A study on degree characterization for hamiltonian-connected graphs*

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Abstract

Consider any undirected and simple graph $G = (V, E)$, where V and E denote the vertex set and the edge set of G, respectively. Let $|G| = |V| = n \geq 3$. The well-known Ore's theorem states that if $\deg_G(u) + \deg_G(v) \geq n + k$ holds for each pair of nonadjacent vertices u and v of G, then G is traceable for $k = -1$, hamiltonian for $k = 0$ and hamiltonian-connected for $k = 1$. In this paper, we investigate any graph G with $\deg_G(u) + \deg_G(v) \geq n - 1$ for any nonadjacent vertex pair $\{u, v\}$ of G in particular. We call it the $(*)$ condition. We derive four graph families, $\mathcal{H}_i$, $1 \leq i \leq 4$, and prove that all graphs satisfying $(*)$ are hamiltonian-connected unless $G \in \mathcal{H}_i$ for some i. We also establish a comprehensive theorem for G satisfying $(*)$, which shows that G is traceable, hamiltonian, pancyclic or hamiltonian-connected unless G belongs to different subsets of $\{\mathcal{H}_i \mid 1 \leq i \leq 4\}$, respectively.

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1 Introduction

For the graph definitions and notations we follow [3], and we consider finite, undirected and simple graphs only. $G = (V, E)$ is a *graph* if V is a finite set and $E \subseteq \{(u, v) \mid (u, v)$ is an unordered pair of $V\}$, where V is called the *vertex set* of G and E the *edge set* of G. We use $|G|$ or $|V|$ for the number of distinct vertices in G. Two vertices u and v are *adjacent*, denoted by $u \sim v$, if $(u, v) \in E$. Given a vertex u of G, the *neighborhood* of u, denoted by $N_G(u)$, is the set $\{v \mid (u, v) \in E\} \subseteq V$. The *degree* of u, denoted by $\deg_G(u)$, is the total number of elements of $N_G(u)$. Let S be a subgraph of G. Define two symbols $N_S(u) = N_G(u) \cap V(S)$, and $\deg_S(u) = |N_S(u)|$. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, v_2, \cdots, v_k \rangle$, where all vertices are different except possibly for $v_0 = v_k$ and every two consecutive vertices are connected by an edge. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, \ldots, v_i, P', v_j, v_{j+1}, \ldots, v_k \rangle$, where P' denotes the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A graph G is *traceable* if it contains a hamiltonian path. A graph G is *connected* if there is a path between any two distinct vertices in G and is *hamiltonian-connected* if there is a hamiltonian path between any two distinct vertices in G. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. We say that G is a *hamiltonian* graph if G contains a hamiltonian cycle. The total number of vertices within a path P is denoted by $|P|$. A graph G is *pancyclic* if for every given integer l with $3 \leq l \leq |G| = n$, there exists a cycle with l distinct vertices in G. Obviously, a hamiltonian graph is traceable, but a traceable graph might not be hamiltonian. Moreover, except for K_2, a hamiltonian-connected/pancyclic graph is a hamiltonian graph, but the converse is not true. See [6, 7] for the relationship among various hamiltonian properties, and [4, 10, 11] for certain most recent studies.

The following well-known theorems set up the corresponding degree-sum conditions for any graph to be traceable, hamiltonian, hamiltonian-connected or pancyclic.

Theorem 1. *(Ore, 1960/63, [12, 13]) Let* $G = (V, E)$ *be a graph with* $|G| = n \geq 3$. *If* $\deg_G(u) + \deg_G(v) \geq n + k$ *for any pair of nonadjacent vertices* u, v *in* V, *then* G *is traceable for* $k = -1$, *hamiltonian for* $k = 0$, *and hamiltonian-connected for* $k = 1$.

Theorem 2. *(Bondy, 1971, [2]) Let $G = (V, E)$ be a graph with $|G| = n \geq 3$. If $\deg_G(u) + \deg_G(v) \geq n$ for any pair of nonadjacent vertices u, v in V, then either G is pancyclic or G is the complete bipartite graph $K_{n/2,n/2}$, where n is an even integer.*

Consider a graph $G = (V, E)$ and two vertices u and v of G. Let $\delta(u, v)$ denote the distance between u and v, which is the total number of edges of a shortest path between u and v, and $\text{diam}(G)$ denote the diameter of G, defined by $\max\{\delta(u, v) \mid u, v \in V\}$. It is easy to see that any graph G satisfying the condition(s) in Theorem 1 must be with $\text{diam}(G) \leq 2$. In addition, the conditions in Theorems 1 and 2 are tight in the sense that under the weakened degree-sum condition, there exist graphs which fail to possess the corresponding hamiltonian property [2, 8]. Nonetheless, when a slight change on the degree condition occurs, researchers are interested in characterizing graphs which perform differently and learning why some graphs lose the original hamiltonicity while others endure well. Some related studies are presented below as a contrast to the above two theorems.

For this purpose, we define some notations on graphs first. Let G_1 and G_2 be two graphs. We say that G_1 and G_2 are *disjoint* if G_1 and G_2 have no vertex in common. The *union* of two disjoint graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. By kG we mean the disjoint union of k copies of the graph G, and I_n denotes nK_1. The *join* of two disjoint subgraphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2. Note that Theorem 4 is adapted for graphs with $\text{diam}(G) \leq 2$.

Theorem 3. *(Aldred et al., 1994, [1]) Let $G = (V, E)$ be a simple graph with $|G| = n \geq 3$. If $\deg_G(u) + \deg_G(v) \geq n - 1$ holds for each pair of nonadjacent vertices u, v in V, then either G is pancyclic or G is isomorphic to one of the following four graphs: (i) $K_1 \vee (K_s + K_t)$ with $s + t = n - 1$; (ii) a subgraph of $K_{(n-1)/2} \vee I_{\frac{n+1}{2}}$; (iii) $K_{n/2,n/2}$; (iv) a cycle with 5 vertices.*

Theorem 4. *(Li, 2006, [9]) Let G be a connected graph with $|G| = n \geq 3$. If $\deg_G(u) + \deg_G(v) \geq n - 1$ holds for any pair of nonadjacent vertices u, v in G, then either G is hamiltonian or G belongs to one of the following two families: (i) $K_1 \vee (K_s + K_t)$ with $s + t = n - 1$; (ii) $\{G \mid K_{p,p+1} \subseteq G \subseteq K_p \vee I_{p+1}\}$ with $n = 2p + 1$.*

Theorem 5. *(Shih et al., 2012, [14]) Let $G = (V, E)$ be a simple graph with $|G| = n \geq 3$. Let H_i be any simple graph with i vertices. If $\deg_G(u) + \deg_G(v) \geq n$ holds for each pair of nonadjacent vertices u, v in V, then either G is hamiltonian-connected or G belongs to one of the two families:*

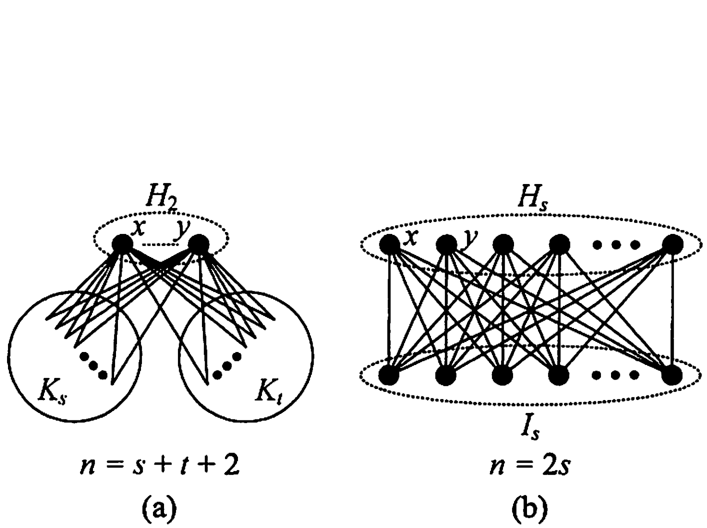

Figure 1: An illustration of graphs of (a) $\mathcal{G}_1$; (b) $\mathcal{G}_2$.

(i) $\mathcal{G}_1 = \{H_2 \vee (K_s + K_t) \mid s, t \geq 1$ *and* $s + t = n - 2\}$; *(ii)* $\mathcal{G}_2 = \{H_s \vee I_s \mid 2s = n\}$.

Graphs in $\mathcal{G}_i$ for $i \in \{1, 2\}$, which appear in Theorem 5, are illustrated in Figure 1. It is easy to show that graphs in $\mathcal{G}_1 \cup \mathcal{G}_2$ satisfy the degree-sum condition in Theorem 5, but there exists no hamiltonian path between x and y. It is also interesting to observe the similarities among the graph families independently obtained in Theorem 3, 4, and 5. Inspired by these works, we intend to relax the condition in Theorem 5 and find out all exceptional graph types. In Section 2, we present our main theorem, Theorem 6, and the proof. A brief conclusion is given in Section 3.

2 Main results

Definition 1. *Let H_i be any simple graph with i vertices, and K_t a complete graph with t vertices for $t \geq 1$. The symbol K_t^- denotes the graph obtained by removing certain edges from K_t such that $\deg_{K_t^-}(v) \geq t - 2$ for all $v \in V(K_t^-)$. We define four graph families, $\{\mathcal{H}_i \mid 1 \leq i \leq 4\}$, in which each graph consists of n vertices with $n \geq 3$.*

- $\mathcal{H}_1$ *contains* $\{G \mid K_{r,r+1} \subseteq G \subseteq (K_r \vee (K_2 + I_{r-1})), \deg_G(u) + \deg_G(v) \geq n - 1$ *for any pair of nonadjacent vertices u and v of* $G\}$. *More specifically, $\mathcal{H}_1$ consists of two types of graphs: (a)* $\mathcal{H}_1^1 = \{H_r \vee I_{r+1} \mid n = 2r+1\}$ *and (b)* $\mathcal{H}_1^2 = \{H_r \vee^* (K_2 + I_{r-1}) \mid n = 2r+1\}$, *where $\vee^*$ is the same as $\vee$, and further, we allow the situation when* $a, b \in V(K_2)$, $|N_{H_r}(a)| \geq r - 1$, $|N_{H_r}(b)| \geq r - 1$, *and* $|N_{H_r}(a) \cup N_{H_r}(b)| = r$.

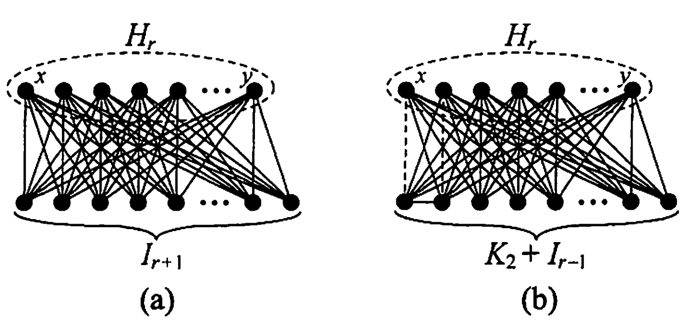

Figure 2: An illustration of graphs of $\mathcal{H}_1$: (a) $\mathcal{H}_1^1 = \{H_r \vee I_{r+1} \mid n = 2r+1\}$ and (b) $\mathcal{H}_1^2 = \{H_r \vee^* (K_2 + I_{r-1}) \mid n = 2r+1\}$.

- *$\mathcal{H}_2$ contains $\{G \mid (K_{t_1} \vee I_2 \vee^- K_{t_2}^-) \subseteq G \subseteq (K_{t_1} \vee K_2 \vee^- K_{t_2})$, $\deg_G(u)+\deg_G(v) \geq n-1$ for any pair of nonadjacent vertices u and v of $G\}$. More specifically, let $V(K_2)$ or $V(H_2)$ be $\{x, y\}$. $\mathcal{H}_2$ consists of three types of graphs: $\mathcal{H}_2^1 = \{K_{t_1} \vee H_2 \vee^- K_{t_2}^- \mid t_1 < t_2, n = t_1+t_2+2\}$; $\mathcal{H}_2^2 = \{K_{t_1} \vee K_2 \vee^- K_{t_2} \mid t_1 \geq t_2, n = t_1+t_2+2$, either $N_{K_{t_2}}(x) = \emptyset$ or $N_{K_{t_2}}(y) = \emptyset\}$; $\mathcal{H}_2^3 = \{K_{t_1} \vee H_2 \vee^- K_{t_2} \mid t_1 \geq t_2, n = t_1+t_2+2$, both $|N_{K_{t_2}}(x)| \geq 1$ and $|N_{K_{t_2}}(y)| \geq 1\}$. Here $G_1 \vee^- G_2$ is the same as $G_1 \vee G_2$ except that for $v \in G_2$ with $\deg_{G_2}(v) = |G_2|-1$, at most one edge between v and G_1 can be removed such that $|N_{G_1}(v)| \geq |G_1|-1$.*
- *$\mathcal{H}_3$ consists of two types of graphs: $\mathcal{H}_3^1 = \{H_r^+ \vee' I_r \mid n = 2r\}$ and $\mathcal{H}_3^2 = \{H_r \vee I_r \mid n = 2r\}$. Here H_r^+ denotes a simple graph consisting of r vertices and at least one edge, and $H_r^+ \vee' I_s$ is the same as $H_r^+ \vee I_s$ except that for only one vertex $y \in H_r^+$ with $\deg_{H_r^+}(y) \geq 1$, exactly one edge between y and I_s is removed.*
- *$\mathcal{H}_4 = \{H_3 \vee (K_1 + 2K_2)\} \cup \{H_3 \vee 3K_2\}$.*

Illustrations of all graphs in the $\mathcal{H}_i$s are shown in Figures 2, 3, 4, and 5. Obviously, each of them satisfies $\deg_G(u)+\deg_G(v) \geq n-1$ for any pair of nonadjacent vertices u, v and there exists no hamiltonian path between x and y. The main theorem is stated below.

Theorem 6. *Let $G = (V, E)$ be a simple graph with $|G| = n \geq 3$ such that $\deg_G(u) + \deg_G(v) \geq n-1$ holds for each pair of nonadjacent vertices u and v in V. Then either G is hamiltonian-connected or G belongs to one of the four families $\mathcal{H}_i$ with $i \in \{1, 2, 3, 4\}$.*

To prove Theorem 6, we make the following assumptions. Let $G = (V, E)$ be a simple graph with $|G| = n \geq 3$ and G satisfy the degree-sum condition in Theorem 6. Let u and v be two arbitrary vertices in G

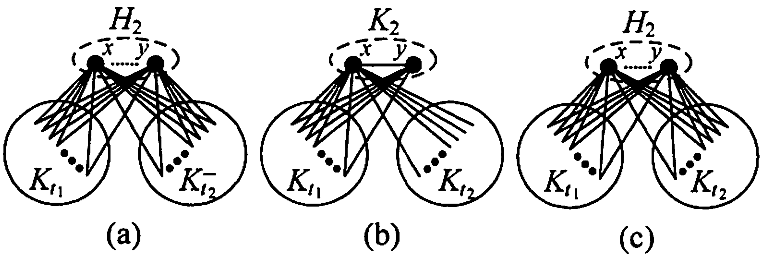

Figure 3: An illustration of graphs of $\mathcal{H}_2$: (a) $\mathcal{H}_2^1 = \{K_{t_1} \vee H_2 \vee^- K_{t_2}^- \mid t_1 < t_2, n = t_1 + t_2 + 2\}$; (b) $\mathcal{H}_2^2 = \{K_{t_1} \vee K_2 \vee^- K_{t_2} \mid t_1 \geq t_2, n = t_1 + t_2 + 2$, either $N_{K_{t_2}}(x) = \emptyset$ or $N_{K_{t_2}}(y) = \emptyset\}$; (c) $\mathcal{H}_2^3 = \{K_{t_1} \vee H_2 \vee^- K_{t_2} \mid t_1 \geq t_2, n = t_1 + t_2 + 2$, both $|N_{K_{t_2}}(x)| \geq 1$ and $|N_{K_{t_2}}(y)| \geq 1\}$.

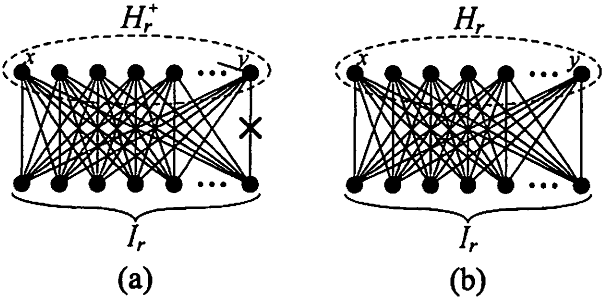

Figure 4: An illustration of graphs of $\mathcal{H}_3$: (a) $\mathcal{H}_3^1 = \{H_r^+ \vee' I_r \mid n = 2r\}$ and (b) $\mathcal{H}_3^2 = \{H_r \vee I_r \mid n = 2r\}$.

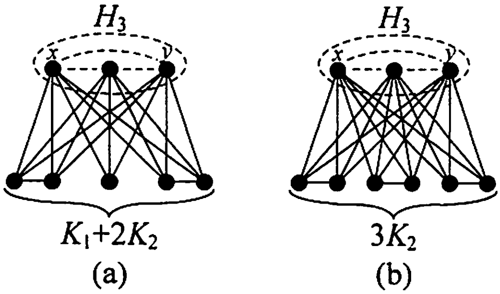

Figure 5: An illustration of graphs of $\mathcal{H}_4$: (a)$\mathcal{H}_4^1 = \{H_3 \vee (K_1 + 2K_2)\}$ and (b) $\mathcal{H}_4^2 = \{H_3 \vee 3K_2\}$.

and P the longest path between u and v in G. If $|P| = n$, then P is a hamiltonian path. It suffices to verify the case when $|P| \le n-1$. In the following discussion, let $P = \langle u = u_1, u_2, \ldots, u_p = v\rangle$ with $p \le n-1$, and $S = G - P$. We define two symbols $\uparrow$ and $\downarrow$ as follows. For $i \le j$, $u_i \uparrow u_j$ denotes the path $\langle u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_j\rangle$; for $i \ge j$, $u_i \downarrow u_j$ denotes the path $\langle u_i, u_{i-1}, u_{i-2}, \ldots, u_{j+1}, u_j\rangle$. For any vertex x of V, the symbol $x \sim S$ means there exists some vertex y in S such that $x \sim y$. By $x \nsim S$, we mean there exists no vertex y in S such that $x \sim y$. For simplicity, let $|G| = n$, $|S| = s$, $|P| = p$, $n = p + s$ and $|N_P(S)| = m$.

Proposition 1. *(1) Let u_i be a vertex in P, where $1 \le i \le p$, such that $u_i \sim x$ for some vertex x in S. If $i \le p-1$, then $u_{i+1} \nsim x$; if $i \ge 2$, then $u_{i-1} \nsim x$. (2) Let u_i and u_j be two distinct vertices in P and x be a vertex in S such that $u_i \sim x$ and $u_j \sim x$, where $1 \le i < j \le p$ and $j - i \ge 2$. If $j \le p-1$, then $u_{i+1} \nsim u_{j+1}$; if $i \ge 2$, then $u_{i-1} \nsim u_{j-1}$.*

Proof. (1) For $i \le p-1$, assume the opposite by letting $u_{i+1} \sim x$, then $\langle u = u_1 \uparrow u_i, x, u_{i+1} \uparrow u_p = v\rangle$ is a path longer than P. It contradicts the assumption that P is the longest path between u and v. Therefore, $u_{i+1} \nsim x$. The proof for the case with $i \ge 2$ is similar.
(2) For $j \le p-1$, assume the opposite by letting $u_{i+1} \sim u_{j+1}$. Thus $\langle u = u_1 \uparrow u_i, x, u_j \downarrow u_{i+1}, u_{j+1} \uparrow u_p = v\rangle$ is a path longer than P, which leads to a contradiction. The proof for the case with $i \ge 2$ is similar. □

The following proposition is derived with the same reasoning as in Proposition 1.

Proposition 2. *Suppose that S is a connected graph. (1) Let u_i be a vertex in P, where $1 \le i \le p$, such that $u_i \sim S$. If $i \le p-1$, then $u_{i+1} \nsim S$, and if $i \ge 2$, then $u_{i-1} \nsim S$. (2) Let u_i and u_j be two distinct vertices in P such that $u_i \sim S$ and $u_j \sim S$, where $1 \le i < j \le p$ and $j - i \ge 2$. If $j \le p-1$, then $u_{i+1} \nsim u_{j+1}$, and if $i \ge 2$, then $u_{i-1} \nsim u_{j-1}$.*

2.1 Proof of Theorem 6: S is not a connected graph

Lemma 1. *If S is not a connected graph, then $G \in \mathcal{H}_1^1$.*

Proof. Suppose S consists of k components with $k \ge 2$, denoted by $S_1, S_2, \ldots, S_k$. Take $w \in S_1$ and $z \in S_2$. Then $w \nsim z$. With Proposition 2(1),

$|N_P(S_i)| \le \lceil \frac{p}{2} \rceil$ for $i \in \{1,2\}$. Thus

$$\begin{aligned} n-1 &\le \deg_G(w) + \deg_G(z) \\ &\le |S_1| - 1 + |N_P(S_1)| + |S_2| - 1 + |N_P(S_2)| \\ &\le |S_1| + |S_2| - 2 + \lceil \frac{p}{2} \rceil + \lceil \frac{p}{2} \rceil \\ &\le |S_1| + |S_2| - 2 + p + 1 \\ &\le n - \sum_{i=3}^{k} |S_i| - 1. \end{aligned}$$

It must be $\sum_{i=3}^{k} |S_i| = 0$, $|P| = p$ being an odd integer, and $\deg_G(w) + \deg_G(z) = n-1$. Namely, S consists of two components, $\deg_G(w) = |S_1| - 1 + \lceil \frac{p}{2} \rceil$, and $\deg_G(z) = |S_2| - 1 + \lceil \frac{p}{2} \rceil$. It implies that both S_1 and S_2 are complete graphs, and each vertex in $S_1 \cup S_2$ is adjacent to $\lceil \frac{p}{2} \rceil$ vertices of P. Let $V(P_{odd}) = \{u_1, u_3, \ldots, u_p\}$ and $V(P_{even}) = \{u_2, u_4, \ldots, u_{p-1}\}$ be two subsets of $V(P)$. Then each vertex in $S = S_1 \cup S_2$ is adjacent to each vertex in P_{odd}. We claim that $|S_1| = |S_2| = 1$. Otherwise, assume the opposite by letting $|S_1| \ge 2$ without loss of generality. Consider two distinct vertices $a, b \in S_1$. Then $a \sim u_1$, $b \sim u_3$, and there is a hamiltonian path Q in S_1 between a and b. The path $\langle u = u_1, a, Q, b, u_3 \uparrow u_p = v \rangle$ is a path longer than P, which is a contradiction. Consequently, $|S_1| = |S_2| = 1$. Furthermore, by Proposition 1(2), there is no edge between any two vertices of P_{even}; applying the degree-sum assumption of Theorem 6 on two vertices of P_{even} results in the subgraph $P_{even} \vee P_{odd}$ of G. Let $r = |P_{odd}|$. Then G is of the form $\{(w + z + P_{even}) \vee P_{odd} \mid P_{even} = I_{r-1}, P_{odd} = H_r\}$. Namely, $G \in \mathcal{H}_1^1$. □

2.2 Proof of Theorem 6: S is a connected, noncomplete graph

Note that S cannot be connected and noncomplete if $s \le 2$. We only consider S with $s \ge 3$ in this subsection. We shall assume that $N_P(S) = \{u_{i_1}, \ldots, u_{i_m}\}$, where $|N_P(S)| = m$.

Proposition 3. *If S is a connected, noncomplete graph, then the following statements hold.*

(1) *For each vertex $x \in S$, $\deg_S(x) \ge s-2$.*

(2) *$u \sim S$ and $v \sim S$.*

(3) *There exists a hamiltonian path of S which connects two distinct vertices x and y with $\deg_S(x) = \deg_S(y) = s-2$.*

(4) *For any vertex x of S with* $\deg_S(x) = s-2$, $N_P(x) = N_P(S)$. *In other words,* $x \sim u_{i_j}$ *for all* $1 \leq j \leq m$.

(5) $s \leq p-2$.

Proof. (1) Let $x \in S$ and $u_i \in P$ with $u_i \sim S$. By Proposition 2, $u_{i+1} \nsim S$ and $\deg_G(u_{i+1}) \leq p-m+1$. Note that $\deg_P(x) \leq m$. Applying the degree assumption of Theorem 6 on x and u_{i+1}, we have

$$\begin{aligned} n-1 &\leq \deg_G(x) + \deg_G(u_{i+1}) \\ &= \deg_P(x) + \deg_S(x) + \deg_G(u_{i+1}) \\ &\leq m + \deg_S(x) + p - m + 1 \\ &= \deg_S(x) + p + 1. \end{aligned}$$

Hence $\deg_S(x) \geq n-1-(p+1) = s-2$.

(2) Assume that $u_p = v \nsim S$. We claim that there exists a vertex $u_i \in P$ with $1 \leq i \leq p-1$ such that $u_i \sim S$. Otherwise, P is not connected with S, which means that G is not traceable. It contradicts Theorem 1. By Proposition 2, $u_{i+1} \nsim S$ and $u_{i+1} \nsim u_{j+1}$ for $1 \leq j \leq p-1$ with $j \neq i$ and $u_j \sim S$. Thus $\deg_G(u_{i+1}) \leq p-m$. Since S is not complete, and by (1), there exists a vertex x in S with $\deg_S(x) = s-2$. Since $x \nsim u_{i+1}$, we have

$$\begin{aligned} \deg_G(x) + \deg_G(u_{i+1}) &\leq (s-2+m) + (p-m) \\ &= s+p-2 \\ &= n-2. \end{aligned}$$

It contradicts the degree-sum assumption of Theorem 6. Consequently, it must be $v \sim S$. With the similar argument, it can be shown that $u \sim S$.

(3) If $s = 3$, then S is a path with 3 vertices. So there is a hamiltonian path between the two vertices with degree 1. If $s = 4$, then either S is a cycle with 4 vertices or S is a cycle with 4 vertices plus an additional edge. It is easy to see that S has the desired hamiltonian path in either case. Finally, consider S with $s \geq 5$. Take two nonadjacent vertices w and z of S. Since $\deg_S(w) + \deg_S(z) \geq (s-2) + (s-2) = 2s-4 \geq s+1$, S is hamiltonian-connected by Theorem 1. In particular, there exists a hamiltonian path between two vertices with degree $s-2$.

(4) Take $x \in S$ with $\deg_S(x) = s-2$. Consider u_{i_1+1}. With Proposition 2 and the statement in (2), $u_{i_1+1} \nsim x$ and $\deg_G(u_{i_1+1}) \leq p-m+1$. For the vertex pair $\{x, u_{i_1+1}\}$, the degree-sum assumption of Theorem 6 implies that $\deg_G(x) \geq (n-1)-(p-m+1) = s-2+m$. It must be $\deg_P(x) = m$, so there exists an edge between x and any vertex in $N_P(S)$.

(5) With (1) and the fact that S is noncomplete, there exist vertices with degree in S equal to $s-2$. By (3), there is a hamiltonian path Q of S between two distinct vertices a and b with $\deg_S(a) = \deg_S(b) = s-2$. Moreover, with (4), $a \sim u$ and $b \sim v$. Consequently, the path $P' = \langle u, a, Q, b, v\rangle$ is a path with $|P'| = s+2$. Since P is the longest path between u and v in G, it must be $s \le p-2$. □

Lemma 2. *If S is a connected, noncomplete graph, then G is hamiltonian-connected.*

Proof. With Proposition 3(1) and the fact that S is noncomplete, let x be any vertex in S with $\deg_S(x) = s-2$. By Proposition 3(2), $m \ge 2$. There are two cases.

Case 1. $m = 2$. By (2) and (4) of Proposition 3, we know $N_P(S) = \{u_1, u_p\}$ and $x \sim u_1$ and $x \sim u_p$. With Proposition 3(5), $s \le p-2$. Take $y \in S$ such that $x \not\sim y$. We have

$$\begin{aligned}\deg_G(x) + \deg_G(y) &= (s-2+2) + (s-2+2)\\ &= s+s\\ &\le s+p-2\\ &= n-2.\end{aligned}$$

It is a contradiction to the degree assumption of Theorem 6. Consequently, no graph which is not hamiltonian-connected appears in this case.

Case 2. $m \ge 3$.
Case 2.1. There exists at least one j with $1 \le j \le m-1$ such that $i_{j+1} - i_j = 2$. Note that S is a noncomplete graph with $s \ge 3$ and, by Proposition 3(1), $\deg_S(z) \ge s-2$ for each vertex z in S. Thus there exist two distinct vertices a and b of S with $\deg_S(a) = \deg_S(b) = s-2$ such that $a \not\sim b$. Since $N_P(a) = N_P(b) = N_P(S)$ by Proposition 3(4), we know $u_{i_j} \sim a$ and $u_{i_{j+1}} \sim b$ for some j with $1 \le j \le m-1$. Since S is connected, let Q be the path between a and b in S. The path $\langle u = u_1 \uparrow u_{i_j}, a, Q, b, u_{i_{j+1}} \uparrow u_p = v\rangle$ is a path longer than P since Q consists of at least two vertices a and b. It is a contradiction.
Case 2.2. For all $1 \le j \le m-1$, $i_{j+1} - i_j \ge 3$. Let j and k be two distinct integers with $j < k$ such that $i_{j+1} - i_j \ge 3$ and $i_{k+1} - i_k \ge 3$. Since $x \not\sim u_{i_j+1}$, by the assumption of Theorem 6,

$$\begin{aligned}n-1 &\le \deg_G(x) + \deg_G(u_{i_j+1})\\ &\le (s-2+m) + (p-m+1)\\ &= s+p-1\\ &= n-1.\end{aligned}$$

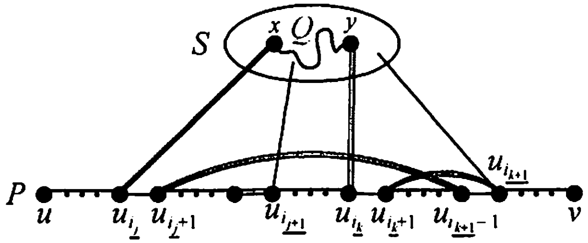

Figure 6: An illustration which leads to a contradiction in Case 2.2 of Lemma 2.

It implies that $\deg_G(u_{i_{\underline{j}}+1}) = p - m + 1$. Thus $u_{i_{\underline{j}}+1} \sim u_{i_{\underline{k+1}}-1}$. Furthermore, since $x \nsim u_{i_{\underline{k}}+1}$, using the same argument, we have $u_{i_{\underline{k}}+1} \sim u_{i_{\underline{k+1}}}$. Since $u_{i_{\underline{k}}} \sim S$, $u_{i_{\underline{k}}} \sim y$ for some vertex $y \in S$. Since S is connected, there exists a path Q in S between x and y. Therefore, the path $\langle u = u_1 \uparrow u_{i_{\underline{j}}}, x, Q, y, u_{i_{\underline{k}}} \downarrow u_{i_{\underline{j}}+1}, u_{i_{\underline{k+1}}-1} \downarrow u_{i_{\underline{k}}+1}, u_{i_{\underline{k+1}}} \uparrow u_p = v\rangle$ is a path longer than P, which is a contradiction. See Figure 6.

The proof is completed. □

2.3 Proof of Theorem 6: S is a connected, complete graph

In this subsection, let S be a complete graph with $s \geq 1$. In other words, $\deg_S(x) = s - 1$ for every vertex $x \in S$. Note that $N_P(S) = \{u_{i_{\underline{1}}}, \ldots, u_{i_{\underline{m}}}\}$, where $|N_P(S)| = m$.

Proposition 4. *If S is a complete graph and $m \geq 2$, then $|N_P(S) \cap \{u, v\}| \geq 1$.*

Proof. To prove $|N_P(S) \cap \{u, v\}| \geq 1$, we assume the opposite. That is, we assume $u \nsim S$ and $v \nsim S$. Equivalently speaking, we assume $i_{\underline{1}} \geq 2$ and $i_{\underline{m}} \leq p - 1$. (Or, $u_{i_{\underline{1}}} \neq u_1$ and $u_{i_{\underline{m}}} \neq u_p$.) For each vertex $u_{i_{\underline{j}}} \in N_P(S)$ and for any vertex $x \in S$, we have

$$\begin{aligned} n - 1 &\leq \deg_G(x) + \deg_G(u_{i_{\underline{j}}+1}) \\ &\leq (s - 1 + m) + (p - m) \\ &\leq n - 1. \end{aligned}$$

It implies $\deg_G(u_{i_{\underline{j}}+1}) = p - m$. We will derive contradictions as follows.
Case 1. $i_{\underline{m}} - i_{\underline{m-1}} = 2$. It implies that $u_{i_{\underline{m-1}}+1} = u_{i_{\underline{m}}-1}$ and $\deg_P(u_{i_{\underline{m-1}}+1}) \leq p - m - 1$, which is a contradiction.
Case 2. $i_{\underline{m}} - i_{\underline{m-1}} \geq 3$. It implies that $u_{i_{\underline{m-1}}+1} \sim u_{i_{\underline{m}}}$ and $u_{i_{\underline{m}}-1} \sim u_{i_{\underline{m}}+1}$.

Let a and b be vertices of S such that $u_{i_{m-1}} \sim a$, $u_{i_m} \sim b$ and Q a path of S between a and b. Thus $P' = \langle u = u_1 \uparrow u_{i_{m-1}}, a, Q, b, u_{i_m}, u_{i_{m-1}+1} \uparrow u_{i_m-1}, u_{i_m+1} \uparrow u_p = v \rangle$ is a path with $|P'| \geq p + \varepsilon$, where $\varepsilon = 1$ if $a = b$ and $\varepsilon = 2$ if $a \neq b$, which is longer than P. It is a contradiction. □

Proposition 5. *If S is a complete graph and $x \in S$ is an arbitrary vertex, then* $\deg_P(x) \geq m - 1$.

Proof. Let $y \in P$ and $y = u_{i_j+1}$ for some $u_{i_j} \in N_P(S)$. It is known that $\deg_G(x) = s - 1 + \deg_P(x)$. With the assumption of Theorem 6,

$$\begin{aligned} n - 1 &\leq \deg_G(x) + \deg_G(y) \\ &= (s - 1 + \deg_P(x)) + \deg_G(y). \end{aligned}$$

It implies $\deg_G(y) \geq (n-1) - (s - 1 + \deg_P(x)) = p - \deg_P(x)$. On the other hand, by Proposition 2 and 4, $\deg_G(y) \leq p - m + 1$. Therefore, $p - m + 1 \geq \deg_G(y) \geq p - \deg_P(x)$, which gives $\deg_P(x) \geq m - 1$. □

If $\deg_P(x) = m - 1$ for some $x \in S$, with the similar discussion as in Lemma 2, we can show that no graph appears for Case 2, where $m \geq 3$. The graphs obtained from Case 1 are $\mathcal{H}_2^2$ and $\mathcal{H}_2^3$. They belong to $\mathcal{H}_2$. With Proposition 5, we shall concentrate on the case with $\deg_P(x) = m$ for all $x \in S$ from now on. Without loss of generality, with Proposition 4, we assume $u = u_1 \sim S$ if $m \geq 2$.

Proposition 6. *If S is a complete graph and $m \geq 2$, then $1 \leq s \leq d - 1$, where $d = \min\{i_{j+1} - i_j \mid 1 \leq j \leq m - 1\}$.*

Proof. Take two vertices u_{i_j} and $u_{i_{j+1}}$ of $N_P(S)$. Note that $\deg_P(x) = m$ for all $x \in S$. Let $a, b \in V(S)$ such that $a \sim u_{i_j}$ and $b \sim u_{i_{j+1}}$. Since S is complete, there exists a hamiltonian path Q of S between a and b. The path $P' = \langle u = u_1 \uparrow u_{i_j}, a, Q, b, u_{i_{j+1}} \uparrow u_p = v \rangle$ is a path between u and v with $|P'| = p - (i_{j+1} - i_j - 1) + s$. Since P is the longest path between u and v, we conclude that $s \leq d - 1$. □

Consider $u_{i_j+1} \in P$. By Proposition 2, $u_{i_j+1} \nsim S$, and $u_{i_j+1} \nsim u_{i_l+1}$ for any $l \neq j$ with $1 \leq l \leq m$. The following proposition is obvious.

Proposition 7. *Suppose $u_{i_j+1} \neq u_{i_{j+1}-1}$, where $1 \leq j \leq m - 1$. (1) If $i_m < p$, then $|N_G(u_{i_j+1})| = p - m$ and $N_G(u_{i_j+1}) = V(P) \setminus \{u_{i_l+1} \mid l = 1, \ldots, m\}$. (2) If $i_m = p$, then $|N_G(u_{i_j+1})| \in \{p - m, p - m + 1\}$ and $N_G(u_{i_j+1}) \subseteq (V(P) \setminus \{u_{i_l+1} \mid l = 1, \ldots, m - 1\})$. Similarly, for $u_{i_j-1} \neq u_{i_{j-1}+1}$, where $2 \leq j \leq m$, since $u_1 \sim S$, $|N_G(u_{i_j-1})| \in \{p - m, p - m + 1\}$ and $N_G(u_{i_j-1}) \subseteq (V(P) \setminus \{u_{i_l-1} \mid l = 2, \ldots, m\})$.*

Proof. If $i_{\underline{m}} = p$, then $|\{u_{i_{\underline{j}}+1} \mid 1 \le j \le m\}| = m - 1$. Otherwise, $|\{u_{i_{\underline{j}}+1} \mid 1 \le j \le m\}| = m$. Take $x \in V(S)$. We know that $x \nsim u_{i_{\underline{j}}+1}$ and $\deg_G(x) = s - 1 + m$. For the vertex pair $\{x, u_{i_{\underline{j}}+1}\}$, the degree-sum condition of Theorem 6 implies that $\deg_G(u_{i_{\underline{j}}+1}) \ge (n-1) - \deg_G(x) = p - m$.

Case 1. $i_{\underline{m}} < p$. With Proposition 2, $u_{i_{\underline{j}}+1} \nsim u_{i_{\underline{l}}+1}$ for any $1 \le l \le m$, then $\deg_G(u_{i_{\underline{j}}+1}) = \deg_P(u_{i_{\underline{j}}+1}) \le p-m$. It must be $\deg_G(u_{i_{\underline{j}}+1}) = p-m$, and $N_G(u_{i_{\underline{j}}+1}) = V(P) \setminus \{u_{i_{\underline{l}}+1} \mid l = 1, \ldots, m\}$.

Case 2. $i_{\underline{m}} = p$. Similarly, with Proposition 2, $\deg_G(u_{i_{\underline{j}}+1}) \le p - m + 1$. It must be $|N_G(u_{i_{\underline{j}}+1})| \in \{p-m, p-m+1\}$ and $N_G(u_{i_{\underline{j}}+1})$ is as in (2). □

Lemma 3. *If S is a complete graph and $m \le 2$, then $G \in \mathcal{H}_1^1 \cup \mathcal{H}_2^1 \cup \mathcal{H}_2^2 \cup \mathcal{H}_3^1$.*

Proof. There are two cases.

Case 1. $m = 1$. Suppose $u_j \sim S$ for some $1 \le j \le p$. Take $u_i \in P$ with $i \ne j$ and $x \in S$. Obviously, $u_i \nsim x$. Using the degree assumption of Theorem 6, we find

$$\begin{aligned} n - 1 &\le \deg_G(x) + \deg_G(u_i) \\ &\le (s - 1 + 1) + (p - 1) \\ &\le n - 1. \end{aligned}$$

Thus any vertex of S must connect to $u_j \in P$ and P must be a complete graph. We obtain the graphs $K_s \vee u_j \vee K_{p-1}$, which are equal to $K_{p-2} \vee K_2 \vee^- K_s$, where $u_j \in V(K_2)$. (With Figure 3(b), readers can take $t_1 = p - 2, V(K_2) = \{x = u_j, y\}$, and $t_2 = s$.) Obviously, they belong to $\mathcal{H}_2^2$.

Case 2. $m = 2$.

Case 2.1. $i_{\underline{2}} < p$.

Case 2.1.1. $i_{\underline{2}} \ge 4$. Note that $u_{i_{\underline{1}}} = u_1 \sim S$ and $u_{i_{\underline{2}}} \sim S$. By Proposition 2, $u_2 \nsim S$ and $u_2 \nsim u_{i_{\underline{2}}+1}$. With Proposition 7, $u_2 \sim u_{i_{\underline{2}}}$ and $u_{i_{\underline{2}}+1} \sim u_{i_{\underline{2}}-1}$. Let a and b be distinct vertices in S such that $u_1 \sim a$ and $u_{i_{\underline{2}}} \sim b$, and Q be a path between a and b in S. Then the path $\langle u = u_1, a, Q, b, u_{i_{\underline{2}}}, u_2 \uparrow u_{i_{\underline{2}}-1}, u_{i_{\underline{2}}+1} \uparrow u_p = v\rangle$ is a path longer than P, which is a contradiction.

Case 2.1.2. $i_{\underline{2}} = 3$. Since $i_{\underline{2}} - i_{\underline{1}} = 2$, by Proposition 6, $s = 1$. Therefore, we assume that $S = \{w\}$. For $p = 4$, using Proposition 7, we obtain $(u_1 + u_3) \vee (w + u_2 + u_4)$, which belongs to $\mathcal{H}_1^1$ with $r = 2$. For $p = 5$, a similar derivation leads to a graph containing $w \vee (u_1 + u_3)$ and $(u_2 + u_4) \vee (u_1 + u_3 + u_5)$ as its subgraphs, in which $u_5 \nsim w$, $|N_G(u_5) \cap \{u_1, u_3\}| \ge 1$ and either $u_1 \sim u_3$ or $u_1 \nsim u_3$. It belongs to $\mathcal{H}_3^1$ for $r = 3$ and $u_5 = y$. Consider the case when $p \ge 6$. Since $u_1 \sim w$ and $u_3 \sim w$, by Proposition 1,

$u_2 \nsim w$ and $u_2 \nsim u_4$. With Proposition 7, $u_2 \sim u_5$ and $u_2 \sim u_6$. Then the path $P' = \langle u = u_1, w, u_3, u_4, u_5, u_2, u_6 \uparrow u_p = v \rangle$ is a path with $|P'| = p + 1$, which is longer than P. It is a contradiction.

Case 2.2. $i_2 = p$. Note that $x \sim u_1$ and $x \sim u_p$ for all $x \in S$. With Proposition 7, we obtain the graphs $K_s \vee H_2 \vee^- K_{p-2}^-$, which belong to $\mathcal{H}_2^1$. □

Lemma 4. *If S is a complete graph and $m \geq 3$, then $G \in \mathcal{H}_i$ for $i \in \{1, 3, 4\}$.*

Proof. By Proposition 4, $|N_P(S) \cap \{u, v\}| \geq 1$. There are three cases.
Case 1. There exist j and k with $1 \leq j < k \leq m-1$ such that $i_{k+1} - i_k \geq 3$ and $i_{j+1} - i_j \geq 3$.
Case 1.1. $i_m = p$ and $m \geq 4$.
Case 1.1.1. $i_{l+1} - i_l \geq 3$ for all $l \neq j, k$ and $1 \leq l \leq m - 1$. Thus the number d in Proposition 6 satisfies $d \geq 3$. Consider $u_{i_1+1} \in P$ and $x \in S$. Obviously, $u_{i_1+1} \nsim x$ by Proposition 2. Applying the degree-sum assumption of Theorem 6, we find

$$\begin{aligned} \deg_G(u_{i_1+1}) &\geq (n-1) - \deg_G(x) \\ &= (n-1) - (s-1+m) \\ &= p - m. \end{aligned}$$

Let $a, b, c \in V(S)$ such that $a \sim u_{i_1}$, $b \sim u_{i_2}$, and $c \sim u_{i_3}$. Let Q_1 be a hamiltonian path between a and b in S, and Q_2 be a hamiltonian path between a and c in S. With Proposition 7, it means that there exists at most one vertex $u^* \in P$, where $u^* \notin \{u_{i_j+1} \mid 2 \leq j \leq m-1\}$, such that $u_{i_1+1} \nsim u^*$. Therefore, we have either $u_{i_1+1} \sim u_{i_2}$ or $u_{i_1+1} \sim u_{i_3}$.
Case 1.1.1.1. $u_{i_1+1} \sim u_{i_2}$. Again, by Proposition 7, either $u_{i_2-1} \sim u_{i_2+1}$ or $u_{i_2-1} \sim u_{i_3+1}$. First, if $u_{i_2-1} \sim u_{i_2+1}$, then $P' = \langle u = u_{i_1}, a, Q_1, b, u_{i_2}, u_{i_1+1} \uparrow u_{i_2-1}, u_{i_2+1} \uparrow u_p = v \rangle$ is a path with $|P'| = p+s$, which is longer than P. See Figure 7. We obtain a contradiction. Second, if $u_{i_2-1} \sim u_{i_3+1}$, then $P'' = \langle u = u_{i_1}, a, Q_2, c, u_{i_3} \downarrow u_{i_2}, u_{i_1+1} \uparrow u_{i_2-1}, u_{i_3+1} \uparrow u_p = v \rangle$ is another path with $|P''| = p + s$, which is longer than P. See Figure 8. We obtain a contradiction.
Case 1.1.1.2. $u_{i_1+1} \sim u_{i_3}$. Again, by Proposition 7, either $u_{i_3+1} \sim u_{i_2-1}$ or $u_{i_3+1} \sim u_{i_3-1}$. First, if $u_{i_3+1} \sim u_{i_2-1}$, then $P' = \langle u = u_{i_1}, a, Q_1, b, u_{i_2} \uparrow u_{i_3}, u_{i_1+1} \uparrow u_{i_2-1}, u_{i_3+1} \uparrow u_p = v \rangle$ is a path with $|P'| = p+s$, which is longer than P. See Figure 9. We obtain a contradiction. Second, if $u_{i_3+1} \sim u_{i_3-1}$, then $P'' = \langle u = u_{i_1}, a, Q_2, c, u_{i_3}, u_{i_1+1} \uparrow u_{i_3-1}, u_{i_3+1} \uparrow u_p = v \rangle$ is another path with $|P''| = p + s$, which is longer than P. See Figure 10. We obtain a contradiction. Consequently, there is no graph present in this case.

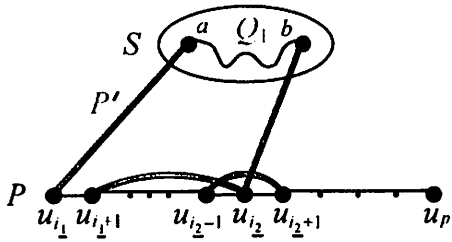

Figure 7: An illustration of P' which leads to a contradiction in Case 1.1.1.1 of Lemma 4.

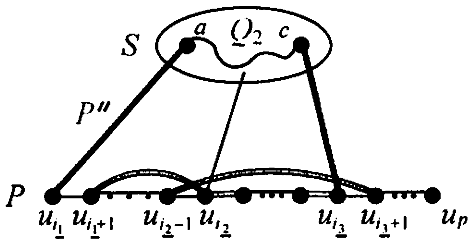

Figure 8: An illustration of P'' which leads to a contradiction in Case 1.1.1.1 of Lemma 4.

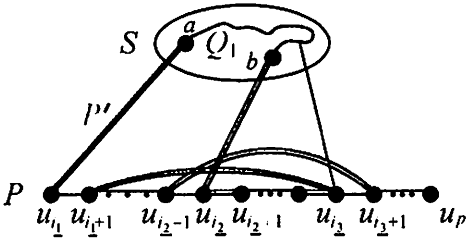

Figure 9: An illustration of P' which leads to a contradiction in Case 1.1.1.2 of Lemma 4.

Case 1.1.2. $i_{l+1} - i_{\underline{l}} = 2$ for some $l \in \{1,\dots,m-1\} \setminus \{j,k\}$. Let $l^* \in \{1,\dots,m-1\} \setminus \{j,k\}$ with $i_{\underline{l^*+1}} - i_{\underline{l^*}} = 2$. In this case, with Proposition 6, $s = 1$. Let $S = \{w\}$. By Proposition 1, $u_{i_{\underline{l^*}}+1} \nsim S$. In addition, $u_{i_{\underline{l^*}}+1} =$

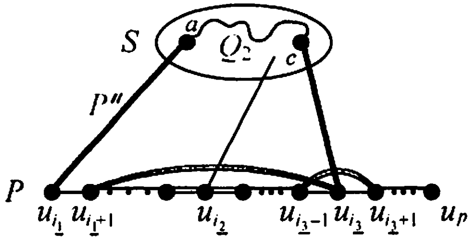

Figure 10: An illustration of P'' which leads to a contradiction in Case 1.1.1.2 of Lemma 4.

$u_{i_{\underline{l^*+1}}-1}$, so $u_{i_{\underline{l^*}}+1} \nsim u_{i_{\underline{j+1}}-1}$ and $u_{i_{\underline{l^*}}+1} \nsim u_{i_{\underline{k+1}}-1}$. We have

$$\begin{aligned} n-1 &\leq \deg_G(w) + \deg_G(u_{i_{\underline{l^*}}+1}) \\ &\leq (s-1+m) + (p-m+1-2) \\ &= s+p-2 \\ &= n-2, \end{aligned}$$

which is a contradiction. Thus no graph which is not hamiltonian-connected appears.

Case 1.2. $i_{\underline{m}} = p$ and $m = 3$.

Case 1.2.1. There exists at least one j^* with $1 \leq j^* \leq 2$ such that $i_{\underline{j^*+1}} - i_{\underline{j^*}} \geq 4$. Without loss of generality, we assume that $i_{\underline{2}} - i_{\underline{1}} \geq 4$. Let $a, b \in V(S)$ such that $a \sim u_{i_{\underline{1}}}$ and $b \sim u_{i_{\underline{2}}}$. Let Q be a path between a and b in S. Note that for $l \in \{1, -1\}$, with Proposition 7, there exists at most one corresponding vertex u^* in P, where $u^* \notin \{u_{i_{\underline{j}}+l} | 1 \leq j \leq 3\}$, such that $u_{i_{\underline{j}}+l} \nsim u^*$ if $u_{i_{\underline{j}}+l}$ exists. We will derive contradictions as follows.

First, in the case that $u_2 \sim u_{i_{\underline{2}}}$. If $u_{i_{\underline{2}}-1} \sim u_{i_{\underline{2}}+1}$, then $\langle u = u_1, a, Q, b, u_{i_{\underline{2}}}, u_2 \uparrow u_{i_{\underline{2}}-1}, u_{i_{\underline{2}}+1} \uparrow u_{i_{\underline{3}}} = v \rangle$ is a longer path than P. We obtain a contradiction. If $u_{i_{\underline{2}}-1} \nsim u_{i_{\underline{2}}+1}$, then $u_{i_{\underline{2}}-1} \sim u_1$. We have the path $P' = \langle u = u_1, u_{i_{\underline{2}}-1} \downarrow u_2, u_{i_{\underline{2}}} \uparrow u_{i_{\underline{3}}} = v \rangle$, which is another path having the same endvertices with P and $V(P) = V(P')$. With Proposition 2(2), we have $u_2 \nsim u_{i_{\underline{3}}-1}$ and $u_{i_{\underline{2}}+1} \nsim u_{i_{\underline{2}}-1}$. It implies $u_2 \sim u_{i_{\underline{2}}-1}$ and $u_{i_{\underline{2}}+1} \sim u_3$. Therefore, we obtain $\langle u = u_1, a, Q, b, u_{i_{\underline{2}}}, u_2, u_{i_{\underline{2}}-1} \downarrow u_3, u_{i_{\underline{2}}+1} \uparrow u_{i_{\underline{3}}} = v \rangle$ being a longer path than P, which is a contradiction.

Second, in the case that $u_2 \nsim u_{i_{\underline{2}}}$. It implies $u_2 \sim u_{i_{\underline{3}}-1}$ and $u_2 \sim u_{i_{\underline{3}}}$. If $u_{i_{\underline{2}}+1} \sim u_3$, then $\langle u = u_1, a, Q, b, u_{i_{\underline{2}}} \downarrow u_3, u_{i_{\underline{2}}+1} \uparrow u_{i_{\underline{3}}-1}, u_2, u_{i_{\underline{3}}} = v \rangle$ is a longer path than P. We obtain a contradiction. If $u_{i_{\underline{2}}+1} \nsim u_3$, then $u_{i_{\underline{2}}+1} \sim u_{i_{\underline{3}}}$. We have the path $\langle u = u_1, a, Q, b, u_{i_{\underline{2}}} \downarrow u_2, u_{i_{\underline{3}}-1} \downarrow u_{i_{\underline{2}}+1}, u_{i_{\underline{3}}} = v \rangle$, which is a longer path than P. We obtain a contradiction.

Case 1.2.2. $i_{\underline{2}} - i_{\underline{1}} = i_{\underline{3}} - i_{\underline{2}} = 3$. By Proposition 6, $1 \leq s \leq 2$. We assume that $V(S) = \{w\}$ if $s = 1$, and $V(S) = \{w, z\}$ if $s = 2$. With Proposition 7, we obtain two graphs $w \vee \{u_1, u_4, u_7\} \vee ((u_2 \vee u_3) + (u_5 \vee u_6))$ and $(w \vee z) \vee \{u_1, u_4, u_7\} \vee ((u_2 \vee u_3) + (u_5 \vee u_6))$. Both graphs belong to $\mathcal{H}_4$. Besides these graphs, the graph G contains a longer path, which contradicts our assumption of the path P.

Case 1.3. $i_{\underline{m}} < p$. Since $u_{i_{\underline{j}}} \sim S$, by Proposition 2, $u_{i_{\underline{j}}+1} \nsim S$. With Proposition 7, $\deg_G(u_{i_{\underline{j}}+1}) = p-m$, and $u_{i_{\underline{j}}+1} \sim u_{i_{\underline{k+1}}}$. Similarly, $u_{i_{\underline{k+1}}+1} \sim u_{i_{\underline{j+1}}-1}$. Let a and b be distinct vertices in S such that $u_{i_{\underline{j}}} \sim a$ and $u_{i_{\underline{j+1}}} \sim b$, and Q a path between a and b in S. Thus the path $\langle u = u_1 \uparrow u_{i_{\underline{j}}}, a, Q, b, u_{i_{\underline{j+1}}} \uparrow u_{i_{\underline{k+1}}}, u_{i_{\underline{j}}+1} \uparrow u_{i_{\underline{j+1}}-1}, u_{i_{\underline{k+1}}+1} \uparrow u_p = v\rangle$ is a path longer than P, which is a contradiction.

Case 2. There exists exactly one k with $1 \leq k \leq m-1$ such that $i_{\underline{k+1}} - i_{\underline{k}} \geq 3$, and $i_{\underline{j+1}} - i_{\underline{j}} = 2$ for all $j \neq k$. Without loss of generality, we assume that $S = \{w\}$ and $i_{\underline{k+1}} - i_{\underline{k}} \geq 3$ with $1 \leq k \leq \lceil \frac{m}{2} \rceil - 1$.

Case 2.1. $i_{\underline{m}} = p$. Note that $w \sim u_{i_{\underline{k+1}}}$ and $w \sim u_{i_{\underline{k+2}}}$.

Case 2.1.1. $i_{\underline{k+1}} - i_{\underline{k}} \geq 5$. With Proposition 7, $u_{i_{\underline{k+1}}+1} \sim u_{i_{\underline{k}}+2}$ and $u_{i_{\underline{k+1}}+1} \sim u_{i_{\underline{k}}+3}$. We have the path $\langle u = u_1 \uparrow u_{i_{\underline{k}}+2}, u_{i_{\underline{k+1}}+1}, u_{i_{\underline{k}}+3} \uparrow u_{i_{\underline{k+1}}}, w, u_{i_{\underline{k+2}}} \uparrow u_{i_{\underline{m}}} = v\rangle$, which is a longer path than P. We obtain a contradiction.

Case 2.1.2. $i_{\underline{k+1}} - i_{\underline{k}} = 4$. We have $w \nsim u_{i_{\underline{k}}+2}$ and, with Proposition 7, $u_{i_{\underline{l}}+1} \sim u_{i_{\underline{k}}+2}$ for each l with $1 \leq l \leq m-1$. We claim that $u_{i_{\underline{k}}+1} \nsim u_{i_{\underline{k}}+3}$. Otherwise, $\langle u = u_1 \uparrow u_{i_{\underline{k}}+1}, u_{i_{\underline{k}}+3}, u_{i_{\underline{k}}+2}, u_{i_{\underline{k+1}}+1}, u_{i_{\underline{k+1}}}, w, u_{i_{\underline{k+2}}} \uparrow u_{i_{\underline{m}}} = v\rangle$ is a longer path than P, which is a contradiction. Therefore, we obtain the graphs $H_r^+ \vee' I_r$, where $V(I_r) = \{w\} \cup \{u_i \in P \mid i \text{ is an even integer}\}$, and $u_{i_{\underline{k}}+2} \in V(H_r^+) = \{u_i \in P \mid i \text{ is an odd integer}\}$ is at least adjacent to one of the other vertices in H_r^+. These graphs belong to $\mathcal{H}_3^1$.

Case 2.1.3. $i_{\underline{k+1}} - i_{\underline{k}} = 3$. Using Proposition 7, we obtain the graphs $H_r \vee^* (K_2 \vee I_{r-1})$, where $V(H_r) = \{u_{i_{\underline{j}}} \in P \mid 1 \leq j \leq m = r\}$, $V(K_2) = \{u_{i_{\underline{k}}+1}, u_{i_{\underline{k}}+2}\}$, and $V(I_{r-1}) = \{w\} \cup (\{u_i \in P \mid u_i \nsim w\} \setminus \{u_{i_{\underline{k}}+1}, u_{i_{\underline{k}}+2}\})$. These graphs belong to $\mathcal{H}_1^2$.

Case 2.2. $i_{\underline{m}} < p$. Since $u_{i_{\underline{j}}+1} = u_{i_{\underline{j+1}}-1}$, by Proposition 2, $u_{i_{\underline{j}}+1} \nsim u_{i_{\underline{k+1}}-1}$. Thus $\deg_G(u_{i_{\underline{j}}+1}) \leq p - m - 1$. For any vertex x in S, we have

$$
\begin{aligned}
n-1 &\leq \deg_G(x) + \deg_G(u_{i_{\underline{j}}+1}) \\
&\leq (s-1+m) + (p-m-1) \\
&\leq n-2,
\end{aligned}
$$

which is a contradiction.

Case 3. $i_{\underline{j+1}} - i_{\underline{j}} = 2$ for all $j \in \{1, \ldots, m-1\}$.

Case 3.1. $i_{\underline{m}} = p$. In this case, p must be an odd integer. The graphs derived are $w \vee H^+_{\lceil \frac{p}{2} \rceil} \vee' I_{\lfloor \frac{p}{2} \rfloor}$ or $w \vee H_{\lceil \frac{p}{2} \rceil} \vee I_{\lfloor \frac{p}{2} \rfloor}$, and they belong to

$\mathcal{H}_3$.
Case 3.2. $i_{\underline{m}} = p-1$. In this case, p must be an even integer. The graphs derived are $w \vee H_{\frac{p}{2}} \vee I_{\frac{p}{2}}$, which belong to $\mathcal{H}_1^1$.
Case 3.3. $i_{\underline{m}} = p-2$, where p is an odd integer. The graphs derived are $I_{\lfloor \frac{p}{2} \rfloor} \vee H^+_{\lceil \frac{p}{2} \rceil} \vee' w$, and they belong to $\mathcal{H}_3^1$.
Case 3.4. $i_{\underline{m}} \leq p-3$. Using the argument similar to the case with $p \geq 6$ of Case 2.1.2 in Lemma 3, a contradiction occurs. Thus no graph appears in this case. □

3 Conclusion

With Lemmas 1–4, Theorem 6 is proved. Note that the exceptional families $\mathcal{G}_1$ and $\mathcal{G}_2$ in Theorem 5 are subsets of $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively. We also note that the exceptional graphs in Theorem 4 belong to $\mathcal{H}_2$ and $\mathcal{H}_1$, and the graph families (i), (ii) and (iii) in Theorem 3 are subsets of $\mathcal{H}_2$, $\mathcal{H}_1$ and $\mathcal{H}_3$, respectively. Graphs in these families satisfy the degree-sum condition such that $\deg_G(x)+\deg_G(y) \geq |G|-1$ holds for every pair of nonadjacent vertices x, y of G, and they are not hamiltonian, not pancyclic, and not hamiltonian-connected. Such an analysis leads to the following comprehensive theorem.

Theorem 7. *Let $G = (V, E)$ be a simple graph with $|G| = n \geq 3$ such that $\deg_G(u) + \deg_G(v) \geq n-1$ holds for each pair of nonadjacent vertices u and v in V. Then*

(i) *G is traceable.*

(ii) *Either G is hamiltonian or G belongs to one of the two families: $\mathcal{H}_1^1$ and $\mathcal{H}_2^2$.*

(iii) *Either G is pancyclic or G belongs to one of the four families: (a) $\mathcal{H}_2^2$; (b) $\mathcal{H}_1^1$; (c) $\mathcal{H}_3^2$ with $H_r = I_r$; (d) $\mathcal{H}_2^1$ with $t_1 = 1$, $H_2 = I_2$ and $K_{t_2}^- = K_2$.*

(iv) *Either G is hamiltonian-connected or G belongs to one of the four families: $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$.*

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