrix of $\Gamma$. The following Theorem is due to Temperley (1964).

**Theorem 3.1** ([16]) *The number of spanning trees of a graph $\Gamma$ with $n$ vertices is given by the formula*

$$\kappa(\Gamma) = \det(\mathbf{J} + \mathbf{Q})/n^2,$$

*where $\mathbf{J}$ denotes the matrix each of whose entries is $+1$.*

Given a graph $\Gamma$, we will let $c(\Gamma)$ denote the number of connected components of $\Gamma$. A cut edge of $\Gamma$ is an edge $e$ such that $c(\Gamma - e) > c(\Gamma)$. Similarly, a cut vertex of $\Gamma$ is a vertex $v$ such that $c(\Gamma - v) > c(\Gamma)$. In particular, a cut edge (resp. a cut vertex) of a connected graph is an edge (resp. a vertex) whose deletion results in a disconnected graph. For any edge $e$ which is not a loop, we also define the graph $\Gamma \cdot e$ to be the subgraph obtained from $\Gamma - e$ by identifying the vertices of $e$. The following is well known, see for example [17, Proposition 2.2.8].

**Theorem 3.2 (Deletion-Contraction Theorem)** *The number of spanning trees of a graph $\Gamma$ satisfies the deletion-contraction recurrence*

$$\kappa(\Gamma) = \kappa(\Gamma - e) + \kappa(\Gamma \cdot e),$$

*where $e \in E(\Gamma)$. In particular, if $e \in E(\Gamma)$ is a cut-edge, then*

$$\kappa(\Gamma) = \kappa(\Gamma \cdot e).$$

The following theorem follows in a straightforward way from the multiplication principle in combinatorics.

**Theorem 3.3** *Let $\Gamma$ be a connected graph and let $v$ be a cut vertex of $\Gamma$ with*

$$\Gamma - v = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_c,$$

*where $\Gamma_i$, $i = 1, 2, \ldots, c$, is the ith connected component of $\Gamma - v$ and $c = c(\Gamma - v)$. Set $\widetilde{\Gamma}_i = \Gamma_i + v$. Then, there holds*

$$\kappa(\Gamma) = \kappa(\widetilde{\Gamma}_1) \times \kappa(\widetilde{\Gamma}_2) \times \cdots \times \kappa(\widetilde{\Gamma}_c).$$

In what follows, we denote by $\pi(n)$ the set of the prime divisors of a positive integer $n$. Given a group $G$, we will write $\pi(G)$ instead of $\pi(|G|)$. If $p \in \pi(G)$, then $\mathrm{Syl}_p(G)$ will denote the set of all Sylow $p$-subgroups of $G$.

**Theorem 3.4** *Let $H_1, H_2, \ldots, H_t$ be nontrivial subgroups of a group $G$ such that*

$$H_i \cap H_j = \{1\}, \quad \text{for each } 1 \leqslant i < j \leqslant t.$$

*Then, there hold:*

(a) $\kappa(G) \geqslant \kappa(H_1)\kappa(H_2)\cdots\kappa(H_t)$. *In particular, if $\pi(G) = \{p_1, \ldots, p_k\}$ and $\mu(P_i) = m_i$, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $1 \leqslant i \leqslant k$, then*

$$\kappa(G) \geqslant \prod_{i=1}^{k} m_i^{m_i - 2} \geqslant \prod_{i=1}^{k} p_i^{p_i - 2}.$$

(b) *If $G = H_1 \cup H_2 \cup \cdots \cup H_t$, then we have*

$$\kappa(G) = \kappa(H_1)\kappa(H_2)\cdots\kappa(H_t).$$

*Proof.* (a) Recall that by Proposition 4.5 in [5], $\mathcal{P}(H_i)$ is an induced subgraph of $\mathcal{P}(G)$. Let $H := \bigcup_{i=1}^{t} H_i$. Now to each spanning tree $\bigcup_{i=1}^{t} T_{H_i}$ of $\bigcup_{i=1}^{t} \mathcal{P}(H_i)$, we associate a spanning tree

$$T_G = \bigcup_{i=1}^{t} T_{H_i} \cup \bigcup_{g \in G \setminus H} \{1, g\},$$

of $\mathcal{P}(G)$, which shows that $\kappa(G) \geqslant \kappa(H_1)\kappa(H_2)\cdots\kappa(H_t)$.

Let $x_i$ be a $p_i$-element of $G$ of order $m_i$ and $Q_i := \langle x_i \rangle$. Then, by Lemma 3.2, $\mathcal{P}(Q_i)$ is a complete graph of order $m_i$, and so Cayley formula implies that $\kappa(Q_i) = m_i^{m_i - 2}$. The result now follows by applying the first part.

(*b*) It is a straightforward verification. □

As immediate consequences of Theorem 3.4, we have the following four corollaries.

**Corollary 3.1** *Let $G = H_1 \times H_2 \times \cdots \times H_n$, where $n$ is a positive integer. Then*

$$\kappa(G) \geqslant \kappa(H_1)\kappa(H_2)\cdots\kappa(H_n).$$

**Corollary 3.2** *If $G = K \rtimes C$ is a semidirect product of $K$ by $C$ (especially, if $G$ is a Frobenius group with kernel $K$ and complement $C$), then*

$$\kappa(G) \geqslant \kappa(K)\kappa(C).$$

**Corollary 3.3** *Let $G$ be a finite group and let $p$ be the smallest prime such that $\kappa(G) < p^{p-2}$. Then $\pi(G) \subseteq \pi((p-1)!)$.*

Given a group $G$, we put $G^{\#} = G \setminus \{1\}$. A group $G$ is called *Element Prime Order* group if every nonidentity element of $G$ has prime order, i.e., $\omega(G) \setminus \{1\} = \pi(G)$. We can consider an EPO-group $G$ as follows:

$$G = \biguplus_{p\in\pi(G)} (\underbrace{\mathbb{Z}_p^{\#} \uplus \cdots \uplus \mathbb{Z}_p^{\#}}_{c_p\text{-times}}) \cup \{1\},$$

where $c_p$ signifies the number of cyclic subgroups of order $p$ in $G$, and hence

$$\mathcal{P}(G) = K_1 \vee \bigoplus_{p\in\pi(G)} c_p K_{p-1}.$$

**Corollary 3.4** *Let $G$ be an EPO-group. Then, there holds*

$$\kappa(G) = \prod_{p\in\pi(G)} p^{(p-2)c_p}.$$

In the same manner as in the proof of Theorem 3.4, we can prove the following theorem:

**Theorem 3.5** *Let $G$ be a group and let $\Omega_1, \Omega_2, \ldots, \Omega_t$ be cliques of $\mathcal{P}(G)$ such that $\Omega_i \cap \Omega_j = \{1\}$, for each $1 \leqslant i < j \leqslant t$. Then, there holds*

$$\kappa(G) \geqslant \prod_{i=1}^{t} |\Omega_i|^{|\Omega_i|-2} \geqslant \omega^{\omega-2},$$

*where $\omega = \omega(\mathcal{P}(G))$. In particular, we have*

$$\kappa(G) \geqslant \prod_{p\in\pi(G)} p^{(p-2)c_p},$$

*where $c_p$ signifies the number of cyclic subgroups of order $p$ in $G$.*

*Some Examples.* (*a*) The power graph $\mathcal{P}(G)$ of an elementary $p$-group $G$ of order $p^n$ consist of

$$(p^n - 1)/(p-1) = p^{n-1} + p^{n-2} + \cdots + p + 1,$$

cliques on $p$ vertices sharing the identity element. Now, using Corollary 3.4, we get

$$\kappa(G) = p^{\frac{p^n-1}{p-1}(p-2)}. \tag{1}$$

(*b*) If $G$ is a non-abelian group of order 21, then $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Therefore, using Corollary 3.2, $\kappa(G) \geqslant \kappa(\mathbb{Z}_7) \cdot \kappa(\mathbb{Z}_3) = 7^5 \cdot 3$. However, $G$ is an EPO-group with $\omega(G) = \{1, 3, 7\}$, and we have

$$\mathcal{P}(G) = K_1 \vee (7K_2 \oplus K_6),$$

for which we conclude that $\kappa(G) = 3^7 \cdot 7^5$.

(*c*) According to Theorem 11.3 in [14], a group is nilpotent if and only if it is the direct product of its Sylow subgroups. Hence, if $G$ is a nilpotent group, then

$$G \cong \prod_{p \in \pi(G)} G_p,$$

where $G_p \in \mathrm{Syl}_p(G)$. Now, by Corollary 3.1, we obtain

$$\kappa(G) \geqslant \prod_{p \in \pi(G)} \kappa(G_p).$$

In particular, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of a natural number $n$, where $k, \alpha_1, \ldots, \alpha_k$ are positive integers and $p_1, \ldots, p_k$ are distinct primes, then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}$, and again, by Corollary 3.1 and Cayley formula, we have:

$$\kappa(\mathbb{Z}_n) \geqslant \prod_{i=1}^{k} \kappa(\mathbb{Z}_{p_i^{\alpha_i}}) = \prod_{i=1}^{k} (p_i^{\alpha_i})^{p_i^{\alpha_i} - 2}.$$

In the next section, we will find an explicit formula for the tree-number $\kappa(\mathbb{Z}_n)$.

# 4 The Tree-number of $\mathcal{P}(\mathbb{Z}_n)$

Let $n$ be a natural number and let $G = \mathbb{Z}_n$ be a finite cyclic group of order $n$. Then for every divisor $d$ of $n$, there exists a unique subgroup of order $d$, and so the number of all elements of order $d$ in $G$ is equal to $\phi(d)$.

Therefore, if $d_1 > d_2 > \cdots > d_k$ are all divisors of $n$ (evidently $d_1 = n$ and $d_k = 1$), then we have:*

$$n = \sum_{i=1}^{k} \phi(d_i).$$

As we mentioned in Lemma 3.1, all non-trivial elements of $G$ with the same orders have the same degrees in $\mathcal{P}(G)$. Let $n_i$, $i = 1, 2, \ldots, k$, denote the degree of all elements of order $d_i$ in $\mathcal{P}(G)$. Let $\mathbf{A}$ be the adjacency matrix of $\mathcal{P}(G)$, and let $\mathbf{Q}$ be the Laplacian matrix. Then the matrix $\mathbf{J} + \mathbf{Q}$ has the following block-matrix structure:

$$\mathbf{J} + \mathbf{Q} = \begin{bmatrix} D_{11} & D_{12} & \ldots & D_{1k} \\ D_{21} & D_{22} & \ldots & D_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k1} & D_{k2} & \ldots & D_{kk} \end{bmatrix} \tag{2}$$

where $D_{ij}$ is a matrix of size $\phi(d_i) \times \phi(d_j)$ with

$$D_{ij} = \begin{cases} m_i \mathbf{I} & if \quad i = j, \\ 0 & if \quad i \neq j, \; d_i | d_j \text{ or } d_j | d_i, \\ \mathbf{J} & \quad \text{otherwise}, \end{cases}$$

where $m_i = n_i + 1$, $i = 1, 2, \ldots, k$. For instance, in the case when $G = \mathbb{Z}_{12}$, the matrix $\mathbf{J} + \mathbf{Q}$ is given by:

$$\left[\begin{array}{cccc|cc|cc|cc|c|c}
m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & m_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & m_2 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & m_2 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & 1 & m_3 & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & m_3 & 1 & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & m_4 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & m_4 & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & m_5 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & m_6
\end{array}\right]. \tag{3}$$

*It is well known that $n = \sum_{d|n} \phi(d)$.

In this section, we set

$$\lambda_i := \frac{m_i}{\phi(d_i)}, \ i = 2, \ldots, k-1, \quad \text{and} \quad \Phi := \prod_{i=2}^{k-1} \lambda_i.$$

To state our first result, we have to introduce a new definition. Actually, the *divisor graph* $D(n)$ of a natural number $n$ is defined as follows: the vertex set of this graph is $\pi_d(n)$, the set of all divisors of $n$, and two divisors $d_i$ and $d_j$ of $n$ are adjacent if and only if $d_i|d_j$ or $d_j|d_i$. As usual, we denote the complement of $D(n)$ by $\overline{D}(n)$. For instance, the subgraph $D(30)\backslash\{1,30\}$ of divisor graph $D(30)$ and its complement are depicted in Fig. 1.

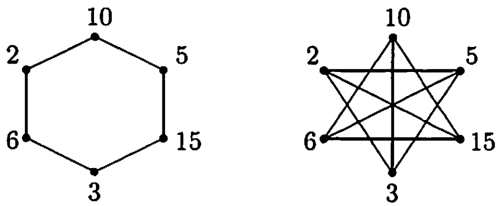


**Fig. 1.** The graph $D(30)\backslash\{1,30\}$ and its complement $\overline{D}(30)\backslash\{1,30\}$.

We are now ready to calculate the number of spanning trees of a power graph associated with a cyclic group of order $n$.

**Theorem 4.1** *Let $n$ be a positive integer and $d_1 > d_2 > \cdots > d_k$ all divisors of $n$. With the notation as explained above, there holds*

$$\kappa(\mathbb{Z}_n) = \prod_{i=1}^{k} m_i^{\phi(d_i)} \Big(\Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_{k-1}^{t_{k-1}}\Big) / (\Phi n^2), \qquad (4)$$

*where $t_i \in \{0,1\}$, $i = 2, 3, \ldots, k-1$, and the summation is over all induced subgraphs $\Lambda$ of $\overline{D}(n) \setminus \{d_1, d_k\}$ to the vertices $\{d_{i_1}, \ldots, d_{i_s}\}$ corresponding to $t_{i_1} = \cdots = t_{i_s} = 0$.*

*Remark* 4. At first glance, it seems that Eq. (4) is a very complicated formula. But, it is worth mentioning that in the right-hand side of Eq. (4), the orders of determinants $\det \mathbf{A}(\Lambda)$ are at most $k-2$, which is pretty small compared with the order of determinant $\det(\mathbf{J}+\mathbf{Q})$ needed to compute the tree-number $\kappa(\mathbb{Z}_n)$. After the proof of this theorem, we will present some examples to illustrate the effectiveness of this formula.

*Proof.* First, we illustrate the proof for a special case and then describe the necessary modifications in the general case. For instance, we consider the cyclic group $G = \mathbb{Z}_{12}$. In this situation, we have

$$n = 12,\ d_1 = 12,\ d_2 = 6,\ d_3 = 4,\ d_4 = 3,\ d_5 = 2,\ d_6 = 1.$$

Then the matrix $\mathbf{J} + \mathbf{Q}$ has the form as Eq. (3), and by expanding the determinant along the $i$-th row, $i = 1, 2, \ldots, \phi(d_1)$ and $n$, we obtain the equivalent expression

$$\det(\mathbf{J}+\mathbf{Q}) = m \cdot \det \left[\begin{array}{cc|cc|cc|c} m_2 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & m_2 & 1 & 1 & \cdot & \cdot & \cdot \\ \hline 1 & 1 & m_3 & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & m_3 & 1 & 1 & \cdot \\ \hline \cdot & \cdot & 1 & 1 & m_4 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & m_4 & 1 \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & 1 & m_5 \end{array}\right], \tag{5}$$

where $m = {m_1}^{\phi(d_1)} \cdot {m_6}^{\phi(d_6)}$.

In order to compute the new determinant on the right-hand side, denoted by $D$, we apply the following row and column operations: We subtract column $j$ from column $j+1$, $j = 1, 3, 5$, and subsequently we add row $i+1$ to row $i$, $i = 1, 3, 5$. It is not difficult to see that, stage by stage, the rows and columns are "emptied" until finally the following determinant

$$D = \prod_{i=2}^{5} {m_i}^{\phi(d_i)} \cdot \det \left[\begin{array}{cc|cc|cc|c} 1 & \cdot & \lambda_2^{-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{1}{m_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \lambda_3^{-1} & \cdot & 1 & \cdot & \lambda_3^{-1} & \cdot & \cdot \\ \frac{1}{m_3} & \cdot & \cdot & 1 & \frac{1}{m_3} & \cdot & \cdot \\ \hline \cdot & \cdot & \lambda_4^{-1} & \cdot & 1 & \cdot & \lambda_4^{-1} \\ \cdot & \cdot & \frac{1}{m_4} & \cdot & \cdot & 1 & \frac{1}{m_4} \\ \hline \cdot & \cdot & \cdot & \cdot & \lambda_5^{-1} & \cdot & 1 \end{array}\right]$$

is obtained. Expanding along the columns 2, 4 and 6, it follows that

$$D = \prod_{i=2}^{5} {m_i}^{\phi(d_i)} \cdot \det \begin{bmatrix} 1 & \lambda_2^{-1} & \cdot & \cdot \\ \lambda_3^{-1} & 1 & \lambda_3^{-1} & \cdot \\ \cdot & \lambda_4^{-1} & 1 & \lambda_4^{-1} \\ \cdot & \cdot & \lambda_5^{-1} & 1 \end{bmatrix}$$

Taking out the common factors $\lambda_i^{-1}$ of $i$-th row, $i = 1, 2, 3, 4$, we obtain

$$D = \Phi^{-1} \prod_{i=2}^{5} m_i{}^{\phi(d_i)} \cdot \det \begin{bmatrix} \lambda_2 & 1 & . & . \\ 1 & \lambda_3 & 1 & . \\ . & 1 & \lambda_4 & 1 \\ . & . & 1 & \lambda_5 \end{bmatrix}$$

$$= \Phi^{-1} \prod_{i=2}^{5} m_i{}^{\phi(d_i)} \cdot \left(\Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \lambda_4^{t_4} \lambda_5^{t_5}\right),$$

where $t_i \in \{0, 1\}$, $i = 2, 3, 4, 5$, and the summation is over all induced subgraphs $\Lambda$ of $\overline{D}(12) \setminus \{d_1, d_6\}$ to the vertices $\{d_{i_1}, \ldots, d_{i_s}\}$ corresponding to $t_{i_1} = \cdots = t_{i_s} = 0$. If this is substituted in Eq. (5) and the products are put together, then we obtain

$$\det(\mathbf{J} + \mathbf{Q}) = \Phi^{-1} \prod_{i=1}^{6} m_i^{\phi(d_i)} \cdot \left(\Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \lambda_4^{t_4} \lambda_5^{t_5}\right).$$

The final assertion immediately follows from Theorem 3.1. This works in general, as we now demonstrate.

As we mentioned already the matrix $\mathbf{J} + \mathbf{Q}$ has the form as Eq. (2), and by developing the determinant several times along the rows $1, 2, \ldots, \phi(d_1)$ and $n$, one gets

$$\det(\mathbf{J} + \mathbf{Q}) = m \cdot \det \begin{bmatrix} D_{22} & D_{23} & \cdots & D_{2,k-1} \\ D_{32} & D_{33} & \cdots & D_{3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k-1,2} & D_{k-1,3} & \cdots & D_{k-1,k-1} \end{bmatrix}, \tag{6}$$

where $m = m_1{}^{\phi(d_1)} \cdot m_k{}^{\phi(d_k)}$.

In what follows, $D$ denotes the new determinant on the right-hand side of Eq. (6). In order to compute this determinant, we apply the following row and column operations: We subtract column $j$ from column $j + r$:

$$\begin{cases} j = 1 + \sum_{l=2}^{h} \phi(d_l), \ h = 1, 2, \ldots, k-2, \\ r = 1, 2, \ldots, \phi(d_{h+1}) - 1, \end{cases}$$

and subsequently we add row $i + s$ to row $i$:

$$\begin{cases} i = 1 + \sum_{l=2}^{h} \phi(d_l), \ h = 1, 2, \ldots, k-2, \\ s = 1, 2, \ldots, \phi(d_{h+1}) - 1. \end{cases}$$

(Note that, when $m > n$, we assume that $\sum_{i=m}^{n} a_i = 0$). Using the above operations, it is easy to see that

$$D = \det \begin{bmatrix} M_{22} & M_{23} & \dots & M_{2,k-1} \\ M_{32} & M_{33} & \dots & M_{3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k-1,2} & M_{k-1,3} & \dots & M_{k-1,k-1} \end{bmatrix},$$

where $M_{ij}$ is a matrix of size $\phi(d_i) \times \phi(d_j)$ with

$$M_{ij} = \begin{cases} m_i \mathbf{I} & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j, \ \ d_i | d_j \text{ or } d_j | d_i, \\ \phi(d_i)\mathbf{E}_{1,1} + \mathbf{E}_{2,1} + \cdots + \mathbf{E}_{\phi(d_i),1} & \text{otherwise,} \end{cases}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{E}_{i,j}$ denotes the square matrix having 1 in the $(i,j)$ position and 0 elsewhere.

Therefore, taking out the common factors and developing the determinant along the columns $j$, with

$$j \neq 1 + \sum_{l=2}^{h} \phi(d_l), \quad h = 1, 2, \dots, k-2,$$

one gets

$$D = \Phi^{-1} \prod_{i=2}^{k-1} {m_i}^{\phi(d_i)} \cdot \det \begin{bmatrix} a_{22} & a_{23} & \dots & a_{2,k-1} \\ a_{32} & a_{33} & \dots & a_{3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1,2} & a_{k-1,3} & \dots & a_{k-1,k-1} \end{bmatrix}, \tag{7}$$

where

$$a_{ij} = \begin{cases} \lambda_i & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j, \ \ d_i | d_j \text{ or } d_j | d_i, \\ 1 & \text{otherwise.} \end{cases}$$

As the reader might have noticed, the following matrix

$$\begin{bmatrix} 0 & a_{23} & \dots & a_{2,k-1} \\ a_{32} & 0 & \dots & a_{3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1,2} & a_{k-1,3} & \dots & 0 \end{bmatrix},$$

is exactly the adjacency matrix of the graph $\Gamma = \overline{D}(n) \setminus \{d_1, d_k\}$. Consequently, we get

$$\det \begin{bmatrix} \lambda_2 & a_{23} & \cdots & a_{2,k-1} \\ a_{32} & \lambda_3 & \cdots & a_{3,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1,2} & a_{k-1,3} & \cdots & \lambda_{k-1} \end{bmatrix} = \Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_{k-1}^{t_{k-1}},$$

where $t_i \in \{0, 1\}$, $i = 2, 3, \ldots, k-1$, and the summation is over all induced subgraphs $\Lambda$ of $\Gamma$ to the vertices $\{d_{i_1}, \ldots, d_{i_s}\}$ corresponding to $t_{i_1} = \cdots = t_{i_s} = 0$. This is substituted in Eq. (7):

$$D = \Phi^{-1} \prod_{i=2}^{k-1} m_i{}^{\phi(d_i)} \cdot \Big( \Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_{k-1}^{t_{k-1}} \Big).$$

Again, if this is substituted in Eq. (6) and the sums are put together, then we obtain

$$\det(\mathbf{J} + \mathbf{Q}) = \Phi^{-1} \prod_{i=1}^{k} m_i{}^{\phi(d_i)} \cdot \Big( \Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_{k-1}^{t_{k-1}} \Big).$$

The required result now follows immediately from Theorem 3.1. □

A computer check has confirmed that our expression is correct for all values of $n$ up to 100 and so we are confident in the correctness of our results.

**Corollary 4.1** *Let $n > 2$ be an integer. Then $\kappa(\mathbb{Z}_n)$ is divisible by $n$.*

*Proof.* Using the above notation and Theorems 4.1, we can see that $\kappa(\mathbb{Z}_n)$ is divisible by

$$\frac{m_1{}^{\phi(d_1)} m_k{}^{\phi(d_k)}}{n^2} = \frac{n^{\phi(n)} n^{\phi(1)}}{n^2} = n^{\phi(n)-1},$$

and since $\phi(n) \geqslant 2$ for all $n > 2$, the result is proved. □

In what follows, we assume that

$$\gamma_i := \frac{n_i}{\phi(d_i)}, \quad i = 2, \ldots, k-1, \quad \text{and} \quad \Psi := \prod_{i=2}^{k-1} \gamma_i.$$

We will denote by $\mathcal{P}(G^{\#})$ the graph obtained by deleting the vertex 1 (the identity element of $G$) from $\mathcal{P}(G)$. For convenience, we will denote $\kappa(\mathcal{P}(G^{\#}))$ as $\kappa(G^{\#})$. In the same manner as in the proof of Theorem 4.1, we can prove the following theorem.

**Theorem 4.2** *Let $n$ be a positive integer and $d_1 > d_2 > \cdots > d_k$ all divisors of $n$. With the above notation, there holds*

$$\kappa(\mathbb{Z}_n^{\#}) = \prod_{i=1}^{k-1} n_i^{\phi(d_i)} \Big( \Psi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \gamma_2^{t_2} \gamma_3^{t_3} \cdots \gamma_{k-1}^{t_{k-1}} \Big) / (\Psi (n-1)^2), \quad (8)$$

*where $t_i \in \{0,1\}$, $i = 2,3,\ldots,k-1$, and the summation is over all induced subgraphs $\Lambda$ of $\overline{D}(n) \setminus \{d_1, d_k\}$ to the vertices $\{d_{i_1}, \ldots, d_{i_s}\}$ corresponding to $t_{i_1} = \cdots = t_{i_s} = 0$.*

*Some Examples.* (1) Suppose that $n = p^f$, where $p$ is a prime number and $f \in \mathbb{N}$. According to Lemma 3.2, the power graph $\mathcal{P}(\mathbb{Z}_n)$ is a complete graph and by Cayley result we obtain

$$\kappa(\mathbb{Z}_n) = n^{n-2}.$$

Alternatively, we may use Theorem 4.1. With the notation as in Theorem 4.1, we have

- $k = f + 1$ and $d_i = p^{f-i+1}$, $i = 1, 2, \ldots, f+1$;
- for every $i$, $m_i = n$;
- The divisor graph $D(n)$ is a complete graph, and so $\overline{D}(n)$ is a null graph.

Substituting the above expressions into Eq. (4), we obtain

$$\begin{aligned}
\kappa(\mathbb{Z}_n) &= \prod_{i=1}^{f+1} n^{\phi(p^{f-i+1})} \Big( \Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda) \lambda_2^{t_2} \lambda_3^{t_3} \cdots \lambda_f^{t_f} \Big) / (\Phi n^2) \\
&= n \prod_{i=1}^{f} n^{\phi(p^{f-i+1})} (\Phi + 0) / (\Phi n^2) \\
&= \frac{1}{n} \prod_{i=1}^{f} n^{p^{f-i}(p-1)} \\
&= \frac{1}{n} n^{(p-1) \sum_{i=1}^{f} p^{f-i}} = \frac{1}{n} n^{p^f - 1} = n^{n-2},
\end{aligned}$$

as required.

(2) Suppose that $n = pq$, where $p < q$ are primes. Once again, with the notation as in Theorem 4.1, we have

- $d_1 = n$, $d_2 = q$, $d_3 = p$ and $d_4 = 1$;
- By Lemma 3.1, we obtain

$$m_1 = n, \quad m_2 = n - p + 1, \quad m_3 = n - q + 1 \quad \text{and} \quad m_4 = n;$$

- The divisor graph $D(n) \setminus \{d_1, d_4\}$ is a null graph, and so $\overline{D}(n) \setminus \{d_1, d_4\}$ is a complete graph.

Substituting the above expressions into Eq. (4), we obtain

$$\begin{aligned}\kappa(\mathbb{Z}_n) &= \prod_{i=1}^{4} m_i{}^{\phi(d_i)}\Big(\Phi + \sum_{\Lambda} \det \mathbf{A}(\Lambda)\lambda_2^{t_2}\lambda_3^{t_3}\Big)/(\Phi n^2) \\ &= \prod_{i=1}^{3} m_i{}^{\phi(d_i)}(\Phi - 1)/(\Phi n) \\ &= n^{(p-1)(q-1)}(n-p+1)^{q-2}(n-q+1)^{p-2}(n-p-q+2).\end{aligned}$$

Similarly, substituting the above expressions into Eq. (8), we obtain

$$\begin{aligned}\kappa(\mathbb{Z}_n^{\#}) &= \prod_{i=1}^{3} n_i{}^{\phi(d_i)}\Big(\Psi + \sum_{\Lambda} \det \mathbf{A}(\Lambda)\lambda_2^{t_2}\lambda_3^{t_3}\Big)/(\Psi (n-1)^2) \\ &= \prod_{i=1}^{3} m_i{}^{\phi(d_i)}(\Psi - 1)/(\Psi (n-1)^2) \\ &= (n-1)^{(p-1)(q-1)-1}(n-p)^{q-2}(n-q)^{p-2}(n-p-q+1).\end{aligned}$$

In particular, we have

$$\kappa(\mathbb{Z}_{2p}) = \frac{1}{2}(2p)^p(2p-1)^{p-2} \quad \text{and} \quad \kappa(\mathbb{Z}_{2p}^{\#}) = \frac{1}{2}(2p-1)^{p-2}(2p-2)^{p-1},$$

where $p$ is an odd prime. For instance, if $n = 6$, then easy computations show that $\kappa(\mathbb{Z}_6) = 540$ and $\kappa(\mathbb{Z}_6^{\#}) = 40$. The power graph $\mathcal{P}(\mathbb{Z}_6)$ is depicted in Fig. 2.

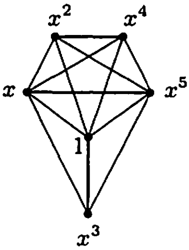


**Fig. 2.** The power graph $\mathcal{P}(\mathbb{Z}_6)$.

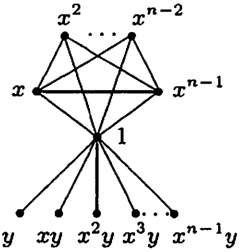


**Fig. 3.** The power graph $\mathcal{P}(D_{2n})$.

(3) Let $D_{2n}$ be the dihedral group of order $2n$, which is defined by

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1}\rangle.$$

The power graph $\mathcal{P}(D_{2n})$ is depicted in Fig. 3. It is easy to see that, the edges $\{1, x^i y\}$, $i = 0, 1, \ldots, n-1$, are cut-edges and by Theorem 3.2, we obtain $\kappa(D_{2n}) = \kappa(\mathbb{Z}_n)$.

We conclude this section with the following corollary, which derived from before observations.

**Corollary 4.2** *Suppose that $|G| = pq$, where $p$ and $q$ are primes (not necessarily distinct). Then one of the following possibilities must occurs:*

(a) $p = q$, $G \cong \mathbb{Z}_{p^2}$, *and* $\kappa(G) = p^{2(p^2-2)}$.

(b) $p = q$, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *and* $\kappa(G) = p^{(p+1)(p-2)}$.

(c) $p \neq q$, $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$, *and*

$$\kappa(G) = n^{(p-1)(q-1)}(n-p+1)^{q-2}(n-q+1)^{p-2}(n-p-q+2),$$

*where* $n = pq$.

(d) $q < p$, $p \equiv 1 \pmod{q}$, $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, *and* $\kappa(G) = p^{p-2}q^{p(q-2)}$.

# 5 The Tree-number of $\mathcal{P}(Q_{4n})$

In this section, we focus our attention on the generalized quaternion groups. Let $Q_{4n}$ denote the generalized quaternion group of order $4n$, which can be presented by

$$Q_{4n} = \langle x, y \mid x^{2n} = 1, y^2 = x^n, x^y = x^{-1} \rangle.$$

First of all, we recall that every element in $Q_{4n}$ can be written uniquely as $x^i y^j$ where $0 \leqslant i \leqslant 2n-1$ and $0 \leqslant j \leqslant 1$, and so the order of $Q_{4n}$ is exactly $4n$. Moreover, for a natural number $n$, the graph $\mathcal{P}(Q_{4n}^{\#})$ consists of the power graph $\mathcal{P}(\mathbb{Z}_{2n}^{\#})$ and $n$ triangles sharing a common vertex (i.e., $x^n$). Indeed $x^n$ is a *cut vertex* in $\mathcal{P}(Q_{4n}^{\#})$.

It is well known that the group $Q_{4n}$ has a unique minimal subgroup if and only if $n$ is a power of 2. Furthermore, in the case when $n$ is a power of 2, the graph $\mathcal{P}(Q_{4n}^{\#})$ has the following form:

$$\mathcal{P}(Q_{4n}^{\#}) = K_1 \vee (K_{2n-2} \oplus \underbrace{K_2 \oplus K_2 \oplus \cdots \oplus K_2}_{n\text{-times}}).$$

As a matter of fact, it consists of a complete graph on $2n-1$ vertices and $n$ triangles sharing a common vertex (the unique involution, i.e., $x^n$). The graph $\mathcal{P}(Q_{4n}^{\#})$, for the case $n$ is a power of 2, is depicted in Fig. 4. Note that, for every element $g \in Q_{4n} \setminus \{1\}$, the subgroup $\langle g \rangle$ contains the unique involution $x^n$, and so $x^n \sim g$ in $\mathcal{P}(Q_{4n}^{\#})$, hence in $\mathcal{P}(Q_{4n}^{\#})$ the unique involution $x^n$ is the *only* vertex of degree $4n-2$.

**Theorem 5.1** *If $n$ is a natural number, then the tree-number of the power graph $\mathcal{P}(Q_{4n}^{\#})$ is given by the formula*

$$\kappa(Q_{4n}^{\#}) = 3^n \cdot \kappa(\mathbb{Z}_{2n}^{\#}).$$

*Proof.* Let $\Gamma = \mathcal{P}(Q_{4n}^{\#})$. As we mentioned already, $\Gamma$ is a connected graph and contains $x^n$ as a cut vertex. Now, we consider the following set of vertices. Let

$$S_0 = \{x, x^2, \ldots, x^{2n-1}\},$$
$$S_i = \{yx^{i-1}, yx^{n+i-1}, x^n\}, \quad i = 1, 2, \ldots, n.$$

Then

$$\Gamma = \Gamma[S_0] + \Gamma[S_1] + \cdots + \Gamma[S_n],$$

in which $\Gamma[S_i] \cap \Gamma[S_j] = (\{x^n\}, \emptyset)$, $0 \leqslant i < j \leqslant n$. Notice that, we have $\Gamma[S_0] = \mathcal{P}(\mathbb{Z}_{2n}^{\#})$. Finally, from Theorem 3.3, it follows that

$$\begin{aligned} \kappa(\Gamma) = \prod_{i=0}^{n} \kappa(\Gamma[S_i]) &= \kappa(\Gamma[S_0]) \prod_{i=1}^{n} \kappa(\Gamma[S_i]) \\ &= \kappa(\mathbb{Z}_{2n}^{\#}) \prod_{i=1}^{n} \kappa(K_3) = \kappa(\mathbb{Z}_{2n}^{\#}) \cdot 3^n, \end{aligned}$$

as claimed. □

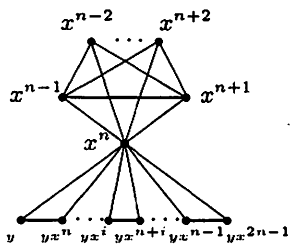


**Fig. 4.** The graph $\mathcal{P}(Q_{4n}^{\#})$.

As an immediate consequence of Theorem 5.1, Lemma 3.2 and Cayley's Theorem, we have the following corollary.

**Corollary 5.1** *If $n$ is a power of 2, then the tree-number of the power graph $\mathcal{P}(Q_{4n}^{\#})$ is given by the formula*

$$\kappa(Q_{4n}^{\#}) = 3^n \cdot (2n-1)^{2n-3}.$$

Now, we return to the power graph $\mathcal{P}(Q_{4n})$.

**Theorem 5.2** *If $n$ is a power of 2, then the tree-number of the power graph $\mathcal{P}(Q_{4n})$ is given by the formula*

$$\kappa(Q_{4n}) = 2^{5n-1} \cdot n^{2n-2}.$$

*Proof.* We follow an algebraic approach, again. In the sequel, for the sake of convenience we will consider the following labeling of the vertices of $\mathcal{P}(Q_{4n})$:

$$\begin{aligned} V &= \{x, x^2, \ldots, x^{n-1}, x^{n+1}, \ldots, x^{2n-1}, y, yx^n, yx, yx^{n+1}, \\ &\qquad \ldots, yx^i, yx^{n+i}, \ldots, yx^{n-1}, yx^{2n-1}, x^n, 1\}. \end{aligned}$$

Now, the Laplacian matrix $\mathbf{Q}$ is given by:

$$\mathbf{Q} = \left[\begin{array}{cccc|cc|c|cc|c|c}
\sigma & -1 & \cdots & -1 & \cdot & \cdot & \cdots & \cdot & \cdot & -1 & -1 \\
-1 & \sigma & \cdots & -1 & \cdot & \cdot & \cdots & \cdot & \cdot & -1 & -1 \\
\vdots & \vdots & \ddots & -1 & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & \cdots & \sigma & \cdot & \cdot & \cdots & \cdot & \cdot & -1 & -1 \\
\hline
\cdot & \cdot & \cdots & \cdot & 3 & -1 & \cdots & \cdot & \cdot & -1 & -1 \\
\cdot & \cdot & \cdots & \cdot & -1 & 3 & \cdots & \cdot & \cdot & -1 & -1 \\
\hline
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\hline
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & 3 & -1 & -1 & -1 \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & -1 & 3 & -1 & -1 \\
\hline
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & \tau & -1 \\
\hline
-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \tau
\end{array}\right],$$

where $\sigma = 2n - 1$ and $\tau = 4n - 1$, and hence we obtain

$$\mathbf{J} + \mathbf{Q} = \left[\begin{array}{cccc|cc|c|cc|c|c}
2n & \cdot & \cdots & \cdot & 1 & 1 & \cdots & 1 & 1 & \cdot & \cdot \\
\cdot & 2n & \cdots & \cdot & 1 & 1 & \cdots & 1 & 1 & \cdot & \cdot \\
\vdots & \vdots & \ddots & \cdot & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdots & 2n & 1 & 1 & \cdots & 1 & 1 & \cdot & \cdot \\
\hline
1 & 1 & \cdots & 1 & 4 & \cdot & \cdots & 1 & 1 & \cdot & \cdot \\
1 & 1 & \cdots & 1 & \cdot & 4 & \cdots & 1 & 1 & \cdot & \cdot \\
\hline
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\hline
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 4 & \cdot & \cdot & \cdot \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdot & 4 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 4n & \cdot \\
\hline
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 4n
\end{array}\right].$$

To compute $\det(\mathbf{J} + \mathbf{Q})$, expand $\det(\mathbf{J} + \mathbf{Q})$ with respect to its two last rows:

$$\det(\mathbf{J}+\mathbf{Q}) = (4n)^2 \cdot \det \left[\begin{array}{cccc|cc|c|cc}
2n & \cdot & \cdots & \cdot & 1 & 1 & \cdots & 1 & 1 \\
\cdot & 2n & \cdots & \cdot & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \cdot & \vdots & \vdots & \cdots & \vdots & \vdots \\
\cdot & \cdot & \cdots & 2n & 1 & 1 & \cdots & 1 & 1 \\
\hline
1 & 1 & \cdots & 1 & 4 & \cdot & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & \cdot & 4 & \cdots & 1 & 1 \\
\hline
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
1 & 1 & \cdots & 1 & 1 & 1 & \cdot & 4 & \cdot \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdot & \cdot & 4
\end{array}\right]. \quad (9)$$

Now, we apply the following row and column operations in the new determinant, which is denoted by $D$:

We subtract row 1 from row $i$, $i = 2, 3, \ldots, 2n-2$, and subsequently we subtract column 1 from column $j$, $j = 2, 3, \ldots, 2n-2$. It is not too difficult to see that, step by step, the rows and columns are "emptied" until finally the determinant

$$D = \det \left[\begin{array}{cccc|cc|c|cc}
2n & -2n & \cdots & -2n & 1 & 1 & \cdots & 1 & 1 \\
-2n & 4n & \cdots & 2n & \cdot & \cdot & \cdots & \cdot & \cdot \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-2n & 2n & \cdots & 4n & \cdot & \cdot & \cdots & \cdot & \cdot \\
\hline
1 & \cdot & \cdots & \cdot & 4 & \cdot & \cdots & 1 & 1 \\
1 & \cdot & \cdots & \cdot & \cdot & 4 & \cdots & 1 & 1 \\
\hline
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
1 & \cdot & \cdots & \cdot & 1 & 1 & \cdots & 4 & \cdot \\
1 & \cdot & \cdots & \cdot & 1 & 1 & \cdots & \cdot & 4
\end{array}\right],$$

is obtained. Again we subtract row 1 from row $i$, $i = 2n-1, 2n, \ldots, 4n-2$, and we obtain

$$D = \det \left[ \begin{array}{cccc|cc|c|cc} 2n & -2n & \cdots & -2n & 1 & 1 & \cdots & 1 & 1 \\ -2n & 4n & \cdots & 2n & . & . & \cdots & . & . \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -2n & 2n & \cdots & 4n & . & . & \cdots & . & . \\ \hline 1-2n & 2n & \cdots & 2n & 3 & -1 & \cdots & . & . \\ 1-2n & 2n & \cdots & 2n & -1 & 3 & \cdots & . & . \\ \hline \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 1-2n & 2n & \cdots & 2n & . & . & . & 3 & -1 \\ 1-2n & 2n & \cdots & 2n & . & . & . & -1 & 3 \end{array} \right].$$

In the following, $R_i$ and $C_j$ respectively designate the row $i$ and the column $j$ of a matrix. Applying $-\frac{1}{2}(R_{2n-1} + R_{2n} + \cdots + R_{4n-2}) \longrightarrow R_1$, to $D$, we conclude that

$$D = \det \left[ \begin{array}{cccc|cc|c|cc} \alpha & \beta & \cdots & \beta & . & . & \cdots & . & . \\ -2n & 4n & \cdots & 2n & . & . & \cdots & . & . \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -2n & 2n & \cdots & 4n & . & . & \cdots & . & . \\ \hline 1-2n & 2n & \cdots & 2n & 3 & -1 & \cdots & . & . \\ 1-2n & 2n & \cdots & 2n & -1 & 3 & \cdots & . & . \\ \hline \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 1-2n & 2n & \cdots & 2n & . & . & . & 3 & -1 \\ 1-2n & 2n & \cdots & 2n & . & . & . & -1 & 3 \end{array} \right],$$

where $\alpha = 2n^2 + n$ and $\beta = -2n - 2n^2$. Finally, by easy computations, we can argue as follows:

$$D = \det \begin{bmatrix} 2n^2 + n & -2n - 2n^2 & \cdots & -2n - 2n^2 \\ -2n & 4n & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ -2n & 2n & \cdots & 4n \end{bmatrix} \cdot 2^{3n}$$

$$(C_1 \longrightarrow C_j,\ j = 2, 3, \ldots, 2n-2.)$$

$$= \det \begin{bmatrix} 2n^2+n & -n & \cdots & -n \\ -2n & 2n & \cdots & . \\ \vdots & \vdots & \ddots & \vdots \\ -2n & . & \cdots & 2n \end{bmatrix} \cdot 2^{3n}$$

$$(\tfrac{1}{2}(R_2 + R_3 + \cdots + R_{2n-2}) \longrightarrow R_1)$$

$$= \det \begin{bmatrix} 4n & . & \cdots & . \\ -2n & 2n & \cdots & . \\ \vdots & \vdots & \ddots & \vdots \\ -2n & . & \cdots & 2n \end{bmatrix} \cdot 2^{3n}$$

$$= (4n)(2n)^{2n-3} \cdot 2^{3n} = 2^{5n-1} \cdot n^{2n-2}.$$

If this is substituted in Eq. (9), then we obtain

$$\det(\mathrm{J} + \mathrm{Q}) = (4n)^2 \cdot 2^{5n-1} \cdot n^{2n-2}.$$

Now, from Theorem 3.1, it follows that

$$\kappa(Q_{4n}) = \frac{\det(\mathrm{J} + \mathrm{Q})}{(4n)^2} = 2^{5n-1} \cdot n^{2n-2},$$

and the proof is complete. □

# 6 Some Applications

## 6.1 Groups With Small Tree-number

We begin with recalling a definition in Graph Theory. A *star* is a tree consisting of one vertex adjacent to all the others.

**Proposition 6.1** *Let $G$ be a finite group. Then the following statements are equivalent:*

(a) *$G$ is an elementary abelian 2-group.*

(b) *the power graph $\mathcal{P}(G)$ is a star graph.*

(c) $\kappa(G) = 1$.

*Proof.* We may assume that $|G| \geqslant 3$, since otherwise all the statements are trivially true.

(a) $\Rightarrow$ (b) Suppose that $G$ is an elementary abelian 2-group. If $x$ and $y$ are two distinct non-trivial elements, then $x \notin \langle y \rangle$ and $y \notin \langle x \rangle$, and so by the definition $x \nsim y$. This shows that the set of involutions of $G$ is an independent set in $\mathcal{P}(G)$. In other words, the power graph $\mathcal{P}(G)$ is a star graph.

(b) $\Rightarrow$ (c) It is a straightforward verification.

(c) $\Rightarrow$ (a) If $\kappa(G) = 1$, then certainly $\mathcal{P}(G)$ is a tree. We observe that if there is an element $x$ of order $o(x) \geqslant 3$, then $1 \sim x \sim x^2 \sim 1$ is a cycle in $\mathcal{P}(G)$ and this contradicts the fact that $\mathcal{P}(G)$ is a tree. Thus each element of $G \setminus \{1\}$ has order 2, and hence $G$ is an elementary abelian 2-group, as required. □

**Theorem 6.1** *Let $G$ be a nontrivial finite group. Then $\kappa(G) < 5^3$ if and only if one of the following occurs:*

(a) $\omega(G) = \{1, 2\}$ *and $G$ is an elementary abelian 2-group.*

(b) $\omega(G) = \{1, 3\}$ *and* $G \cong \mathbb{Z}_3$ *or* $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(c) $\omega(G) = \{1, 2, 3\}$ *and* $G \cong \mathbb{S}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, *or* $\mathbb{A}_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$

(d) $\omega(G) = \{1, 2, 4\}$ *and* $G \cong \mathbb{Z}_4$ *or* $D_8$.

*Proof.* We need only prove the necessity. Let $G$ be a finite group such that $\kappa(G) < 5^3$. Then, from Theorem 3.4, we have

$$125 > \kappa(G) \geqslant \prod_{p \in \pi(G)} p^{p-2},$$

which implies that $G$ is a $\{2, 3\}$-group with spectrum

$$\omega(G) \subseteq \{2^\alpha \cdot 3^\beta \mid \alpha \geqslant 0, \beta \geqslant 0\}.$$

Let $G_p \in \mathrm{Syl}_p(G)$, where $p \in \{2, 3\}$. If $\mu(G_2) \geqslant 8$ (resp. $\mu(G_3) \geqslant 9$), then by Theorem 3.4, $\kappa(G) \geqslant 8^6 > 125$ (resp. $\kappa(G) \geqslant 9^7 > 125$), which is again a contradiction. Hence, we conclude that $\omega(G) \subseteq \{1, 2, 3, 4, 6, 12\}$. If $G$ contains an element of order 6, say $x$, then $\Omega = \{1, x, x^2, x^4, x^5\}$ is a clique in $\mathcal{P}(G)$, and by Theorem 3.5, it follows that $\kappa(G) \geqslant 5^3$, which is a contradiction. This forces $\omega(G) \subseteq \{1, 2, 3, 4\}$. We now consider five cases separately.

*Case* 1. $\omega(G) = \{1, 2\}$. In this case, $G$ is an elementary abelian 2-group and by Proposition 6.1, $\kappa(G) = 1$.

*Case* 2. $\omega(G) = \{1,3\}$. In this case, $G$ is an elementary 3-group of order $3^n$, and by Eq. (1), we have $\kappa(G) = 3^{\frac{3^n-1}{2}}$. Consequently $3^{\frac{3^n-1}{2}} < 125$, which forces $n \leqslant 2$.

- If $n = 1$, then $G \cong \mathbb{Z}_3$ and $\kappa(G) = 3$.
- If $n = 2$, then $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\kappa(G) = 81$.

*Case* 3. $\omega(G) = \{1,2,3\}$. In this case, using a result of B. H. Neumann [13], $G = K \rtimes C$ is a Frobenius group with kernel $K$ and complement $C$, where either $K \cong \mathbb{Z}_3^t$, $C \cong \mathbb{Z}_2$ or $K \cong \mathbb{Z}_2^{2t}$, $C \cong \mathbb{Z}_3$.

*Subcase* 3.1. $G = \mathbb{Z}_3^t \rtimes \mathbb{Z}_2$. By Theorem 3.4 and Eq. (1), we have

$$125 > \kappa(G) \geqslant \kappa(\mathbb{Z}_3^t) = 3^{\frac{3^t-1}{2}},$$

which implies that $t \leqslant 2$. Thus $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2 \cong \mathbb{S}_3$ or $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. In the last case, we see that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is the only Sylow 3-subgroup of $G$, and thus $G \setminus \{1\}$ has 8 elements of order 3 and 9 element of order 2. Therefore, we have

$$\mathcal{P}(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) = K_1 \vee (3K_1 \oplus K_2) \quad \text{and} \quad \mathcal{P}((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) = K_1 \vee (9K_1 \oplus 4K_2),$$

and so $\kappa(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) = 3$ and $\kappa((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) = 3^4$.

*Subcase* 3.2. $G = \mathbb{Z}_2^{2t} \rtimes \mathbb{Z}_3$. Let $x_1, x_2, \ldots, x_m$ be elements of $G$ of order 3 such that $\langle x_i \rangle \cap \langle x_j \rangle = \{1\}$. Then, by Theorem 3.4, we deduce that

$$125 > \kappa(G) \geqslant \prod_{i=1}^{m} \kappa(\langle x_i \rangle) = 3^m,$$

which yields that $m \leqslant 4$. This means that, the group $G$ has at most 8 elements of order 3. The only group of this type is $G = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \cong \mathbb{A}_4$. Clearly, $\mathcal{P}(\mathbb{A}_4) = K_1 \vee (3K_1 \oplus 4K_2)$ and we have $\kappa(\mathbb{A}_4) = 3^4$.

*Case* 4. $\omega(G) = \{1,2,4\}$. In this case $G$ is a 2-group with exponent 4. Suppose that $G$ contains a pair of elements, say $x_1$ and $x_2$, of order 4 such that $\langle x_1 \rangle \cap \langle x_2 \rangle = \{1\}$. Then it follows by Theorem 3.4 that $\kappa(G) \geqslant \kappa(\langle x_1 \rangle) \cdot \kappa(\langle x_2 \rangle) = 4^2 \cdot 4^2 = 256$, which is a contradiction. Now assume that $x_1, x_2, \ldots, x_m$ are elements of order 4 such that $|\langle x_i \rangle \cap \langle x_j \rangle| = 2$. Then, $\Omega_i = \{1, x_i, x_i^3\}$, $i = 1, 2, \ldots, m-1$, and $\Omega_m = \langle x_m \rangle$ are cliques in $\mathcal{P}(G)$ with $\Omega_i \cap \Omega_j = \{1\}$, for each $1 \leqslant i < j \leqslant m$, and by Theorem 3.5, we deduce that

$$125 > \kappa(G) \geqslant \kappa(\langle x_m \rangle) \cdot \prod_{i=1}^{m-1} |\Omega_i|^{|\Omega_i|-2} = 4^2 \cdot 3^{m-1},$$

which forces $m \leqslant 2$. This means that, the group $G$ has at most 2 elements of order 4. The only groups $G$ with these conditions are: $\mathbb{Z}_4$ or $D_8$, and we have $\kappa(\mathbb{Z}_4) = \kappa(D_8) = 2^4$.

*Case* 5. $\omega(G) = \{1, 2, 3, 4\}$. Let $x$ and $y$ be two elements of order 3 and 4, respectively. If there exists another element $z$ of order 3 such that $\langle x\rangle \cap \langle z\rangle = 1$, then by Theorem 3.4 we have

$$125 > \kappa(G) \geqslant \kappa(\langle x\rangle) \cdot \kappa(\langle z\rangle) \cdot \kappa(\langle y\rangle) = 3 \cdot 3 \cdot 4^2 = 144,$$

which is a contradiction. Therefore, the Sylow 3-subgroup $G_3 \in \mathrm{Syl}_3(G)$ is a normal subgroup of $G$ of order 3. Moreover $\langle y\rangle$ acts fixed-point-freely by conjugation on $G_3$, and so $G_3 : \langle y\rangle$ is a Frobenius group. But then, we must have $|\langle y\rangle| \mid |G_3| - 1$, which is impossible. Thus, in this case there does not exist a candidate for $G$. This completes the proof. □

The next result is a simple consequence of Theorem 6.1.

**Corollary 6.1** *The following statements hold:*

(a) $h_{\mathcal{F}}(\mathbb{Z}_3) = h_{\mathcal{F}}(\mathbb{S}_3) = 2$.

(b) $h_{\mathcal{F}}(\mathbb{Z}_3 \times \mathbb{Z}_3) = h_{\mathcal{F}}((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) = h_{\mathcal{F}}(\mathbb{A}_4) = 3$.

(c) $h_{\mathcal{F}}(\mathbb{Z}_4) = h_{\mathcal{F}}(D_8) = 2$.

## 6.2 A New Characterization of $\mathbb{A}_5$

The following result is needed to prove Theorem 1.1.

**Lemma 6.1 (I. M. Isaacs)** *Let $G$ be a finite non-abelian simple group and let $p$ be a prime dividing the order of $G$. Then $G$ has at least $p^2 - 1$ elements of order $p$, or equivalently, there is at least $p + 1$ cyclic subgroups of order $p$ in $G$.*

*Proof.* Let $U$ be a subgroup of order $p$ contained in the center of a Sylow $p$-subgroup of $G$, and let $N = N_G(U)$. Since $G$ is non-abelian simple, $U$ is not normal in $G$ and thus $N < G$. Write $n = |G : N|$. By the elementary "$n$ factorial theorem" we know that $|G : \mathrm{core}_G(N)|$ divides $n!$, and since $G$ is simple and $\mathrm{core}_G(N) \leqslant N < G$, we have $\mathrm{core}_G(N) = 1$. Thus $|G|$, and so $p$ divides $n!$, which forces $n \geqslant p$. But $N$ contains a full Sylow $p$-subgroup of $G$ so $n$ is not $p$, and thus $n \geqslant p + 1$. Then $G$ contains at least $p + 1$ conjugates of $U$. □

*Proof of Theorem 1.1.* Let $G$ be a finite simple group with $\kappa(G) = \kappa(\mathbb{A}_5)$. First of all, we recall, $\mathbb{A}_5 \setminus \{1\}$ has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2, and hence $\omega(\mathbb{A}_5) = \{1, 2, 3, 5\}$. Thus, the power graph $\mathcal{P}(\mathbb{A}_5)$ is the union of complete subgraphs $K_5$, $K_3$ and $K_2$ with exactly one common vertex, i.e., the identity element 1, or equivalently

$$\mathcal{P}(\mathbb{A}_5) = K_1 \vee (15K_1 \oplus 10K_2 \oplus 6K_4).$$

Now, by Corollary 3.4, we have

$$k(\mathbb{A}_5) = \kappa(K_2)^{15}\kappa(K_3)^{10}\kappa(K_5)^6 = (2^0)^{15}(3^1)^{10}(5^3)^6 = 3^{10}5^{18}.$$

Therefore, $G$ is a finite simple group with $\kappa(G) = 3^{10}5^{18}$. Clearly, $G$ is non-abelian, since otherwise $G \cong \mathbb{Z}_p$ for some prime $p$, and so $\kappa(G) = \kappa(\mathbb{Z}_p) = p^{p-2}$, which is a contradiction. Now, we claim that $\pi(G) = \{2, 3, 5\}$. Since $\kappa(G) = 3^{10}5^{18} < 17^{15}$, from Corollary 3.3, it follows that

$$\pi(G) \subseteq \pi(16!) = \{2, 3, 5, 7, 11, 13\}.$$

Suppose now that $p \in \pi(G)$ and $p \geqslant 7$. Let $s_p$ be the number of elements of order $p$. Then, $s_p = c_p\phi(p) = c_p(p-1)$, where $c_p$ is the number of cyclic subgroups of order $p$ in $G$. By Lemma 6.1, $c_p \geqslant p+1$, because $G$ is a non-abelian simple group. Therefore, from Theorem 3.4 $(a)$, we deduce that

$$\kappa(G) \geqslant \kappa(\mathbb{Z}_p)^{p+1} = p^{(p-2)(p+1)} \geqslant 7^{40} > 3^{10}5^{18},$$

which is a contradiction. Our claim follows.

By results collected in [18, Table 1], $G$ is isomorphic to one of the groups $\mathbb{A}_5$, $\mathbb{A}_6$ or $U_4(2)$. In all cases, $G$ does not contain an element of order 15 (see [6]). Hence, a Sylow 3-subgroup $G_3$ acts fixed point freely on the set of elements of order 5 by conjugation, and so $|G_3|$ divides $s_5 = c_5\phi(5) = 4c_5$. Now, if $|G_3| \geqslant 9$, then $c_5 \geqslant 9$, and so

$$\kappa(G) > \kappa(\mathbb{Z}_5)^9 = (5^3)^9 > 3^{10}5^{18},$$

which is a contradiction. Therefore, $|G_3| = 3$, which forces $G \cong \mathbb{A}_5$. This completes the proof. □

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## 6.3 Appendix

Using the obtained results in previous sections we can compute the tree-numbers $\kappa(G)$ and $\kappa(G^{\#})$ of a finite group $G$, with $|G| \leqslant 15$. These results are shown below in Table 1. In this table, $M$ denotes a non-abelian group of order 12 with the following presentation:

$$M = \langle x, y \mid x^4 = y^3 = 1, yx = xy^2 \rangle.$$

**Table 1.** *The tree-numbers of small finite groups*

| $n$ | Group of order $n$ | $\kappa(G)$ | $\kappa(G^{\#})$ |
|---|---|---|---|
| 1 | 1 | 1 | – |
| 2 | $\mathbb{Z}_2$ | 1 | 1 |
| 3 | $\mathbb{Z}_3$ | 3 | 1 |
| 4 | $\mathbb{Z}_4$ | $2^4$ | 3 |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | 1 | 0 |
| 5 | $\mathbb{Z}_5$ | $5^3$ | $2^4$ |
| 6 | $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ | $2^2 \cdot 3^3 \cdot 5$ | $2^3 \cdot 5$ |
| | $\mathbb{S}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ | 3 | 0 |
| 7 | $\mathbb{Z}_7$ | $7^5$ | $2^4 \cdot 3^4$ |
| 8 | $\mathbb{Z}_8$ | $2^{18}$ | $7^5$ |
| | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $2^6 \cdot 3$ | 0 |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 1 | 0 |
| | $D_8$ | $2^4$ | 0 |
| | $Q_8$ | $2^{11}$ | $3^3$ |
| 9 | $\mathbb{Z}_9$ | $3^{14}$ | $2^{18}$ |
| | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | $3^4$ | 0 |
| 10 | $\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2$ | $2^4 \cdot 3^6 \cdot 5^5$ | $2^{11} \cdot 3^6$ |
| | $D_{10} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ | $5^3$ | 0 |
| 11 | $\mathbb{Z}_{11}$ | $11^9$ | $2^8 \cdot 5^8$ |
| 12 | $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ | $2^{14} \cdot 3^6 \cdot 5 \cdot 131$ | $2^4 \cdot 3^2 \cdot 7 \cdot 11^3 \cdot 173$ |
| | $\mathbb{Z}_6 \times \mathbb{Z}_2$ | $2^6 \cdot 3^5 \cdot 5^2 \cdot 17$ | $2^8 \cdot 5^3$ |
| | $\mathbb{A}_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ | $3^4$ | 0 |
| | $D_{12} \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2$ | $2^2 \cdot 3^3 \cdot 5$ | 0 |
| | $M$ | $2^{11} \cdot 3^2 \cdot 5 \cdot 7$ | $2^3 \cdot 3^3 \cdot 5$ |
| 13 | $\mathbb{Z}_{13}$ | $13^{11}$ | $2^{20} \cdot 3^{10}$ |
| 14 | $\mathbb{Z}_{14} \cong \mathbb{Z}_7 \times \mathbb{Z}_2$ | $2^6 \cdot 7^7 \cdot 13^5$ | $2^{11} \cdot 3^6 \cdot 13^5$ |
| | $D_{14} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_2$ | $7^5$ | 0 |
| 15 | $\mathbb{Z}_{15} \cong \mathbb{Z}_5 \times \mathbb{Z}_3$ | $3^{10} \cdot 5^8 \cdot 11 \cdot 13^3$ | $2^{17} \cdot 3^3 \cdot 5 \cdot 7^7$ |

# References


[1] J. Abawajy, A. Kelarev and M. Chowdhury. Power graphs: a survey, *Electron. J. Graph Theory Appl.*, **1** (2) (2013), 125–147.

[2] S. Abe and N. Iiyori. A generalization of prime graphs of finite groups, *Hokkaido Math. J.*, **29** (2) (2000), 391–407.

[3] N. Biggs. *Algebraic Graph Theory*, Second Edition. Cambridge University Press, Cambridge, 1993.

[4] A. Cayley. A theorem on trees, *Quart. J. Math.*, **23** (1889), 376–378.

[5] I. Chakrabarty, S. Ghosh and M. K. Sen. Undirected power graphs of semigroups, *Semigroup Forum*, **78** (3)(2009), 410–426.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson. *Atlas of Finite Groups*, Clarendon Press, oxford, 1985.

[7] S. Dolfi. On independent sets in the class graph of a finite group, *J. Algebra*, 303 (1)(2006), 216–224.

[8] A. V. Kelarev and S. J. Quinn. A combinatorial property and power graphs of groups, *Contributions to general algebra*, 12 (1999), 229–235.

[9] A. V. Kelarev and S. J. Quinn. Directed graph and combinatorial properties of semigroups, *J. Algebra*, **251** (1)(2002), 16–26.

[10] C. Krattenthaler. Advanced determinant calculus: a complement, *Linear Algebra Appl.*, 411 (2005), 68–166.

[11] A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi. On the noncommuting graph associated with a finite group, *Siberian Math. J.*, **46** (2)(2005), 325-332.

[12] A. R. Moghaddamfar, S. Rahbariyan and W. J. Shi. Certain properties of the power graph associated with a finite group, *J. Algebra Appl.*, **13** (7)(2014), 1450040 (18 pages).

[13] B. H. Neumann. Groups whose elements have bounded orders, *J. London Math. Soc.*, **S1-12** (3)(1937), 195–198.

[14] J. S. Rose. *A Course on Group Theory*, Cambridge University Press, 1978.

[15] Y. Segev and G. Seitz. Anisotropic groups of type $A_n$ and the commuting graph of finite simple groups, *Pacific J. Math.*, 202 (1)(2002), 125–225.

[16] H. N. V. Temperley. On the mutual cancellation of cluster integrals in Mayer's fugacity series, *Proc. Phys. Soc.*, 83 (1964), 3–16.

[17] D. B. West. *Introduction to Graph Theory*, Second Edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.

[18] A. V. Zavarnitsine. Finite simple groups with narrow prime spectrum, *Sib. Elektron. Mat. Izv.*, **6** (2009), 1–12.