# A relationship between Minors and Linkages

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**ABSTRACT.** Linkage is very important in very large scale integration (VLSI) physical design. In this paper, we mainly study the relationship between minors and linkages. Thomassen conjectured that every $(2k+2)$-connected graph is $k$-linked. For $k \geq 4$, $K_{3k-1}$ with $k$ disjoint edges deleted is a counterexample to this conjecture, however, it is still open for $k = 3$. Thomas and Wollan proved that every 6-connected graph on $n$ vertices with $5n - 14$ edges is 3-linked. Hence they obtain that every 10-connected graph is 3-linked. Chen et al. showed that every 6-connected graph with $K_9^-$ as a minor is 3-linked, and every 7-connected graph with $K_9^-$ as a minor is $(2,5)$-linked. Using a similar method, we prove that every 8-connected graph with $K_{12}^-$ as a minor is 4-linked, and every $(2k+1)$-connected graph with $K_{2k+3}^-$ as a minor is $(2, 2k-1)$-linked. Our results extend Chen et al.'s conclusions, improve Thomas and Wollan's results, and moreover, they give a class of graphs that satisfy Thomassen's conjecture for $k = 4$.

**Keywords**: Minor; Linkage; $k$-Linked; $(2,k)$-Linked.

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## 1 Introduction

All graphs considered in this paper are finite, undirected, and simple (without loops or multiple edges). The sets of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. Let $X$ be a subset of $V(G)$.

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We use $G[X]$ to denote the subgraph of $G$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ that have both ends in $X$. The *neighbors* of $v$ in $G[X]$, denoted by $N_{G[X]}(v)$, is the set of vertices in $G[X]$ which are adjacent to $v$. When $G[X] = G$, we simply write $N(v)$ instead of $N_{G[X]}(v)$. A *minor* of $G$ is any graph obtained from $G$ by deleting edges and (or) vertices and contracting edges. According to Bondy and Murty [2], we use $K_n$ to denote the *complete graph* with $n$ vertices, and $K_n^-$ the subgraph of $K_n$ with exactly one edge deleted. Let $s_1$, $s_2$, ..., $s_k$ be $k$ positive integers. A graph $G$ is said to be $(s_1, s_2, \ldots, s_k)$*-linked* if it has at least $\sum_{i=1}^{k} s_i$ vertices and for any $k$ disjoint vertex sets $S_1$, $S_2$, ..., $S_k$ with $|S_i| = s_i$, $G$ contains vertex disjoint connected subgraphs $F_1$, $F_2$, ..., $F_k$ such that $S_i \subseteq V(F_i)$. The $(2, 2, \ldots, 2)$-linked graphs are called $k$*-linked*, that is, for any $2k$ distinct vertices $x_1$, $y_1$, $x_2$, $y_2$, ..., $x_k$, $y_k$, there exists $k$ vertex disjoint paths $P_1$, $P_2$, ..., $P_k$ such that $P_i$ joins $x_i$ and $y_i$, for $1 \leq i \leq k$.

The layout is first modeled as a routing graph, where each node represents a tile and each edge denotes the boundary between two adjacent tiles. A number of basic models for very large scale integration (VLSI) layout are based on the construction of vertex disjoint paths between terminals on a multi-layer grid. So linkage is very important in VLSI physical design.

The research on linkage has a long history, and has attracted more and more graph theorists. In 1980, Thomassen [9] conjectured that

**Conjecture 1.1** (Thomassen [9]). Every $(2k+2)$-connected graph is $k$-linked.

It has been observed that $K_{3k-1}$ with $k$ disjoint edges deleted is a counterexample to this conjecture for $k \geq 4$, however, it is still open for $k = 3$. In 2005, Chen, Gould, Kawarabayashi, Pfender and Wei [3] proved that

**Theorem 1.1** (Chen, Gould, Kawarabayashi, Pfender and Wei [3]). *Every 6-connected graph with $K_9^-$ as a minor is 3-linked.*

**Theorem 1.2** (Chen, Gould, Kawarabayashi, Pfender and Wei [3]). *Every 7-connected graph with $K_9^-$ as a minor is $(2,5)$-linked.*

In 2008, Thomas and Wollan [8] proved that

**Theorem 1.3** (Thomas and Wollan [8]). *Every 6-connected graph on $n$ vertices with $5n - 14$ edges is 3-linked.*

By Theorem 1.3, they obtain the following corollary.

**Corollary 1.1** (Thomas and Wollan [8]). *Every 10-connected graph is 3-linked.*

By applying a similar method to the proofs of Theorems 1.1 and 1.2, we obtain the following two main results.

**Theorem 1.4.** *Every 8-connected graph with $K_{12}^-$ as a minor is 4-linked.*

**Theorem 1.5.** *Every $2k+1$-connected graph with $K_{2k+3}^-$ as a minor is $(2, 2k-1)$-linked.*

Theorem 1.4 extends Theorem 1.1 and improves Theorem 1.3. Moreover, it gives a class of graphs that satisfy Conjecture 1.1 for $k = 4$. Theorem 1.5 is the extension of Theorem 1.2.

By Theorems 1.1 and 1.4, it's natural to propose the following conjecture.

**Conjecture 1.2.** Every $2k$-connected graph with $K_{3k}^-$ as a minor is $k$-linked.

In Sections 2 and 3, proofs of Theorems 1.4 and 1.5 will be given, respectively.

# 2 Proof of Theorem 1.4

Let $G$ be a connected graph and $H$ a minor of $G$. Let $S$, $A$, $B \subseteq V(G)$ and $C_1$, $C_2$, ..., $C_{|H|}$ a partition of $V(G)$, such that each $G[C_i]$ is connected, and contracting each $C_i$ yields $H$. Let $l = |A \cap B|$. If $S \subseteq A$, $V(G) = A \cup B$, and there are no edges between $A \setminus B$ and $B \setminus A$, then $(A, B)$ is an *$S$-cut* of size $l$. If $C_i \subseteq B \setminus A$ for some $1 \leq i \leq |H|$, then the $S$-cut $(A, B)$ is called an *$S^H$-cut*.

The following theorem proved by Hall [4] is very important in our main proof.

**Theorem 2.1** (Hall's Theorem, Hall [4]). *A bipartite graph $G[X, Y]$ has a matching which covers every vertex in $X$ if and only if*

$$|N(S)| \geq |S|$$

*for all $S \subseteq X$.*

In order to prove Theorem 1.4, we introduce the following theorem which is stronger than Theorem 1.4.

**Theorem 2.2.** *Let $G$ be a graph and $S = \{x_1, x_2, y_1, y_2, z_1, z_2, d_1, d_2\} \subseteq V(G)$. Let $G^*$ be the graph obtained from $G$ by adding all missing edges in $G[S]$. Suppose that there is a partition $C_1$, $C_2$, ..., $C_{12}$ of $V(G)$, such that each $G^*[C_i]$ is connected, and contracting each $C_i$ in $G^*$ yields $H = K_{12}^-$. If $G^*$ has no $S^H$-cut of size smaller than 8, then there are four vertex disjoint paths in $G$ connecting $(x_1, x_2)$, $(y_1, y_2)$, $(z_1, z_2)$, $(d_1, d_2)$, respectively.*

*Proof.* If not, then let $G$ be a counterexample with the minimum number of edges. Let $S$, $C_1$, $C_2$, ..., $C_{12}$ be as in the theorem. Then by the choice of $G$, $G[S]$ contains no edges.

We say that for each $1 \leq i \leq 12$, $G[C_i]$ contains no edges. Since if for some $i$, $G[C_i]$ contains edges, then without loss of generality, suppose that $uv \in E(C_1)$. As $G[S]$ contains no edges, suppose that $v \notin S$. By the choice of $G$, there has to be an $S^H$-cut $(A, B)$ of size 8 with $u, v \in A \cap B$, otherwise the contraction of $uv$ would yield a smaller counterexample. As $|A \cap B| = 8$ and $u$, $v \in C_1$, at least five of the twelve $C_i$ sets contain no vertices of $A \cap B$. Without loss of generality, we may assume that $C_i \cap A \cap B \neq \emptyset$ for $1 \leq i \leq k$, and $C_i \cap A \cap B = \emptyset$ for $i > k$, where $k$ is an integer with $1 \leq k \leq 7$. As $S \subseteq A$, and $G^*[C_i]$ is connected, $C_i \subseteq B \setminus A$ or $C_i \subseteq A \setminus B$ for each $i > k$. Since $C_i \subseteq B \setminus A$ for at least one $i > k$, it is in fact true that $C_i \subseteq B \setminus A$ for all $i > k$, otherwise, contracting each $C_i$ in $G^*$ doesn't yield a $K_{12}^-$. As there is no $S^H$-cut of size less than 8 in $G^*$, there are eight vertex disjoint paths from $S$ to $A \cap B$ in $G[A]$. Label the vertices of $S' = A \cap B$ with $x_1'$, $x_2'$, $y_1'$, $y_2'$, $z_1'$, $z_2'$, $d_1'$, $d_2'$ according to the starting vertices in $S$ of these paths. Let $C_i' = C_i \cap B$ for $1 \leq i \leq 12$. Then $G[B]$, $S'$, $C_1'$, $C_2'$, ..., $C_{12}'$ satisfy all the conditions of the theorem, and $G[B]$ is smaller than $G$, as there is at least one vertex in $S \setminus B$ (note that $v \notin S$). By the choice of $G$, we can find four vertex disjoint paths in $G[B]$ connecting $(x_1', x_2')$, $(y_1', y_2')$, $(z_1', z_2')$, $(d_1', d_2')$, respectively. This, together with the eight paths in $G[A]$, produce the desired paths in $G$, a contradiction.

Now we have that $G[C_i]$ contains no edges for each $1 \leq i \leq 12$. Thus, $C_i \subseteq S$ or $|C_i| = 1$. If $C_i = S$ for some $1 \leq i \leq 12$, then $|V(G)| = 19$. Therefore, $12 \leq |V(G)| \leq 19$. Suppose without loss of generality that $|V(C_i)| \geq |V(C_j)|$ for $1 \leq i < j \leq 12$.

**Case 2.1.** $|V(G)| = 12$.

*Proof.* In this case, $|C_i| = 1$ for each $1 \leq i \leq 12$. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4\}$. Then either $\{x_1v_1x_2, y_1v_2y_2, z_1v_3z_2, d_1v_4d_2\}$ or $\{x_1v_2x_2, y_1v_3y_2, z_1v_4z_2, d_1v_1d_2\}$ is the desired set of vertex disjoint paths, a contradiction.

□

If $|V(G)| \geq 13$, we have that each vertex in $S$ has at least two neighbors in $V(G) \setminus S$. Otherwise, suppose $x_1$ has at most one neighbor in $V(G) \setminus S$. If $x_1$ has no neighbors in $V(G) \setminus S$, then $(A = S, B = V(G) \setminus x_1)$ is an $S^H$-cut of size 7. On the other hand, if $x_1$ has exactly one neighbor in $V(G) \setminus S$, say $x_1v_1 \in E(G)$, then $C_i \setminus x_1 \neq \emptyset$ for all $1 \leq i \leq 12$. Since $|V(G) \setminus S| \geq 5$, $G \setminus x_1$ with $S' = (S \setminus x_1) \cup \{v_1\}$ is a smaller counterexample, a contradiction to the minimality of $G$.

**Case 2.2.** $|V(G)| = 13$.

*Proof.* In this case, $|C_1| = 2$. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4, v_5\}$.

If $C_1 = \{x_1, x_2\}$ (the cases $C_1 = \{y_1, y_2\}$ and $C_1 = \{z_1, z_2\}$ are analogous), then there exists a matching from $C_1$ into $V(G) \setminus S$. Since each vertex in $S$ has at least two neighbors in $V(G)\setminus S$, suppose that $\{x_1v_1, x_2v_2\}$ is such a matching. If $v_1v_2 \in E(G)$, then either $\{x_1v_1v_2x_2, y_1v_3y_2, z_1v_4z_2, d_1v_5d_2\}$ or $\{x_1v_1v_2x_2, y_1v_4y_2, z_1v_5z_2, d_1v_3d_2\}$ is the desired set of vertex disjoint paths, a contradiction. Then $v_1v_2 \notin E(G)$. As $G^*$ contracts to a $K_{12}^-$, $v_3$ has a neighbor in $C_1$. Without loss of generality, suppose that $x_1v_3 \in E(G)$. Now $\{x_1v_3v_2x_2, y_1v_1y_2, z_1v_4z_2, d_1v_5d_2\}$ is the desired set of vertex disjoint paths, a contradiction.

Now suppose that $C_1 = \{x_1, y_1\}$ (the other cases are analyzed by a similar argument). As above, there exists a matching from $C_1$ into $V(G)\setminus S$. Suppose that $\{x_1v_1, y_1v_2\}$ is such a matching. Then at most one of the edges in a path in $\{x_1v_1x_2, y_1v_2y_2, z_1v_3z_2, d_1v_4d_4\}$ is missing, but now this edge can be replaced by a path of length 2 through $v_5$ to produce the desired set of vertex disjoint paths, a contradiction.

$\square$

**Case 2.3.** $|V(G)| = 14$.

*Proof.* In this case, $|C_1| \geq 2$. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

If $|C_1| = 3$, without loss of generality suppose that $x_1, y_1, z_1 \notin C_1$, then there ia a matching from $\{x_2, y_2, z_2, d_1, d_2\}$ into $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Since if not, we consider the bipartite graph $G_1 = G[\{x_2, y_2, z_2, d_1, d_2\}, V(G)\setminus S]$, then by Theorem 2.1, there exists a set $S_1 \subseteq \{x_2, y_2, z_2, d_1, d_2\}$, such that $|N_{G_1}(S_1)| < |S_1|$. Now $(A = S \cup N_{G_1}(S_1), B = (S \setminus S_1) \cup (V(G) \setminus S))$ is an $S^H$-cut of size $|S\setminus S_1| + |N_{G_1}(S_1)|$, which is smaller than 8 in $G^*$, a contradiction. Without loss of generality, suppose that $\{x_2v_2, y_2v_3, z_2v_4, d_1v_5, d_2v_6\}$ is this matching. Now $G^*[x_1, y_1, z_1, v_1, v_2, v_3, v_4, v_5, v_6]$ is a $K_9$ or a $K_9^-$, and therefore 4-linked. Then there are four vertex disjoint paths in $G^*$ connecting $(x_1, v_2), (y_1, v_3), (z_1, v_4), (v_5, v_6)$, respectively. As the edges $x_1y_1$, $y_1z_1$, $x_1z_1$ are not used in this path system, this is in fact a path system in $G$. Together with the matching, we get the desired set of vertex disjoint paths, a contradiction.

Then $|C_1| = |C_2| = 2$. If $x_1$, $y_1$, $z_1 \notin C_1 \cup C_2$, then the same argument as above applies. Without loss of generality, we may assume that $C_1 \cup C_2 = \{y_1, y_2, z_1, z_2\}$. If $x_jv_k$ or $d_jv_k \notin E(G)$ for some $1 \leq j \leq 2$ and some $1 \leq k \leq 6$, say $x_1v_1 \notin E(G)$, then $G[x_2, d_1, v_1, v_2, v_3, v_4, v_5, v_6]$ is a $K_8$, a very similar argument can be used to find the desired vertex disjoint paths. Thus, we may assume that $x_jv_k$, $d_jv_k \in E(G)$ for $1 \leq j \leq 2$ and $1 \leq k \leq 6$. Still we get that there is a matching from $\{y_1, y_2, z_1, z_2\}$

into $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, suppose that $y_1v_1, y_2v_2, z_1v_3, z_2v_4$ is the matching. If $v_1v_2$, $v_3v_4 \in E(G)$, then $\{x_1v_5x_2, y_1v_1v_2y_2, z_1v_3v_4z_2, d_1v_6d_2\}$ is the desired set of vertex disjoint paths, a contradiction. Hence, we may assume that $v_1v_2 \notin E(G)$. As $G^*$ contracts to a $K_{12}^-$, $v_5$ and $v_6$ are adjacent to both $C_1$ and $C_2$. If $v_5y_1 \in E(G)$ (and similarly if $v_5y_2 \in E(G)$), then $\{x_1v_1x_2, y_1v_5v_2y_2, z_1v_3v_4z_2, d_1v_6d_2\}$ is the desired set of vertex disjoint paths, a contradiction. Hence $v_5z_1$, $v_5z_2 \in E(G)$, but now $\{x_1v_4x_2, y_1v_1v_3v_2y_2, z_1v_5z_2, d_1v_6d_2\}$ are the desired vertex disjoint paths, a contradiction.

□

**Case 2.4.** $|V(G)| = 15$.

*Proof.* In this case, $|C_1| \geq 2$. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

If $|C_1| \geq 3$, then without loss of generality, suppose that $x_1$, $y_1 \notin C_1 \cup C_2$. As above, there is a matching from $\{x_2, y_2, z_1, z_2, d_1, d_2\}$ into $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Without loss of generality, suppose that $\{x_2v_2, y_2v_3, z_1v_4, z_2v_5, d_1v_6, d_2v_7\}$ is the matching. As $G^*[x_1, y_1, v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a $K_9$ or a $K_9^-$, it's 4-linked. Then there are four vertex disjoint paths in $G^*$ connecting $(x_1, v_2)$, $(y_1, v_3)$, $(v_4, v_5)$, $(v_6, v_7)$, respectively. As the edge $x_1y_1$ is not used in this path system, this is in fact a path system in $G$. Together with the matching, we get the desired set of vertex disjoint paths, a contradiction.

Then $|C_1| = |C_2| = |C_3| = 2$. If $x_1$, $y_1 \notin C_1 \cup C_2 \cup C_3$, the same argument as above applies. Without loss of generality, we may assume that $C_1 \cup C_2 \cup C_3 = \{y_1, y_2, z_1, z_2, d_1, d_2\}$. If $x_jv_k \notin E(G)$ for some $1 \leq j \leq 2$ and some $1 \leq k \leq 7$, say $x_1v_1 \notin E(G)$, then $G[x_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a $K_8$, and thus 4-linked, and a very similar argument can be used to find the paths. Thus, we may assume that $x_jv_k \in E(G)$ for $1 \leq j \leq 2$ and $1 \leq k \leq 7$. As above, there is a matching from $\{y_1, y_2, z_1, z_2, d_1, d_2\}$ into $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. Suppose that $\{y_1v_1, y_2v_2, z_1v_3, z_2v_4, d_1v_5, d_2v_6\}$ is the matching. If $v_1v_2$, $v_3v_4$, $v_5v_6 \in E(G)$, then $\{x_1v_7x_2, y_1v_1v_2y_2, z_1v_3v_4z_2, d_1v_5v_6d_2\}$ is the desired set of vertex disjoint paths, a contradiction. Hence, we may assume that $v_1v_2 \notin E(G)$. Now $v_7y_1$, $v_7y_2 \notin E(G)$, otherwise, either $\{x_1v_1x_2, y_1v_7v_2y_2, z_1v_3v_4z_2, d_1v_5v_6d_2\}$ or $\{x_1v_2x_2, y_1v_1v_7y_2, z_1v_3v_4z_2, d_1v_5v_6d_2\}$ is the desired set of vertex disjoint paths, a contradiction. We also have that at least one of $v_7z_1$ and $v_7z_2$ is not in $E(G)$, otherwise, $\{x_1v_4x_2, y_1v_1v_3v_2y_2, z_1v_7z_2, d_1v_5v_6d_2\}$ are the desired vertex disjoint paths, a contradiction. Similarly, we obtain that if $v_7d_1 \in E(G)$, then $v_7d_2 \notin E(G)$. As $G^*$ contracts to a $K_{12}^-$, $v_7$ is adjacent to $C_1$, $C_2$, and $C_3$. But $v_7$ is adjacent to at most two vertices in $C_1 \cup C_2 \cup C_3$, a contradiction.

□

**Case 2.5.** $|V(G)| = 16$.

*Proof.* In this case, $|C_1| \geq 2$. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.

If $|C_1| \geq 3$, then $|C_4| = 1$ and $G[C_4 \cup \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}]$ is a $K_9$ or a $K_9^-$. Thus the same argument as above applies.

Then $|C_1| = |C_2| = |C_3| = |C_4| = 2$. We claim that there is a matching from $S$ into $V(G) \setminus S$. Since if not, we consider the bipartite graph $G_1 = G[S, V(G) \setminus S]$, then by Theorem 2.1, there exists a set $S_1 \subseteq S$, such that $|N_{G_1}(S_1)| < |S_1|$. Now $(A = S \cup N_{G_1}(S_1), B = (S \setminus S_1) \cup (V(G) \setminus S))$ is an $S^H$-cut of size $|S \setminus S_1| + |N_{G_1}(S_1)|$, which is smaller than 8 in $G^*$, a contradiction. Without loss of generality, suppose that $\{x_1v_1, x_2v_2, y_1v_3, y_2v_4, z_1v_5, z_2v_6, d_1v_7, d_2v_8\}$ is such a matching. One of the edges $v_1v_2$, $v_3v_4$, $v_5v_6$, $v_7v_8$ is missing, otherwise the four paths are easy to find. This implies that each $v_i$ has at least four neighbors in $S$ and $|N_{C_j}(v_i)| \geq 1$, for $1 \leq j \leq 4$. It's easy to see that $x_2v_1 \notin E(G)$, otherwise, $\{x_1v_1x_2, y_1v_3v_4y_2, z_1v_5v_2v_6z_2, d_1v_7v_8d_2\}$, $\{x_1v_1x_2, y_1v_3v_2v_4y_2, z_1v_5v_6z_2, d_1 v_7v_8d_2\}$, or $\{x_1v_1x_2, y_1v_3v_4y_2, z_1v_5v_6z_2, d_1v_7v_2v_8d_2\}$ is the desired path system, a contradiction. Similarly, we could get that $x_1v_2$, $y_1v_4$, $y_2v_3$, $z_1v_6$, $z_2v_5$, $d_1v_8$, $d_2v_7 \notin E(G)$.

Suppose that $x_1v_3$, $x_2v_3 \in E(G)$. If $y_1v_1 \in E(G)$ or $y_1v_2 \in E(G)$, then a path system can easily be found. Therefore $y_1v_1$, $y_1v_2 \notin E(G)$. Thus, $y_1v_5$, $y_1v_6$, $y_1v_7$, or $y_1v_8 \in E(G)$. Without loss of generality, assume that $y_1v_5 \in E(G)$. Now $z_1v_1 \notin E(G)$, otherwise $\{x_1v_3x_2, y_1v_5v_4y_2, z_1v_1v_6z_2, d_1v_7v_2v_8 d_2\}$ is the desired path system, a contradiction. Similarly, $z_1v_2 \notin E(G)$. If $z_1v_4 \in E(G)$, then if $d_1v_1 \in E(G)$, $y_2v_2, y_2v_7 \notin E(G)$, otherwise either $\{x_1v_3x_2, y_1v_5v_2y_2, z_1v_4v_6z_2, d_1v_1v_8d_2\}$ or $\{x_1v_3x_2, y_1v_5v_7y_2, z_1v_4v_6z_2, d_1v_1v_8d_2\}$ is the desired path system, a contradiction. As $v_2$ has at least four neighbors in $S$, $x_2v_2$, $z_2v_2$, $d_1v_2$, $d_2v_2 \in E(G)$. Now $y_2v_1 \notin E(G)$, since otherwise $\{x_1v_3x_2, y_1v_5v_1y_2, z_1v_4v_6z_2, d_1v_2d_2\}$ is the desired path system, a contradiction. As $v_1$ has at least four neighbors in $S$, $x_1v_1$, $z_2v_1$, $d_1v_1$, $d_2v_1 \in E(G)$. Now it's easy to obtain that $y_2v_5$, $y_2v_6$, $y_2v_8 \notin E(G)$, since otherwise $\{x_1v_3x_2, y_1v_5y_2, z_1v_4v_6z_2, d_1v_2d_2\}$, $\{x_1v_3x_2, y_1v_5v_7v_6y_2, z_1v_4v_1z_2, d_1v_2d_2\}$, or $\{x_1v_3x_2, y_1v_5v_8y_2, z_1v_4v_6z_2, d_1v_2d_2\}$ is the desired path system, a contradiction. Now $y_2$ has at most one neighbor in $V(G) \setminus S$, a contradiction. Then $d_1v_1 \notin E(G)$. Similarly, we have that $d_2v_1 \notin E(G)$, now $v_1$ has at most three neighbors in $S$, a contradiction.

Then $z_1v_4 \notin E(G)$. If $z_1v_7 \in E(G)$, then $d_1v_1, d_1v_2 \notin E(G)$. Otherwise either $\{x_1v_3x_2, y_1v_5v_4y_2, z_1v_7v_6z_2, d_1v_1v_8d_2\}$ or $\{x_1v_3x_2, y_1v_5v_4y_2, z_1v_7v_6z_2, d_1v_2v_8d_2\}$ is the desired path system, a contradiction. As each of $v_1$ and $v_2$ has at least four neighbors in $S$, $y_2v_1$, $z_2v_1$, $d_2v_1$, $y_2v_2$, $z_2v_2$, $d_2v_2 \in E(G)$. Now $d_1v_4$ and $d_1v_6 \notin E(G)$, otherwise, either $\{x_1v_3x_2, y_1v_5v_1y_2, z_1v_7v_6z_2, d_1v_4v_8d_2\}$ or $\{x_1v_3x_2, y_1v_5v_1y_2, z_1v_7v_2z_2, d_1v_6v_8d_2\}$ is the desired path system, a contradiction. If $d_1v_3 \in E(G)$, then $x_1v_4, x_2v_4 \notin E(G)$, otherwise, either $\{x_1v_4v_2x_2, y_1v_5v_1y_2, z_1v_7v_6z_2, d_1v_3v_8d_2\}$ or $\{x_1v_1v_4x_2, y_1v_5v_2y_2, z_1v_7 v_6z_2, d_1v_3v_8d_2\}$ is the desired path system, a contradiction. Now $v_4$ has at most three neighbors in $S$, a contradiction. Thus, $d_1v_3 \notin E(G)$. As $d_1$ has at least two neighbors in $V(G) \setminus S$, $d_1v_5 \in E(G)$. But now $x_1v_4$, $x_2v_4 \notin E(G)$, otherwise, either $\{x_1v_4v_2x_2, y_1v_3v_1y_2, z_1v_7v_6z_2, d_1v_5v_8d_2\}$ or $\{x_1v_1v_4x_2, y_1v_3v_2y_2, z_1v_7v_6z_2, d_1v_5v_8d_2\}$ is the desired path system, a contradiction. Now $v_4$ has at most three neighbors in $S$, a contradiction. Thus, $d_1v_5 \notin E(G)$. Now $d_1$ has only one neighbor in $V(G) \setminus S$, a contradiction. Then $z_1v_7 \notin E(G)$. Similarly, we could obtain that $z_1v_8 \notin E(G)$. As $z_1$ has at least two neighbors in $V(G) \setminus S$, $z_1v_3 \in E(G)$.

We claim that each of $d_1$ and $d_2$ is adjacent to exactly one of $v_1$ and $v_2$. Since if $d_1$ (the case $d_2$ is analogous) is adjacent to both $v_1$ and $v_2$, then $x_1v_7$, $y_1v_7 \notin E(G)$. Otherwise, either $\{x_1v_7v_2x_2, y_1v_5v_4y_2, z_1v_3v_6z_2, d_1v_1v_8d_2\}$ or $\{x_1v_3x_2, y_1v_7v_4y_2, z_1v_5v_1v_6z_2, d_1v_2v_8d_2\}$ is the desired path system, a contradiction. Similarly, we could obtain that $x_2v_7 \notin E(G)$. Now $v_7$ has at most three neighbors in $S$, a contradiction. As $x_2v_1$, $y_1v_1$, $z_1v_1$, $x_1v_2$, $y_1v_2$, $z_1v_2 \notin E(G)$, if $d_1v_j \notin E(G)$, then $d_2v_j \in E(G)$ ($j \in \{1,2\}$). Otherwise $v_1$ or $v_2$ has at most three neighbors in $S$, a contradiction.

Without loss of generality, assume that $d_1v_1$, $d_2v_2 \in E(G)$. Now we could obtain that $x_1v_7$, $y_1v_7$, $y_1v_8 \notin E(G)$, otherwise $\{x_1v_7v_2x_2, y_1v_5v_4y_2, z_1v_3v_6z_2, d_1v_1v_8d_2\}$, $\{x_1v_3x_2, y_1v_7v_4y_2, z_1v_5v_2z_2, d_1v_1v_8d_2\}$, or $\{x_1v_3x_2, y_1v_8v_4y_2, z_1v_5v_1z_2, d_1v_7v_2d_2\}$ is the desired path system, a contradiction. Similarly, we could obtain that $x_2v_8 \notin E(G)$. Since each of $v_7$ and $v_8$ has at least four neighbors in $S$, $x_2v_7$, $y_2v_7$, $z_2v_7$, $x_1v_8$, $y_2v_8$, $z_2v_8 \in E(G)$. Now we have that $x_1v_4 \notin E(G)$, otherwise, $\{x_1v_4v_2x_2, y_1v_5v_7y_2, z_1v_3v_6z_2, d_1v_1v_8d_2\}$ is the desired path system, a contradiction. Similarly, $x_2v_4 \notin E(G)$. As $v_4$ has at least four neighbors in $S$, $y_2v_4$, $z_2v_4$, $d_1v_4$, $d_2v_4 \in E(G)$. But now $\{x_1v_1v_7x_2, y_1v_5v_2y_2, z_1v_3v_6z_2, d_1v_4d_2\}$ is the desired path system, a contradiction.

Thus, $z_1v_3 \notin E(G)$. Now $z_1$ has only one neighbor in $V(G) \setminus S$, a contradiction.

Then $x_1v_3$ and $x_2v_3$ can't both be edges. By symmetrical arguments, we have that $N(x_1) \cap N(x_2) = N(y_1) \cap N(y_2) = N(z_1) \cap N(z_2) = N(d_1) \cap N(d_2) = \emptyset$. Therefore, each $v_i$ has exactly four neighbors in $S$. Suppose that $v_1v_2 \notin E(G)$ and $N(v_1) = \{x_1, y_1, z_1, d_1\}$. Then $x_1v_3 \notin E(G)$, otherwise, $\{x_1v_3v_2x_2, y_1v_1v_4y_2, z_1v_5v_6z_2, d_1v_7v_8d_2\}$ is the desired path system, a contradiction. Hence $x_2v_3 \in E(G)$. Now $y_1v_2 \notin E(G)$, otherwise, $\{x_1v_1v_3x_2, y_1v_2v_4y_2, z_1v_5v_6z_2, d_1v_7v_8d_2\}$ is the desired path system, a contradiction. Thus $y_2v_2 \in E(G)$. Now $x_2v_4 \notin E(G)$, otherwise, $\{x_1v_1v_4x_2, y_1v_3v_2y_2, z_1v_5v_6z_2, d_1v_7v_8d_2\}$ is the desired path system, a contradiction. Then $x_1v_4 \in E(G)$. Now $y_2v_5 \notin E(G)$, otherwise, $\{x_1v_4v_2x_2, y_1v_3v_5y_2, z_1v_1v_6z_2, d_1v_7v_8d_2\}$ is the desired path system, a contradiction. Hence $y_1v_5 \in E(G)$. But now, $\{x_1v_4v_3x_2, y_1v_5v_2y_2, z_1v_1v_6z_2, d_1v_7v_8d_2\}$ is the desired path system, a contradiction. This completes the case $|V(G)| = 16$.

□

**Case 2.6.** $|V(G)| > 16$.

*Proof.* In this case, $V(G) \setminus S \supseteq \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$. It's easy to see that $G[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ is a $K_9$ or a $K_9^-$, thus it's 4-linked. We claim that there is a matching from $S$ to eight vertices of $V(G) \setminus S$. Since if not, we consider the bipartite graph $G_1 = G[S, V(G) \setminus S]$, then by Theorem 2.1, there exists a set $S_1 \subseteq S$, such that $|N_{G_1}(S_1)| < |S_1|$. Now $(A = S \cup N_{G_1}(S_1), B = (S \setminus S_1) \cup (V(G) \setminus S))$ is an $S^H$-cut of size $|S \setminus S_1| + |N_{G_1}(S_1)|$, which is smaller than 8 in $G^*$, a contradiction. Now the desired path system can easily be found, a contradiction.

□

The completion of the cases completes the proof of the theorem.

□

It's easy to see that Theorem 1.4 is a corollary of Theorem 2.2.

# 3 Proof of Theorem 1.5

In order to prove Theorem 1.5, we introduce the following theorem which is stronger than Theorem 1.5.

**Theorem 3.1.** *Let $G$ be a graph, and $S = \{x_1, x_2, y_1, y_2, y_3, \ldots, y_{2k-1}\} \subseteq V(G)$. Let $G^*$ be the graph obtained from $G$ by adding all missing edges in $G[S]$. Suppose that there is a partition $C_1$, $C_2$, ..., $C_{2k+3}$ of $V(G)$, such that each $G^*[C_i]$ is connected, and contracting each $C_i$ in $G^*$ yields $H = K_{2k+3}^-$. Further suppose that $G^*$ has no $S^H$-cut of size smaller than $2k+1$. Then there are two vertex disjoint connected subgraphs in $G$ containing $\{x_1, x_2\}$ and $\{y_1, y_2, y_3, \ldots, y_{2k-1}\}$, respectively.*

*Proof.* If not, then let $G$ be a counterexample with the minimum number of edges. Let $S$, $C_1$, $C_2$, ..., $C_{2k+3}$ be as in the theorem. As in the proof of Theorem 2.2, it's easy to get that for each $1 \le i \le 2k+3$, $C_i \subseteq S$ or $|C_i| = 1$. If $C_i = S$ for some $1 \le i \le 2k+3$, then $|V(G)| = 4k+3$. Therefore, $2k+3 \le |V(G)| \le 4k+3$. Without loss of generality, suppose that $|V(C_i)| \ge |V(C_j)|$ for $1 \le i < j \le 2k+3$.

If $|V(G)| = 2k+3$, then $|C_i| = 1$ for each $1 \le i \le 2k+3$. Let $V(G) \setminus S = \{v_1, v_2\}$. Then either $G[x_1, x_2, v_1]$, $G[y_1, y_2, y_3, \ldots, y_{2k-1}, v_2]$ or $G[x_1, x_2, v_2]$, $G[y_1, y_2, y_3, \ldots, y_{2k-1}, v_1]$ is the desired set of connected subgraphs, a contradiction.

As in the proof of Theorem 2.2, we have that if $|V(G)| \ge 2k+4$, then each vertex in $S$ has at least two neighbors in $V(G) \setminus S$.

If $|V(G)| = 2k+4$, then $|C_1| = 2$. Let $V(G) \setminus S = \{v_1, v_2, v_3\}$. It's easy to see that $N(x_1) \cap N(x_2) \cap V(G) \setminus S \neq \emptyset$. Since $|N(x_1) \cap (V(G) \setminus S)| \geq 2$ and $|N(x_2) \cap (V(G) \setminus S)| \geq 2$. Without loss of generality, assume that $x_1v_1$, $x_2v_1 \in E(G)$. We claim that $G[y_1, y_2, y_3, \ldots, y_{2k-1}, v_2, v_3]$ is connected. Since each $y_i$ is connected to at least one of $v_2$ and $v_3$. Then if $v_2v_3 \in E(G)$, this is clear. Otherwise, observe that $|C_i| = 1$ for $2 \leq i \leq 2k+3$, then there is a $y_j$ with $y_jv_2$, $y_jv_3 \in E(G)$.

If $|V(G)| = 2k+5$, then let $V(G) \setminus S = \{v_1, v_2, v_3, v_4\}$. If $N(x_1) \cap N(x_2) \cap (V(G) \setminus S) \neq \emptyset$, say $x_1v_1$, $x_2v_1 \in E(G)$, then $G[x_1, x_2, v_1]$ and $G[y_1, y_2, y_3, \ldots, y_{2k-1}, v_2, v_3, v_4]$ are connected subgraphs. Thus, suppose that $N(x_1) \cap N(x_2) \cap (V(G) \setminus S) = \emptyset$, say $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4\}$. Note that this implies that neither $x_1$ nor $x_2$ is in a $C_i$ by itself, so at least $2k-3$ of the vertices in $\{y_1, y_2, \ldots, y_{2k-1}\}$ have at least three neighbors in $V(G) \setminus S$ and at least $2k-4$ of the vertices in $\{y_1, y_2, \ldots, y_{2k-1}\}$ have four neighbors of $V(G) \setminus S$. Without loss of generality, we may assume that $v_1v_3$, $v_1v_4$, $v_2v_3 \in E(G)$ (potentially $v_2v_4 \notin E(G)$). As there are at most two vertices in $\{y_1, y_2, y_3, \ldots, y_{2k-1}\}$ with less than three neighbors in $V(G) \setminus S$, we can pick $1 \leq j < k \leq 4$ such that $G[x_1, x_2, v_j, v_k]$ is connected, and every $y_i$ has a neighbor in $\{v_1, v_2, v_3, v_4\} \setminus \{v_j, v_k\}$. But now $G[V(G) \setminus \{x_1, x_2, v_j, v_k\}]$ is connected, a contradiction.

If $|V(G)| \geq 2k+6$, then let $V(G) \setminus S = \{v_1, v_2, v_3, \ldots, v_{n-2k-1}\}$. If $N(x_1) \cap N(x_2) \cap (V(G) \setminus S) \neq \emptyset$, say $x_1v_1$, $x_2v_1 \in E(G)$, then $G[x_1, x_2, v_1]$ and $G[y_1, y_2, y_3, \ldots, y_{2k-1}, v_2, v_3, \ldots, v_{n-2k-1}]$ are connected subgraphs. Thus, suppose that $N(x_1) \cap N(x_2) = \emptyset$.

As each vertex in $S$ has at least two neighbors in $V(G) \setminus S$, $|N(x_1) \cup N(x_2)| \geq 4$. If $|N(x_1) \cup N(x_2)| = 4$, then $|N(x_1)| = |N(x_2)| = 2$. Suppose that $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4\}$. Without loss of generality, we may assume that $v_1v_3$, $v_1v_4$, $v_2v_3 \in E(G)$ (potentially $v_2v_4 \notin E(G)$). If every $y_i$ has a neighbor in $\{v_1, v_2, \ldots, v_{n-2k-1}\} \setminus \{v_1, v_3\}$, then $G[x_1, x_2, v_1, v_3]$ and $G[y_1, y_2, \ldots, y_{2k-1}, v_2, v_4, \ldots, v_{n-2k-1}]$ are the desired connected subgraphs. Therefore, there is some $y_i$ with $N(y_i) = \{v_1, v_3\}$, say $i = 1$. Similarly, we may assume that $N(y_2) = \{v_1, v_4\}$ and $N(y_3) = \{v_2, v_3\}$. But now $(A = S \cup \{v_1, v_2, v_3, v_4\}, B = \{y_4, y_5, \ldots, y_{2k-1}, v_1, v_2, \ldots, v_{n-2k-1}\})$ is an $S^H$-cut of size $2k$ in $G^*$, a contradiction.

Now suppose that $|N(x_1) \cup N(x_2)| \geq 5$, say $N(x_1) \supseteq \{v_1, v_2\}$ and $N(x_2) \supseteq \{v_3, v_4, v_5\}$. Without loss of generality, we may assume that $v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4 \in E(G)$ (potentially $v_2v_5 \notin E(G)$). By similar arguments as above, $N(y_1) = \{v_1, v_3\}$, $N(y_2) = \{v_1, v_4\}$, $N(y_3) = \{v_1, v_5\}$, $N(y_4) = \{v_2, v_3\}$, and $N(y_5) = \{v_2, v_4\}$. Furthermore, we have that $N(x_1) = \{v_1, v_2\}$ and $N(x_2) = \{v_3, v_4, v_5\}$.

If $|V(G)| = 2k+6$, there is a vertex $u \in S$, such that $|N(u)| \geq 4$, a contradiction. If $|V(G)| > 2k+6$, $(A = S \cup \{v_1, v_2, v_3, v_4, v_5\}, B = \{y_6, y_7, \ldots, y_{2k-1}, v_1, v_2, \ldots, v_{n-2k-1}\})$ is an $S^H$-cut of size $2k-1$ in $G^*$, a contradiction. This completes the proof. □

It's easy to see that Theorem 1.5 is a corollary of Theorem 3.1.

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