List vertex arboricity of graphs without forbidden minors*

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Abstract

The notions of L-tree-coloring and list vertex arboricity of graphs are introduced in the paper, while a sufficient condition for a plane graph admitting an L-tree-coloring are given. Further, it is proved that every graph without K_5-minors or $K_{3,3}$-minors has list vertex arboricity at most 3, and this upper bound is sharp.

Keywords: vertex arboricity, K_5-minor-free, $K_{3,3}$-minor-free, minor closed.

1 Introduction

All graphs in this paper are undirected, finite and simple. A graph is *planar* if it can be drawn on the plane in such a way that no edges cross each other. Such a drawing of a planar graph is called a *plane* graph. A cycle C in a plane graph is *separating* if both the interior and exterior of C contains vertices of G. A plane graph G is a *near-triangulation* if the boundary of every face, except possibly the outer face, is a cycle on three vertices, and is *triangulation* if the boundary of every face is a cycle on three vertices.

A graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges. The theory of graph minors began with the well-known Wagner's theorem [4] that a graph is planar if and only if its minors do not include the complete graph K_5 nor the complete bipartite graph $K_{3,3}$.

A *k-tree-coloring* of G is a function φ from the vertex set $V(G)$ to $\{1,2,\dots,k\}$ so that the graph induced by $\varphi^{-1}(i)$ is a union of trees for every $1 \le i \le k$. The minimum integer k so that G admits a k-tree-coloring is the *vertex arboricity* of

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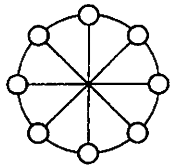

Figure 1: Wagner graph

G, denoted by $va(G)$. Chartrand, Kronk and Wall [3] showed that $va(G) \leq 3$ for any planar graph G.

Naturally, we can consider the list version of vertex arboricity. Let $L(v)$ be a *list* of colors assigned to each vertex $v \in V(G)$. An *L-tree-coloring* of G is a function $\varphi : V(G) \rightarrow \bigcup_v L(v)$ so that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and the graph induced by $\varphi^{-1}(i)$ is a union of trees for every $i \in \bigcup_v L(v)$. A graph G is *list k-tree-colorable* if G has an L-tree-coloring as long as one assign to each vertex $v \in V(G)$ an arbitrary list $L(v)$ with size k. The minimum integer k so that G is list k-tree-colorable is the *list vertex arboricity* of G, denoted by $va_l(G)$. Clearly, $va(G) \leq va_l(G)$, but whether there is a gap between these two parameters is unknown.

In this paper, we first give a sufficient condition for a plane graph admitting an L-tree-coloring (see Theorem 6), and further, prove that $va_l(G) \leq 3$ if G is K_5-minor-free, or $K_{3,3}$-minor-free (see Theorem 11).

2 Main results and their proofs

By $G_1 \cap G_2$ (resp. $G_1 \cup G_2$), we denote the graph with vertex set $V(G_1) \cap V(G_2)$ (resp. $V(G_1) \cup V(G_2)$) and edge set $E(G_1) \cap E(G_2)$ (resp. $E(G_1) \cup E(G_2)$). If G_1 and G_2 are subgraphs of G so that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete graph on k-vertices, then we say that G is the *clique k-sum* of G_1 and G_2.

An H-minor-free graph G is *edge-maximal* if the graph derived form G by joining any two nonadjacent vertices has at least one H-minor. A planar graph G is *edge-maximal* if joining any two nonadjacent vertices of G will disturb the planarity. *Wagner graph* is a 3-regular graph with 8 vertices and 12 edges named after Klaus Wagner, see Figure 1.

Lemma 1. *(Wagner [4]) A graph is K_5-minor-free if and only if it can be obtained by clique 0-, 1-, 2-, 3-summing starting from planar graphs and the Wagner graph.*

Lemma 2. *(Wagner [4]) A graph is $K_{3,3}$-minor-free if and only if it can be obtained by clique 0-, 1-, 2-summing starting from planar graphs and K_5.*

The following corollaries directly follow from Lemmas 1 and 2, respectively.

Corollary 3. *If G is an edge-maximal K_5-minor-free graph, then it can be obtained by clique 2-, 3-summing starting from edge-maximal planar graphs and the Wagner graph.* □

Corollary 4. *If G is an edge-maximal $K_{3,3}$-minor-free graph, then it can be obtained by clique 2-summing starting from edge-maximal planar graphs and K_5.* □

Lemma 5. *Let G be a near-triangulation with outer face $C = v_1v_2 \dots v_nv_1$. Assume that $L(u)$ is a list of at least two colors for $u \in V(C)$, and at least three colors for $u \in V(G) \setminus V(C)$. If φ is an L-tree-coloring of $\{v_1, v_2\}$, then φ can be extended to an L-tree-coloring of G.*

Proof. We prove it by induction on n. First, the conclusion is trivial when $n = 3$, so we assume that it holds for any near-triangulation with order less than n and consider near-triangulation G with order $n \geq 4$.

If C contains a chord v_iv_j with $1 \leq i < j \leq n$, then let G_1 be the subgraph induced by $\{v_i, v_{i+1}, \dots, v_j\}$ and let G_2 be the subgraph induced by $\{v_j, v_{j+1} \dots, v_n, v_1, \dots, v_i\}$.

Since G is a near-triangulation, G_1 is a near-triangulation with outer face $C_1 = v_iv_{i+1} \dots v_jv_i$ and G_2 is a near-triangulation with outer face $C_2 = v_jv_{j+1} \dots v_nv_1 \dots v_i v_j$. Without loss of generality, assume that $v_1, v_2 \in V(C_1)$.

By the induction hypothesis, φ can be extended to an L-tree-coloring λ_1 of G_1, and then the coloring on $\{v_i, v_j\}$ can be extended to an L-tree-coloring λ_2 of G_2. Combining the L-tree-colorings λ_1 with λ_2, we obtain a coloring λ of G. If λ is not an L-tree-coloring of G, then there is a monochromatic cycle in G that is incident with the chord v_iv_j. This implies a monochromatic cycle in either G_1 or G_2, which are all impossible since λ_1 and λ_2 are L-tree-colorings. Therefore, λ is an L-tree-coloring of G to which φ is extended.

Hence we assume that C contains no chord.

Let $v_1, u_1, u_2, \dots, u_k$ and v_{n-1} be the neighbors of v_n in that clockwise order around v_n. Since G is a near-triangulation and C is chordless, $C' = v_1v_2 \dots v_{n-1}u_k u_{k-1} \dots u_1v_1$ is a cycle.

Let G' be the subgraph induced by the vertices of C'. Clearly, G' is a near-triangulation with outer face C'. Let $a \in L(v_n) \setminus \{\varphi(v_1)\}$ and let $L'(u_i) = L(u_i) \setminus \{a\}$ for every $1 \leq i \leq k$. Further, let $L'(w) = L(w)$ for every $w \in V(G') \setminus \{u_1, \dots, u_k\}$. This follows that $|L'(w)| \geq 2$ for every $w \in V(C')$ and $|L'(w)| \geq 3$ for every $w \in V(G') \setminus V(C')$. Hence by the induction hypothesis, φ can be extended to an L'-tree-coloring λ' of G'. At last, we color v_n with a and get a coloring λ of G. Since λ' is an L'-tree-coloring and at most one neighbor of v_n is colored with a under λ', λ is an L-tree-coloring of G, as required. □

The following theorem is an immediate corollary of the above lemma.

Theorem 6. *Let G be a plane graph with outerface C. If L is a list of colors so that $|L(v)| = 2$ for every $v \in V(C)$ and $|L(v)| = 3$ for every $v \in V(G) \setminus V(C)$, then G has an L-tree-coloring.* □

Lemma 7. *Let G be a near-triangulation and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_3 and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. We prove it by induction on the order n of a near-triangulation. First, the conclusion is trivial when $n = 3$, so we assume that it holds for any near-triangulation with order less than n and consider near-triangulation G with order $n \geq 4$.

If $H = uvwu$ (i.e., a cycle on three vertices) is not separating, then we may redraw the graph G and add some necessary edges so that the resulting drawing, also denoted by G, is a triangulation with outer face H. Let $u, w_1, w_2, \ldots, w_k$ and v be the neighbors of w in that clockwise order around w. Since G is a triangulation, $uw_1 \ldots w_kvu$ is a cycle C. Let G' be the subgraph induced by the vertices of C'. Clearly, G' is a near-triangulation with outer face C'. Let $L'(w_i) = L(w_i) \setminus \{\varphi(w)\}$ for every $1 \leq i \leq k$ and $L'(x) = L(x)$ for every $x \in V(G') \setminus \{w_1, \ldots, w_k\}$. Since $|L'(x)| \geq 2$ for every $x \in V(C')$ and $|L'(x)| \geq 3$ for every $x \in V(G') \setminus V(C')$, the coloring φ of $\{u, v\}$ can be extended to an L-tree-coloring of G' by Lemma 5. Clearly, this L-tree-coloring of G' along with the coloring φ of the vertex w form an L-tree-coloring of G.

If $H = uvwu$ is separating, then the subgraph G_1 induced by the vertices inside or on H is a triangulation with outer face H. By the induction hypothesis, the coloring φ on H can be extended to an L-tree-coloring λ_1 of G_1. On the other hand, the subgraph G_2 induced by the vertices outside or on H is a near-triangulation with a non-separating cycle H on three vertices, so by the same argument as the one in the previous paragraph, the coloring φ on H can be extended to an L-tree-coloring λ_2 of G_2. Combining the L-tree-colorings λ_1 with λ_2, we obtain an L-tree-coloring of G. □

Lemma 8. *Let G be a near-triangulation and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. If $H = xy$ is an edge on the outer face of G, then the conclusion follows from Lemma 5. If $H = xy$ is not an edge on the outer face of G, then there is a vertex z so that $H' = xyzx$ forms a K_3. Clearly, φ can be extended to an L-tree-coloring φ' of H' via coloring z with a color different from $\varphi(x)$ and $\varphi(y)$. By Lemma 7, φ' can be extended to an L-tree-coloring of G. □

Similar conclusion also holds for the Wagner graph and the complete graph K_5. The proof of the following lemma is quite basic, so we omit it here.

Lemma 9. *Let G be the Wagner graph (resp. the complete graph K_5) and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 or K_3, and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.* □

Lemma 10. *Let G be an edge-maximal K_5-minor-free graph (resp. $K_{3,3}$-minor-free graph) and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 or K_3 (resp. isomorphic to K_2), and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. If G is an edge-maximal planar graph, then G can be drawn as a triangulation, so the conclusion holds by Lemmas 7 and 8 (resp. by Lemma 8). If G is the Wagner graph (resp. the complete graph K_5), then it holds by Lemma 9. In the following, we assume that G is neither an edge-maximal planar graph nor the Wagner graph (resp. the complete graph K_5), and prove it by induction on the order of G.

By Corollary 3, $G = G_1 \cup G_2$, where G_1 is an edge-maximal K_5-minor-free graph (resp. edge-maximal $K_{3,3}$-minor-free graph) and G_2 is an edge-maximal planar graph or a Wagner graph (resp. K_5) so that $G_1 \cap G_2 = H'$ that is isomorphic to K_2 or K_3 (resp. K_2).

If $H \subseteq G_1$, then by the induction hypotheses, φ can be extended to an L-tree-coloring λ_1 of G_1. Whereafter, the L-tree-coloring of H' can be extended to an L-tree-coloring λ_2 of G_2 by Lemmas 7, 8 and 9 (resp. by Lemmas 8 and 9). Combining the coloring λ_1 with λ_2, we obtain an L-tree-coloring of G.

If $H \subseteq G_2$, then by Lemmas 7, 8 and 9 (resp. by Lemmas 8 and 9), φ can be extended to an L-tree-coloring λ_2 of G_2. Whereafter, the L-tree-coloring of H' can be extended to an L-tree-coloring λ_1 of G_1 by the induction hypotheses. Combining the coloring λ_1 with λ_2, we obtain an L-tree-coloring of G. □

Theorem 11. *If G is a K_5-minor-free graph, or a $K_{3,3}$-minor-free graph, then $va_l(G) \leq 3$. Moreover, this upper bound 3 is best possible.*

Proof. Since every K_5-minor-free graph (resp. $K_{3,3}$-minor-free graph) is a subgraph of an edge-maximal K_5-minor-free graph (resp. edge-maximal $K_{3,3}$-minor-free graph), this conclusion directly follows from Lemma 10.

Since there exists planar graph with vertex arboricity exactly 3 (see [2]) and every planar graph is K_5-minor-free and $K_{3,3}$-minor-free, the upper bound 3 in this theorem is best possible. □

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, GTM 244, 2008.

[2] G. Chartrand, H. V. Kronk. The point-arboricity of planar graphs. *J. Lond. Math. Soc*, 44 (1969), 612–616.

[3] G. Chartrand, H. V. Kronk, C. E. Wall, The point-arboricity of a graph. *Israel J. Math.*, 6 (1968), 169–175.

[4] K. Wagner, über eine Eigenschaft der ebenen Komplexe, *Math. Ann.*, 114 (1937), 570–590,