Some results and remarks on harmonic mean and mean cordial labelings

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Abstract

We give more results in mean cordial and harmonic mean labelings such as: upper bounds for the number of edges of graphs of given orders for both labelings with direct results, labeling all trees of order ≤ 9 to be harmonic mean with the restriction of using the floor function of the definition, and labeling all graphs of order ≤ 5 that are harmonic mean graphs without using the label $q+1$ in labeling the vertices. Also we give mean cordial labelings for some families of graphs.

Keywords: Graph labeling, Cordial, Harmonic, Mean

AMS subject classification: 05C78

0 Introduction

A Graph $G(V,E)$ with p vertices and q edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\ldots,q+1$ in such a way that when each edge $e = uv$ is labeled with $f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case f is called Harmonic mean labeling of G.

The concept of *harmonic mean labeling* was introduced by Somasundaram, Ponraj & Sandhya [4]. They investigate the harmonic mean labeling of several standard graphs such as paths, cycles, combs, ladders, triangular snakes, quadrilateral snakes, etc. Besides they investigate the harmonic mean labeling for a polygonal chain, square of path and dragon. Also they enumerate all harmonic mean graphs of order ≤ 5.

In [3] Ponraj, Sivakumar and Sundaram introduced the definition of the *mean cordial labeling* as follows: Let f be a map from $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a mean cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i,j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x

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$(x = 0,1,2)$. A graph with a mean cordial labeling is called a mean cordial graph. They investigate the mean cordial labeling of Paths, Cycles, Stars, Complete graphs, Combs and some more standard graphs.

1 Harmonic mean labeling

Lemma 1.1: Let u and v be two adjacent vertices in a harmonic mean graph with the labels x and y respectively. Now the label of the edge , $f^*(uv) \leq \min\{2x, 2y\}$.

Proof: Let G be a harmonic mean graph. Suppose without any loss of generality that: $y > x$, so it can be written as $y = x + j$ for some positive integer j.

$f^*(uv) = \left\lfloor \frac{2x \cdot y}{x+y} \right\rfloor$ or $\left\lceil \frac{2x \cdot y}{x+y} \right\rceil$, but $\frac{2x \cdot y}{x+y} = \frac{2x\cdot(x+j)}{x+x+j} = x + \frac{xj}{2x+j}$. Suppose that: $\frac{xj}{2x+j} \geq x \Rightarrow 2x^2 \leq 0$, a contradiction, hence $\frac{xj}{2x+j} < x \Rightarrow f^*(uv) \leq 2x < 2y$.

Theorem 1.2: Let $G(p,q)$ be a graph with $p \geq 5$. If $q \geq \frac{p^2-3p+10}{2}$, then the graph G can't be a harmonic mean graph.

Proof: Suppose that $G(p,q)$ is a harmonic mean graph and let $q = \frac{p^2-3p+10}{2}$ (*) then we have two cases:

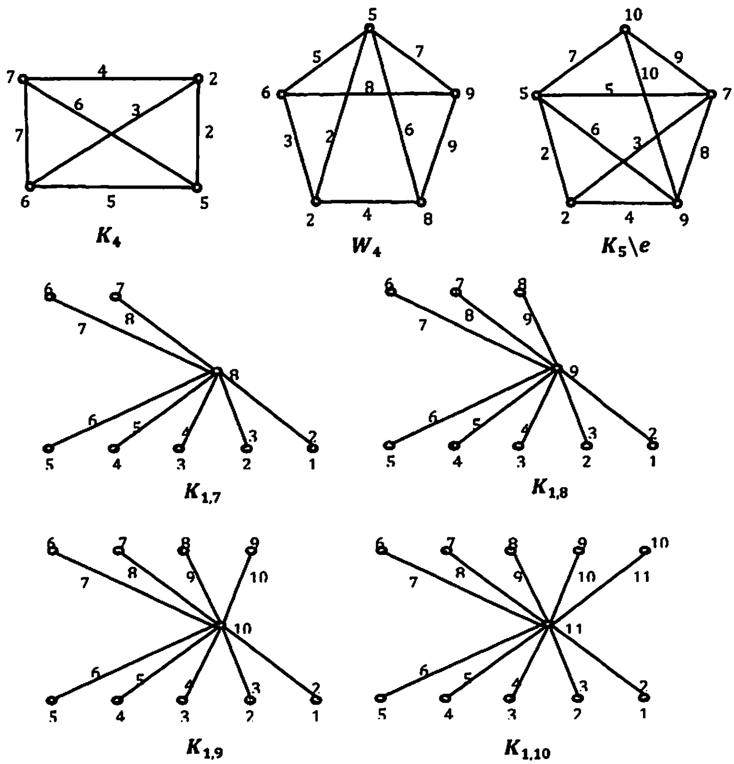
Case 1) None of the edge labels of the set $\{q+1, q, q-1, \dots, q - \left[\frac{(p-1)(p-2)}{2} - 1\right] \underbrace{=}_{from\,(*)} 5\}$ is eliminated in the labeling of the edges of the graph G. We realize that the number of these edge labels is $\frac{(p-1)(p-2)}{2} + 1$ and they need at least p vertices to be covered, and by *Lemma 1.1* the minimum vertex label could be found among these vertices is greater than 2, i.e. the vertex labels 1 and 2 are not used in labeling the graph, hence we can't get the two edge labels 1 and 2, i.e. there is a repeated edge label in the graph, a contradiction.

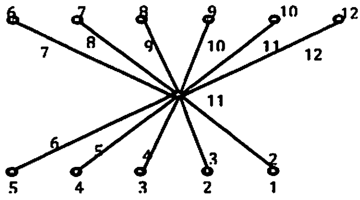
Case 2) One of the edge labels of the set $\{q+1, q, q-1, \dots, q - \left[\frac{(p-1)(p-2)}{2} - 1\right] \underbrace{=}_{from\,(*)} 5\}$ is eliminated. In this case we join to this set the label: $q - \left[\frac{(p-1)(p-2)}{2} - 0\right] \underbrace{=}_{from\,(*)} 4$, consequently we have again $\frac{(p-1)(p-2)}{2} + 1$ edge labels and to cover them we need at least p vertices and by *Lemma 1.1*, the minimum vertex label among them is greater than 1, hence the edge label 1 can't be obtained, i.e. we lost two edge labels, which means that there is a repeated edge label in the graph, a contradiction.

Result 1.3: By realizing that: $\frac{p(p-1)}{2} - \frac{p^2-3p+10}{2} = p - 5$, we deduce that: For a complete graph $K_n, n \geq 6$, all the graphs: $K_n \backslash e, K_n \backslash 2e, K_n \backslash 3e, \ldots$ and $K_n \backslash (n-5)e$, are all not harmonic mean graphs, (*where $K_n \backslash je$ is the graph K_n after deleting j edges*).

Remark 1.4: In [4]:

- the authors say that: K_n is a harmonic mean graph if and only if $n \leq 3$, but we label K_4 to be a harmonic mean graph *see Figure 1.*
- they say that: $K_{1,n}$ is a harmonic mean graph if and only if $n \leq 7$, but in *Figure 1,* we provide a harmonic mean labeling for each of: $K_{1,7}, K_{1,8}, K_{1,9}, K_{1,10}$ and $K_{1,11}$.
- they say that each of the graphs: $K_4 \cup K_1$, $K_5 \backslash e$ and W_4 are not harmonic mean graphs. We give a harmonic mean labeling to each one of them in *Figure 1 ($K_4 \cup K_1$ is labeled similar to K_4 with adding isolated vertex of label* 1, 3, *or* 4).

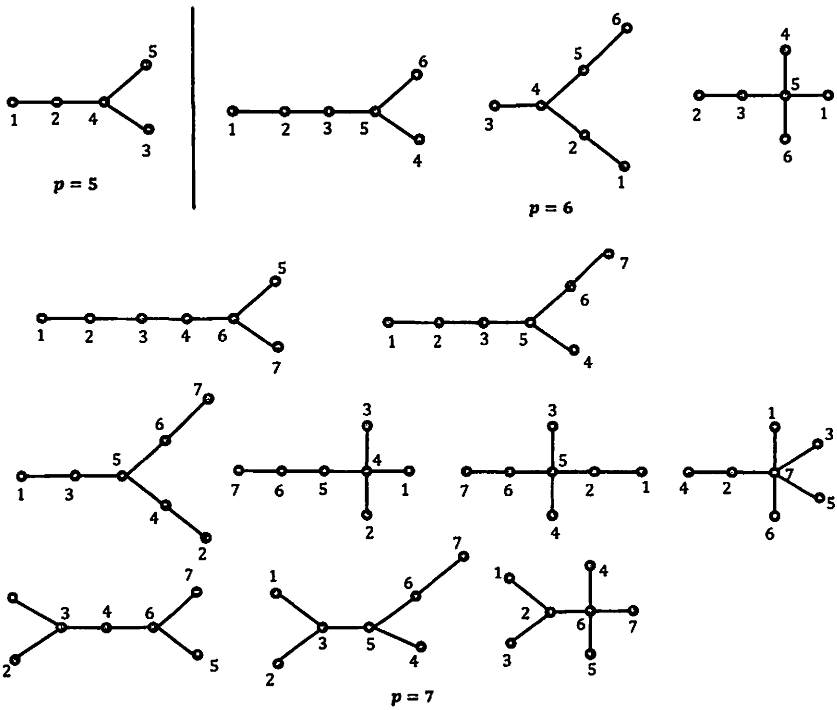


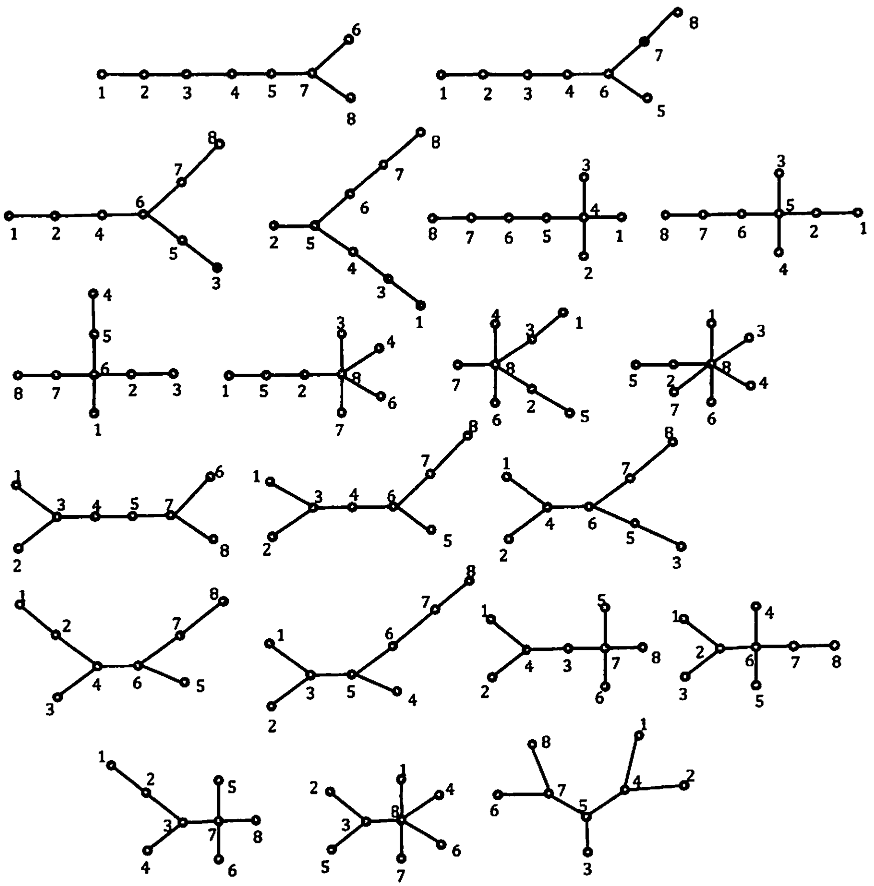


$K_{1,11}$

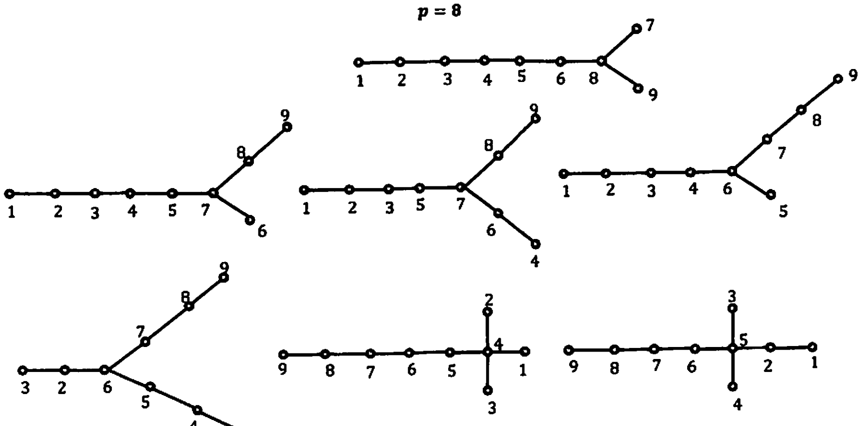
Figure 1

Survey 1.5: We give in the following survey all trees of order ≤ 9, except $K_{1,6}$, $K_{1,7}$ and $K_{1,8}$. All these trees are labeled according to the definition of harmonic mean labeling, but with the condition $f(uv) = \left\lfloor \frac{2f(u)\,f(v)}{f(u)+f(v)} \right\rfloor$, instead of $f(uv) = \left\lceil \frac{2f(u)\,f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)\,f(v)}{f(u)+f(v)} \right\rfloor$. We will prove that: $K_{1,6}$, $K_{1,7}$ and $K_{1,8}$ can't be labeled with this condition. (*The harmonic mean labeling behavior of path, star and bistar have been already investigated, so we don't give their labelings*).

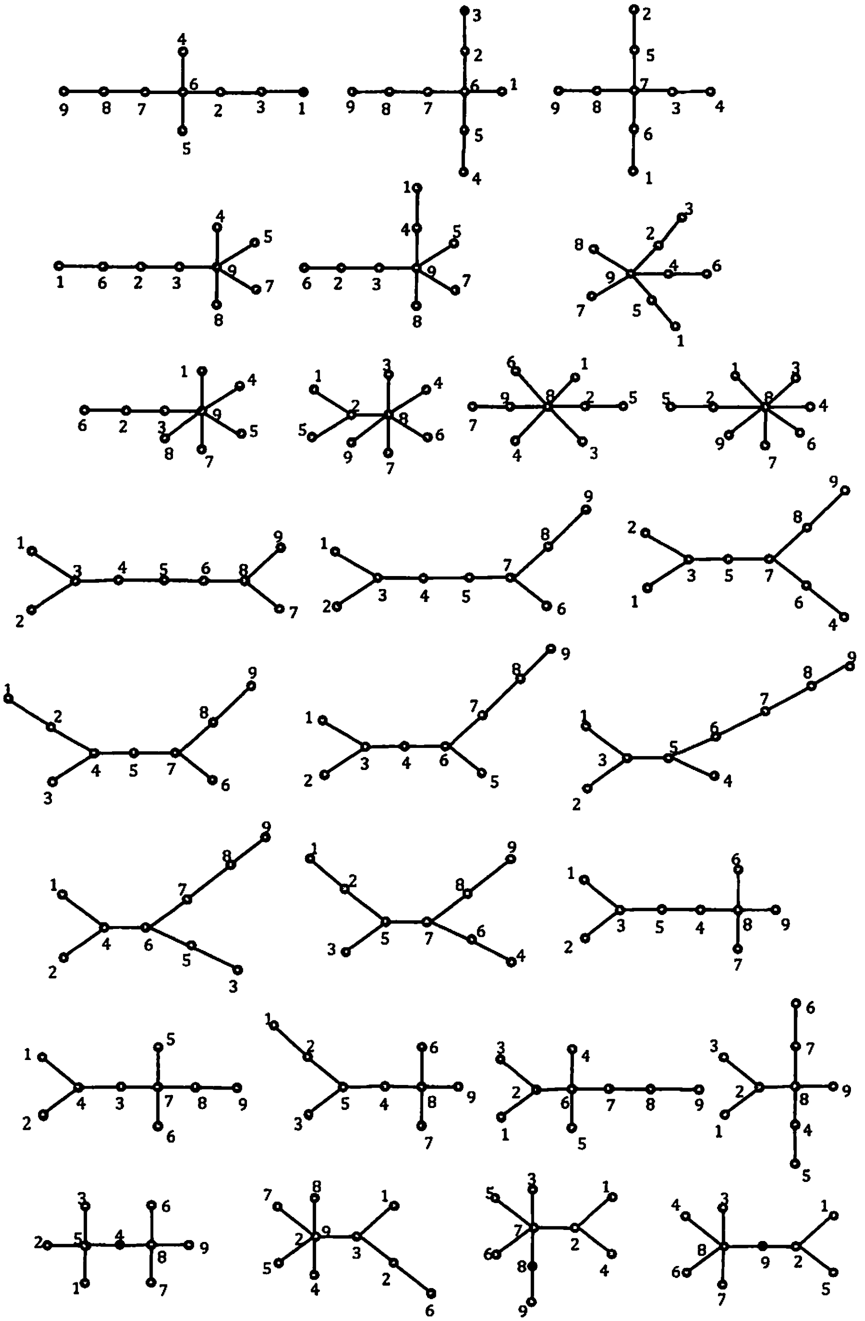




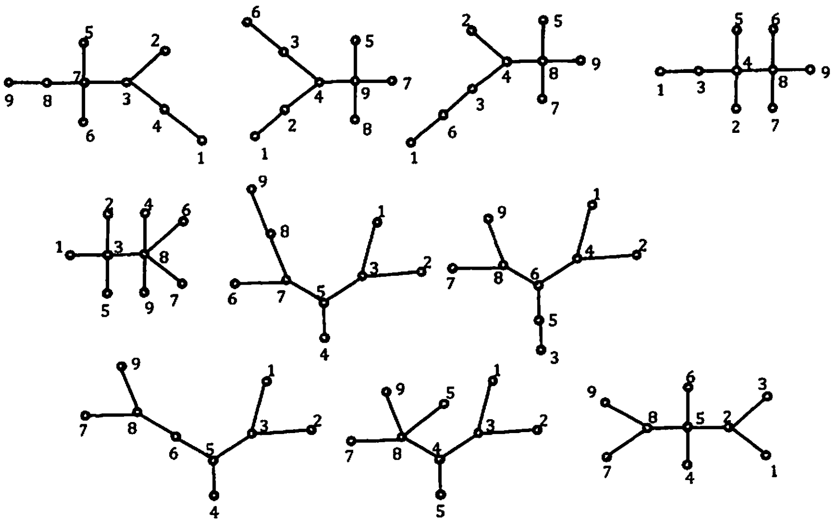
$p = 8$



$p = 9$



$p = 9$



$p = 9$

Survey 1

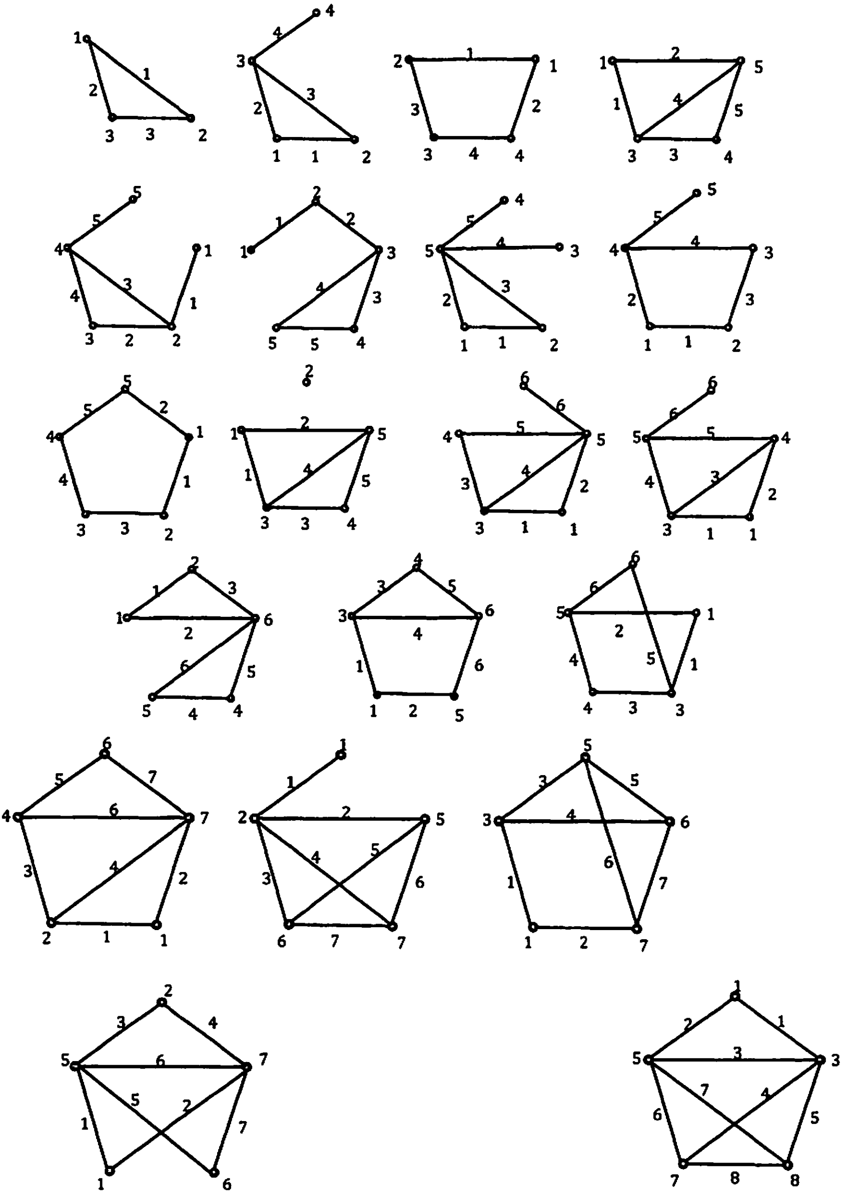
We prove that $K_{1,6}$ can't be labeled with this condition and similar argument could be applied for $K_{1,7}$ and $K_{1,8}$.

Suppose that $K_{1,6}$ could be labeled with the mentioned condition: the maximum possible edge label is 6 and the minimum possible one is 1, which means that all labels 1,2,3,4,5,6 appear as edge labels. However to get the edge labeled 6 the vertex labeled 6 and the vertex labeled 7 must be adjacent i.e. we have these two cases:

Case 1) The vertex labeled 7 is the central vertex: here the labels 1,2,3,4,5,6 are the labels of the pendant vertices, but $\left\lfloor\frac{2\cdot7\cdot5}{7+5}\right\rfloor = \left\lfloor\frac{2\cdot7\cdot4}{7+4}\right\rfloor = 5$, which means a repeated edge label "5": a contradiction.

Case 2) The vertex labeled 6 is the central vertex: here the labels $\{1,2,3,4,5,7\}$ are the labels of the pendant vertices, similar to case 1) we get: $\left\lfloor\frac{2\cdot6\cdot4}{6+4}\right\rfloor = \left\lfloor\frac{2\cdot6\cdot3}{6+3}\right\rfloor = 4$, hence we get again a repeated edge label.

Survey 1.6: We present in the following: a survey of all graphs of order equal to or less than 5, which could be labeled as harmonic mean graphs with additional constrain: "*the label* $q+1$ *is not used in the labeling of vertices*" . Of course all graphs in which $q < p$ are not included. We give a labeling to all other graphs, except $K_4, K_4 \cup K_1, W_4$ and $K_5 \backslash e$, We will prove that they can't be labeled with this condition.



Survey 2

In each one of the graphs $K_4, K_4 \cup K_1, W_4$ and $K_5\backslash e$, we have to use the vertex label "1" if they admit such type of labeling (*the existence of vertex label "1" is the only way to get the edge label "1"*), but from *Lemma 1.1* we can deduce that the vertex labeled 1 can't be of degree > 2 (*any incident edge to the vertex labeled "1" has a label* ≤ 2 *from Lemma 1.1, i.e. it has either the label "1" or "2"*).
Realize that none of the graphs $K_4, K_4 \cup K_1, W_4$ and $K_5\backslash e$, has none isolated vertex of degree ≤ 2).

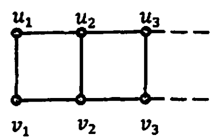
2 Mean cordial labeling

Theorem 2.1: Let $G(p,q)$ be a mean cordial graph, then:
The maximum number of edges could be found in a mean cordial graph of order p is given as follows:

$$\left.\begin{array}{l} \dfrac{p^2-3p+12}{6}, p \equiv 0\ (mod\ 3) \\ \dfrac{p^2+p+10}{6}, p \equiv 1\ (mod\ 3) \\ \dfrac{p^2-p+10}{6}, p \equiv 2\ (mod\ 3) \end{array}\right\} (*)$$

Proof: The maximum number of vertices labeled 0 in the mean cordial graph is $\left\lceil\frac{p}{3}\right\rceil$, and since every edge labeled 0 could only be obtained when it is incident to two vertices labeled 0, it follows that the maximum number of edges labeled 0 could not be greater than: $\binom{\lceil\frac{p}{3}\rceil}{2}$. Suppose now that we could have $\binom{\lceil\frac{p}{3}\rceil}{2}+1$ edges labeled 1 and 2, which means the number of all edges is: $3\binom{\lceil\frac{p}{3}\rceil}{2}+2$. This can be expressed as the three cases $(*)$ mentioned in the theorem above.

Theorem 2.2: The Ladder $L(p=2t,q), t=2,3,4,\ldots$, is a mean cordial graph if and only if $t \equiv 2\ (mod\ 3)$.
Proof: Suppose first $t = 3t_1 + 2$, we prove that the ladder in this case is mean cordial as follows: let the ladder be symbolized as in the figure below:



We define the labeling of the vertices:

$f(u_i) = 0$ and $f(v_i) = 0, 1 \le i \le t_1 + 1$

$f(u_i) = 1, t_1 + 2 \le i \le 2t_1 + 1$ and $f(v_i) = 1, t_1 + 2 \le i \le 2t_1 + 2$

$f(u_i) = 2, 2t_1 + 2 \le i \le 3t_1 + 2$ and $f(v_i) = 2, 2t_1 + 3 \le i \le 3t_1 + 2$

Clearly the number of vertices with the labels 0, 1 and 2 is $2t_1 + 2$, $2t_1 + 1$ and $2t_1 + 1$ respectively.
We check now the number of labeled edges of each type: the number of edges labeled 0 is $3t_1 + 1$, these edges are all edges of the subgraph "sub-ladder" of the vertices with indices i, $1 \le i \le t_1 + 1$, the number of edges labeled 1 is the number of edges of the subgraph of the vertices with indices i, $t_1 + 2 \le i \le 2t_1 + 1$, which is $3t_1 - 2$ plus 3 which is the number of the edges: $u_{t_1+1}u_{t_1+2}$, $v_{t_1+1}v_{t_1+2}$ and $v_{2t_1+1}v_{2t_1+2}$, the number of edges labeled 2 is the number of edges of the subgraph of the vertices with indices i, $2t_1 + 3 \le i \le 3t_1 + 2$, which is $3t_1 - 2$ plus 4 which is the number of the edges: $u_{2t_1+1}u_{2t_1+2}$, $u_{2t_1+2}u_{2t_1+3}$, $v_{2t_1+2}v_{2t_1+3}$ and $u_{2t_1+2}v_{2t_1+2}$, hence $e_f(0) = e_f(1) = 3t_1 + 1$, $e_f(2) = 3t_1 + 2$ i.e. the ladder is mean cordial in this case
We prove that the ladder is not mean cordial when $t \equiv 0$ and $1\ (mod\ 3)$,

Case 1) $t \equiv 0\ (mod\ 3)$: suppose that $t = 3t_1$, the maximum number of vertices labeled 0 is $2t_1$ and the number of edges is $q = 9t_1 - 2$. The minimum number of edges labeled 0 in the ladder to be mean cordial is $3t_1 - 1$, but $2t_1$ vertices labeled 0 can maximally give $3t_1 - 2$ edges labeled 0.
Case 2) $t \equiv 1\ (mod\ 3)$: suppose that $t = 3t_1 + 1$, the maximum number of vertices labeled 0 is $2t_1 + 1$ and the number of edges is $q = 9t_1 + 1$. The minimum number of edges labeled 0 in the ladder to be mean cordial is $3t_1$, but $2t_1 + 1$ vertices labeled 0 can maximally give $3t_1 - 1$ edges labeled 0.

Theorem 2.3: The bistar $B_{t,t}$(*Two copies of* $K_{1,t}$ *attached together with an edge between their centers*) is a mean cordial graph.
Proof: We will label the two centers of the bistar with the labels 0 and 1, for the pendant vertices we will label those which are adjacent to the vertex labeled 0 according to the following sequences:
the number of vertices labeled 0 is given by the sequence

1,2,3,3,4,5,5,6,7,7,8,9,9, ... ; $t = 2, 3, 4, 5, \ldots$

the number of vertices labeled 1 is given by the sequence

1,1,1,2,2,2,3,3,3,4,4,4,5, ... ; $t = 2, 3, 4, 5, \ldots$

Similarly we label the pendant vertices which are adjacent to the vertex labeled 1 according to the following sequences:
the number of vertices labeled 2 is given by the sequence

2,3,3,4,5,5,6,7,7,8,9,9,10,11,11, ... ; $t = 2, 3, 4, 5, \ldots$

the number of vertices labeled 1 is given by the sequence

$$0,0,1,1,1,2,2,2,3,3,3,4,4,4,5, \ldots; \; t = 2,3,4,5, \ldots$$

By counting the vertices labeled 0 and 2 we obtain the same number of edges of labels 0 and 2. The central edge is labeled 1 and every pendant vertex labeled 1 gives us an edge labeled 1 emerging from the vertex labeled 0. It is easy now to verify the number of edges of each label:

$e_f(0) = 1,2,3,3,4,5,5,6,7,7,8,9,9, \ldots; \; t = 2,3,4,5, \ldots$

$e_f(1) = 2,2,3,4,4,5,6,6,7,8,8,9,10, \ldots; \; t = 2,3,4,5, \ldots$

$e_f(2) = 2,3,3,4,5,5,6,7,7,8,9,9,10,11,11, \ldots; \; t = 2,3,4,5, \ldots$

(e.g. for $t = 5$ we have no. of edges labeled $0, 1, 2$ is $3, 4, 4$ respectively)

Theorem 2.4: The corona [2] $C_n \odot K_1$ is a mean cordial graph if and only if $n \equiv 1$ and $2 \; (mod\; 3)$.

Proof:

Case 1) $n \equiv 2 \; (mod\; 3)$: suppose $n = 3t + 2, t = 2,3,4, \ldots$

we give the vertices of a path of order $t + 1$ from the cycle and their pendant vertices the label 0, the remaining $2t + 1$ vertices of the cycle take the label 1, but their $2t + 1$ pendant vertices take the label 2. Hence we get $2t + 2$ vertices labeled 0 and $2t + 1$ vertices for each of the two labels 1 and 2. Regarding the edges we get clearly $2t + 1$ edges labeled 0. Also we get $2t + 2$ edges labeled 1 on the cycle and $2t + 1$ pendant edges labeled 2.

Case 2) $n \equiv 1 \; (mod\; 3)$: suppose $n = 3t + 1, t = 2,3,4, \ldots$

we give the vertices of a path of order $t + 1$ from the cycle the label 0 and we give t vertices of their pendants the label 0, the last pendant vertex gets the label 1. For the remaining $2t$ vertices of the cycle we give the label 2 to "the first" one and the label 1 to the next $2t - 1$ vertices, each of the $2t$ pendant vertices takes the label 2. Hence we get $2t + 1$ vertices for each of the two labels 0 and 2 and $2t$ vertices labeled 1. Regarding the edges we get clearly $2t$ edges labeled 0. Also we get $2t + 1$ edges labeled 1 and $2t + 1$ pendant edges labeled 2.

Finally we prove that $C_n \odot K_1$ is not mean cordial when $n \equiv 0 \; (mod\; 3)$. Suppose that $C_n \odot K_1$ is a mean cordial graph with $n \equiv 0 \; (mod\; 3)$. Whatever the way we label $\frac{n}{3}$ of the vertices with the label 0, we can't get more than $\frac{n}{3} - 1$ edges labeled 0. Anyway $\frac{n}{3} - 1$ edges labeled 0 lead us to contradiction since $C_n \odot K_1$ is a mean cordial graph, it must contain $n/3$ edges of each label.

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The Authors are thankful to the referee for his valuable remarks.