The Fibonacci Polynomials in Rings

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Abstract In this paper, we study the Fibonacci polynomials modulo m such that $x^2 = x + 1$ and then we obtain miscellaneous properties of these sequences. Also, we extend the Fibonacci polynomials to the ring of complex numbers. We define the Fibonacci Polynomial-type orbits $F^R_{(a,b)}(x) = \{x_i\}$ where R be a 2-generator ring and (a, b) is a generating pair of the ring R. Furthermore, we obtain the periods of the Fibonacci Polynomial-type orbits $F^R_{(a,b)}(x)$ in finite 2-generator rings of order p^2.

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1 Introduction and Preliminaries

Fibonacci numbers are one of the most well-known numbers, and it has many important applications to diverse fields such as mathematics, computer science, physics, biology and statistics. The Fibonacci numbers f_n are the terms of the sequence $0,1,1,2,3,5,8,13,\ldots$ where $f_n = f_{n-1} + f_{n-2}$ with the initial values $f_0 = 0$ and $f_1 = 1$. Generalized Fibonacci sequence have been intensively studied for many years and have become into an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order (tribonacci, k-nacci) sequences are generally viewed as sequences of integers.

Most of the study of Fibonacci sequences is done with groups. The notion of Wall number was first proposed by D. D. Wall [1] in 1960 and gave some theorems and properties concerning Wall number of the Fibonacci sequence. K. Lü and J. Wang [13] contributed to the study of the Wall number for the k-step Fibonacci sequence. In the mid eighties, Wilcox [7] extended the problem to abelian groups. Knox [9] proved that the periods of k-nacci (k-step Fibonacci sequences in dihedral groups were equal to $2k + 2$. Deveci, Karaduman and Campbell [16] examined the behavior of the period of the k-nacci sequence in some finite binary polyhedral groups. Recently, k-nacci sequences have been investigated; see for example [12, 14, 15]. However, very little is done with the rings. Let R be a ring identity I. Consider the sequence $\{M_n\}$ of elements of R, recursively defined by

$$M_{n+2} = A_1 M_{n+1} + A_0 M_n \quad \text{for} \quad n \geq 0 \qquad (1.1)$$

D. J. DeCarli begin by considering a special case of (1.1), denoted by $\{F_n\}$ and defined by

$$F_{n+2} = A_1 F_{n+1} + A_0 F_n \qquad \text{for} \qquad n \geq 0$$

That is, D. J. DeCarli [6]gave a generalized Fibonacci sequence over an arbitrary ring in 1970. Special cases of Fibonacci sequence over an arbitrary ring have been considered by R. G. Buschman [4], A. F. Horadam [2] and N. N. Vorobyov [3] where this ring was taken to be the set of integers. O. Wyler [5]also worked with such a sequence over a particular commutative ring with identity. Taşyurdu and Gültekin obtain the period of generalized Fibonacci sequence in finite rings with identity of order p^2 by using equality recursively defined by $F_{n+2} = A_1 F_{n+1} + A_0 F_n$, for $n \geq 0$, where $F_0 = 0$ (the zero of the ring), $F_1 = 1$ (the identity of the ring) and A_0, A_1 are generator elements of finite rings with identity of order p^2 [17]. Classification of all finite rings of order p^2 with p a prime has been studied by B. Fine [10].

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots$ is periodic after the initial element a and has period 4 and the sequence $a, b, c, d, e, f, b, c, d, e, f, b, c, d, e, f \ldots$ is simply periodic with period 6. The minimum period length of $(F_i \, mod n)_{i=-\infty}^{\infty}$ sequence is denote by $k(n)$ and is called Wall number of n [1].

Consequence 1.1 (Renault [11]).

$$F_{k(n)} \equiv 0 \pmod n \tag{1.2}$$

$$F_{k(n)-1} \equiv F_{k(n)+1} \equiv F_{k(n)+2} \equiv 1 \pmod n \tag{1.3}$$

Definition 1.2. The Fibonacci polynomials are defined by recurrence relation

$$F_n(x) = \begin{cases} 0 & if\ n = 0 \\ 1 & if\ n = 1 \\ xF_{n-1}(x) + F_{n-2}(x) & if\ n \geq 2 \end{cases} \tag{1.4}$$

Note the the Fibonacci polynomials are generated by a matrix
$Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$,

$$(Q_2)^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix} \tag{1.5}$$

which can be proved by mathematical induction. The first few Fibonacci polynomials are displayed below as well as the array of their coefficients [8].

Theorem 1.3. For any prime p, up to isomorphism, the finite 2-generator rings which is not field of order p^2 are given by the following presentations [10]:

$D = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle \cong \mathbb{Z}_p + \mathbb{Z}_p$

$E = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$

$F = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a \rangle$

$G = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a \rangle$

$H = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle \cong \mathbb{Z}_p + C_p(0)$

$I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0 \rangle$

$J = \langle a, b \mid pa = pb = 0, a^2 = b^2 = 0 \rangle \cong C_p \times C_p(0)$.

Definition 1.4. The sequence $\{M_n\}$ of elements of R is defined by

$$M_{n+2} = A_1 M_{n+1} + A_0 M_n \quad \text{for} \quad n \geq 0 \tag{1.6}$$

where R is a ring with identity I and M_0, M_1, A_0, A_1 are arbitrary elements of R [6].

Definition 1.5. A special case of equality (1.6) is denoted by $\{F_n\}$ and defined by

$$F_{n+2} = A_1 F_{n+1} + A_0 F_n \quad \text{for} \quad n \geq 0$$

where $F_0 = 0$ (the zero of the ring), $F_1 = 1$ (the identity of the ring) and A_0, A_1 are arbitrary elements of R [6].

2 The Fibonacci Polynomials Modulo m

Reducing the sequence of the Fibonacci Polynomials by a modulus m such that $x^2 = x + 1$, we can get a repeating sequence, denoted by $\{F_n(x)(mod\ m)\} = \{F_0(x)(mod\ m), F_1(x)(mod\ m), \ldots F_i(x)(mod\ m), \ldots\}$.
It has the same recurrence relation as in (1.4).

Theorem 2.1. The sequence $\{F_n(x)(mod\ m)\}$ is simply periodic.

Proof. Let $= \{a_1x + b_1, a_2x + b_2 \mid 0 \leq a_1, a_2, b_1, b_2 \leq m - 1\}$, then $|S| = m^4$. The sequence $\{F_n(x)(mod\ m)\}$ repeats since there are only a finite number m^4 of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms, which implies that the sequence $\{F_n(x)(mod\ m)\}$ is periodic. Since the sequence is periodic, there exist natural numbers i and j, with $i > j$ such that $F_{i+1}(x)(mod\ m) = F_{j+1}(x)(mod\ m)$, $F_{i+2}(x)(mod\ m) = F_{j+2}(x)(mod\ m)$. From the definition of the sequence $\{F_n(x)\}$, we can easily derive that $F_{n-2}(x) = F_n(x) - xF_{n-1}(x)$.

Therefore, we obtain $F_i(x)(mod\ m) = F_j(x)(mod\ m)$, and hence,
$F_{i-1}(x)(mod\ m) = F_{j-1}(x)(mod\ m)$,
$F_{i-2}(x)(mod\ m) = F_{j-2}(x)(mod\ m), \ldots,$
$F_{i-j+1}(x)(mod\ m) = F_1(x)(mod\ m)$,
$F_{i-j}(x)(mod\ m) = F_0(x)(mod\ m)$.
So we get that the sequence is simply periodic.

We next denote the period of the sequence $\{F_n(x)(mod\ m)\}$ by $h^{F(x)}(m)$.

Example. The sequence$\{F_n(x)(mod\ 3)\}$ is
$\{0,1,x,x+2,x+1,0,x+1,2x+1,x,2,0,2,2x,2x+1,2x+2,0,2x+2,$
$x+2,2x,1,0,1,x,\ldots\}$
and thus $h^{F(x)}(3) = 20$.

For a given matrix $A = [P_{ij}]$ with P_{ij}'s being polynomials, $A(mod\ m)$ means that every entry of A is reduced modulo m, that is, $A(mod\ m) = (P_{ij}(mod\ m))$. Let $\langle A\rangle_m = \{(A)^n(mod\ m) \mid n \geq 0\}$. If $\gcd(\det A, m) = 1$, $\langle A\rangle_m$ is a cyclic group. We denote the cardinal of the set $\langle Q_2\rangle_m$ by $|\langle Q_2\rangle_m|$. Since $detQ_2 = -1$, it is clear that the set is a cyclic group for every positive integer m.

It is easy to see from (1.5) that $h^{F(x)}(p) = |\langle Q_2\rangle_p|$ for every prime p if $x^2 = x + 1$.

Theorem 2.2. If $m = \prod_{i=1}^{t} p_i^{e_i}$, $(t \geq 1)$ where p_i's are distinct primes, then $h^{F(x)}(m)$=lcm$\left[h^{F(x)}(p_1^{e_1}), h^{F(x)}(p_2^{e_2}), \dots, h^{F(x)}(p_t^{e_t})\right]$.

Proof. Since $h^{F(x)}(p_i^{e_i})$ is the length of the period of the sequence $\{F_n(x)(mod\, p_i^{e_i})\}$, the sequence $\{F_n(x)(mod\, p_i^{e_i})\}$ repeats only after blocks of length $k.h^{F(x)}(p_i^{e_i})$, $(k \in \mathbb{N})$. Since also $h^{F(x)}(m)$ is the length of the period $\{F_n(x)(mod\, m)\}$, the sequence $\{F_n(x)(mod\, p_i^{e_i})\}$ repeats after $h^{F(x)}(m)$ terms for all values i. This implies that $h^{F(x)}(m)$ is of the form $k.h^{F(x)}(p_i^{e_i})$ for all values of i. We thus prove that $h^{F(x)}(m)$ equals the least common multiple of $h^{F(x)}(p_i^{e_i}) l_p^j(u_i^{e_i})$'s.

Example. The sequences $\{F_n(x)(mod\, 2)\}$ and $\{F_n(x)(mod\, 6)\}$ are as follows, respectively:

$$\{0,1,x,x,1,0,1,x,\dots\}$$

and

$\{0,1,x,x+2,4x+1,0,4x+1,5x+4,x,5,0,5,5x,5x+4,2x+5,0,\ 2x+5,x+2,5x,1,0,1,x,\dots\}$.

Then, we obtain $h^{F(x)}(2) = 5$ and $h^{F(x)}(6) = 20$. Also we know that $h^{F(x)}(3) = 20$. Thus it is verified that $h^{F(x)}(6)$ =lcm$\left[h^{F(x)}(2), h^{F(x)}3\right]$.

Theorem 2.3. Let p be a prime and let u be the largest positive integer such that such that $h^{F(x)}(p) = h^{F(x)}(p^u)$. Then we have $h^{F(x)}(p^v) = p^{v-u} h^{F(x)}(p)$ for every $v \geq u$. In particular, if $h^{F(x)}(p) \neq h^{F(x)}(p^2)$, then we have $h^{F(x)}(p^v) = p^{v-1} h^{F(x)}(p)$ for every $v \geq 2$.

Proof. Let k be a positive integer and I be the 2×2 identity matrix. If $(Q_2)^{h^{F(x)}(p^{k+1})} \equiv I(mod\, p^{k+1})$, then $(Q_2)^{h^{F(x)}(p^{k+1})} \equiv I(mod\, p^k)$. This yields that $h^{F(x)}(p^k)$ divides $h^{F(x)}(p^{k+1})$. Also, writing $(Q_2)^{h^{F(x)}(p^k)} = I + (q_{i,j}^{(k)}.p^k)$ we obtain

$$(Q_2)^{h^{F(x)}(p^k).p} = \left(I + \left(q_{i,j}^{(k)}.p^k\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(q_{i,j}^{(k)}.p^k\right)^i \equiv I(mod\, p^{k+1})$$

by the binomial expansion. This means that $h^{F(x)}(p^{k+1})$ divides $h^{F(x)}(p^k).p$. Therefore, $h^{F(x)}(p^{k+1}) = h^{F(x)}(p^k)$ or $h^{F(x)}(p^{k+1}) = h^{F(x)}(p^k).p$, and the latter holds if, and only if, there is a $q_{i,j}^{(k)}$ which is not divisible by p. Since $h^{F(x)}(p^u) \neq h^{F(x)}(p^{u+1})$, there is an $q_{i,j}^{(u+1)}$ which is not divisible by p, thus, $h^{F(x)}(p^{u+1}) \neq h^{F(x)}(p^{u+2})$. To complete the proof we use an inductive method on u.

Example. Since $\{F_n(x)(mod\ 4)\} = \{0,1,x,x+2,1,2x+2,3,x+2,3x,1,0,1,x,...\}$, $h^{F(x)}(4) = 10$. Also we know that $h^{F(x)}(2) = 5$. Thus it is verified that $h^{F(x)}(4) = 2.h^{F(x)}(2)$.

3. The Fibonacci Polynomials in The Set of Complex Numbers $\mathbb{C}$

Define the sequence of the Fibonacci polynomials in the set of complex numbers $\mathbb{C}$ as shown:

$$F_n = \begin{cases} 0, & \text{if } \ n = 0, \\ 1, & \text{if } \ n = 1, \\ (i+1)F_{n-1} + F_{n-2} & \text{if } \ n \geq 2. \end{cases} \tag{3.1}$$

Letting

$$M = \begin{bmatrix} i+1 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $n \geq 1$, by (1.5) we may write

$$M^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \tag{3.2}$$

It is easy to see that we obtain the sequence in (3.1) if we choose $x = i + 1$ in (1.4). Let $h^{\mathbb{C}}(m)$ denote the period of the sequence of $\{F_n(mod\ m)\}$ which is obtained by reducing the sequence of the Fibonacci polynomials in the set of complex numbers $\mathbb{C}$ modulo m and let $|\langle M\rangle_m|$ denote the order of the cyclic group $\langle M\rangle_m$ which is generated by reducing the matrix m modulo. Therefore, it is clear that the rules produced for the period $h^{\mathbb{C}}(m)$ and the cyclic group $\langle M\rangle_m$ are of the same form of the results obtained in the above.

It is important to note that $h^{\mathbb{C}}(m)$ may not be equal to $h^{F(x)}(m)$. We can give the following Example for this situation.

Example. The sequences $\{F_n(mod\ 2)\}$ and $\{F_n(x)(mod\ 2)\}$ are as follows, respectively:

$$0,1,i+1,1,0,1,i+1,...$$

$$0,1,x,x,1,0,1,...$$

and thus $h^{\mathbb{C}}(2) = 4$ and $h^{F(x)}(2) = 5$.

4 The Fibonacci Polynomials in Some Finite Rings

Definition 4.1. Let R be a 2-generator ring and (a,b) be a generating pair of the ring R. We define the Fibonacci Polynomial-type orbit $F^R_{(a,b)}(x) = \{x_i\}$ of (a,b) by

$$x_0 = a, \quad x_1 = b, \qquad x_{n+1} = bx_n + x_{n-1} \qquad n \geq 1.$$

Similarly, we define the Fibonacci Polynomial-type orbit $F^R_{(b,a)}(x) = \{x_i\}$ of (b, a) by

$$x_0 = b, \quad x_1 = a, \qquad x_{n+1} = ax_n + x_{n-1} \qquad n \geq 1.$$

Proposition 4.2. A Fibonacci Polynomial-type orbit of a finite 2-generator is periodic.

Proof. Let R be a finite 2-generator ring and n be the order of R. Since there are n^2 distinct 2-tuples of elements of R, at least one of the 2-tuples appears twice in a Fibonacci Polynomial-type orbit. Therefore, the subsequence following this 2-tuple repeats. Because of the repeating the sequence is periodic.

We next denote the period of the sequence $F^R_{(a,b)}(x)$ by $PF^R_{(a,b)}(x)$.

Definition 4.3. Let R be a finite 2-generator ring. If there exist a Fibonacci Polynomial-type orbit of the ring R such that every element of the ring R appears in the sequence, then the ring R is called Fibonacci Polynomial-type sequenceable.

Example. Let us consider the finite ring with identity

$$D = \langle a, b\, ; pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle = \mathbb{Z}_p + \mathbb{Z}_p$$

which has two generators a and b. For $p = 3$ the orbit $F^D_{(a,b)}(x)$ is

$$a, b, a + b, 2b, a, 2b, a + 2b, b, a, b, \ldots$$

and hence is Fibonacci Polynomials-type sequenceable. Also, $PF^D_{(a,b)}(x) = 8$.

Proposition 4.4. For any prime $p \neq 2$,

$$D = \langle a, b\, ; pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle = \mathbb{Z}_p + \mathbb{Z}_p$$

with generators a and b. The periods of the Fibonacci Polynomial-type orbits $F^D_{(a,b)}$ and $F^D_{(b,a)}$ are $k(p)$.

Proof. Let us consider the Fibonacci Polynomial-type orbit $F^D_{(a,b)}$. The sequence $F^D_{(a,b)}$ is as follows:

$$x_0 = a, x_1 = b, x_2 = a + b, x_3 = 2b, x_4 = a + 3b, x_5 = 5b,$$
$$x_6 = a + 8b, x_7 = 13b, x_8 = a + 21b, \ldots, x_{2n} = a + f_{2n}b,$$
$$x_{2n+1} = f_{2n+1}b,$$
$$\begin{aligned} x_{2n+2} &= bx_{2n+1} + x_{2n} = b(f_{2n+1}b) + a + f_{2n}b = f_{2n+1}b^2 + a + f_{2n}b \\ &= f_{2n+1}b + a + f_{2n}b = b(f_{2n+1} + f_{2n}) + a = a + f_{2n+2}b, \\ x_{2n+3} &= bx_{2n+2} + x_{2n+1} = b(a + f_{2n+2}b) + f_{2n+1}b \\ &= ba + f_{2n+2}b^2 + f_{2n+1}b = f_{2n+2}b + f_{2n+1}b = (f_{2n+2} + f_{2n+1})b \\ &= f_{2n+3}b, \ldots. \end{aligned}$$

Using the above, the sequence becomes:

$a + f_0 b, f_1 b, a + f_2 b, f_3 b, \ldots, a + f_{2n} b, f_{2n+1} b, a + f_{2n+2} b, f_{2n+3} b, \ldots,$
where f_n denote the n th term of the ordinary 2-step Fibonacci sequence. Now let's determine the period of this sequence up to prime number p. It can be seen that the coefficient of the term b of each element x_n of this sequence is term of ordinary Fibonacci sequences. Hence, the period of the sequence is determined by the coefficient of the term b which f_n is the n th term of the ordinary 2-step Fibonacci sequence. Thus, we obtain $PF^{D}_{(a,b)}(x) = k(p)$.

There is a similar proof for the sequence $F^{D}_{(b,a)}(x)$.

Proposition 4.5. For any prime $p \neq 2$.

$$E = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$$

with generators a and b. The periods of the Fibonacci Polynomial-type orbits $F^{E}_{(a,b)}$ and $F^{E}_{(b,a)}$ are $k(p)$.

Proof. Let us consider the Fibonacci Polynomial-type orbit $F^{E}_{(a,b)}$. The sequence $F^{E}_{(a,b)}$ is as follows:

$x_0 = a, x_1 = b, x_2 = a + b, x_3 = 3b, x_4 = a + 4b, x_5 = 8b,$
$x_6 = a + 12b,\ x_7 = 21b,\ x_8 = a + 33b,\ \ldots\ ,\ x_{2n} = a + (f_{2n+1} - 1)b,$
$x_{2n+1} = f_{2n+2} b,$

$$\begin{aligned} x_{2n+2} &= b x_{2n+1} + x_{2n} = b(f_{2n+2} b) + a + (f_{2n+1} - 1)b \\ &= f_{2n+2} b^2 + a + f_{2n+1} b - b = f_{2n+2} b + a + f_{2n+1} b - b \\ &= (f_{2n+2} + f_{2n+1}) b + a - b = f_{2n+3} b + a - b = a + (f_{2n+3} - 1)b, \\ x_{2n+3} &= b x_{2n+2} + x_{2n+1} = b(a + (f_{2n+3} - 1)b) + f_{2n+2} b \\ &= ba + f_{2n+3} b^2 - b^2 + f_{2n+2} b = b + f_{2n+3} b - b + f_{2n+2} b \\ &= (f_{2n+3} + f_{2n+2}) b = f_{2n+4} b, \ldots . \end{aligned}$$

Using the above, the sequence becomes:

$a + (f_1 - 1)b, f_2 b, a + (f_3 - 1)b, f_4 b, a + (f_5 - 1)b, \ldots,$
$a + (f_{2n+1} - 1)b, f_{2n+2} b, a + (f_{2n+3} - 1)b, f_{2n+4} b, \ldots$

where f_n denote the n th term of the of ordinary 2-step Fibonacci sequence. Notice that each element x_n of this sequence has the form

$$x_n = \begin{cases} a + (f_{n+1} - 1)b & \text{if } n \text{ is even} \\ f_{n+1} b & \text{if } n \text{ is odd} \end{cases}$$

That is, two consecutive terms of this sequences are $a + (f_{n+1} - 1)b$ and $f_{n+1} b$. From Consequence 1.1., we have $a + (f_{n+1} - 1)b \equiv a$ and $f_{n+1} b \equiv b$ where $f_{n+1} \equiv 1$, it is clear that $PF^{E}_{(a,b)}(x) = k(p)$.

There is a similar proof for the sequence $F^{E}_{(b,a)}(x)$.

Proposition 4.6. For any prime $p \neq 2$,

$$F = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a \rangle$$

with generators a and b. The periods of the Fibonacci Polynomial-type orbits $F^F_{(a,b)}(x)$ and $F^F_{(b,a)}(x)$ are $k(p)$.

Proof. Let us consider the Fibonacci Polynomial-type orbit $F^F_{(a,b)}(x)$. The sequence $F^F_{(b,a)}(x)$ is as follows:

$x_0 = a,\ x_1 = b,\ x_2 = a + b,\ x_3 = a + 2b,\ x_4 = 2a + 3b,\ x_5 = 3a + 5b,$
$x_6 = 5a + 8b,\ x_7 = 8a + 13b,\ x_8 = 13a + 21b,\ \ldots,\ x_n = f_{n-1}a + f_n b,$
$x_{n+1} = f_n a + f_{n+1} b,$

$$\begin{aligned} x_{n+2} &= bx_{n+1} + x_n = b(f_n a + f_{n+1} b) + f_{n-1}a + f_n b \\ &= f_n ba + f_{n+1} b^2 + f_{n-1} a + f_n b = f_n a + f_{n+1} b + f_{n-1} a + f_n b \\ &= (f_n + f_{n-1})a + (f_{n+1} a + f_n)b = f_{n+1} a + f_{n+2} b, \ldots \\ x_{n+3} &= bx_{n+2} + x_{n+1} = b(f_{n+1} a + f_{n+2} b) + f_n a + f_{n+1} b \\ &= f_{n+1} ba + f_{n+2} b^2 + f_n a + f_{n+1} b = f_{n+1} a + f_{n+2} b + f_n a + f_{n+1} b \\ &= (f_{n+1} + f_n)a + (f_{n+2} a + f_{n+1})b = f_{n+2} a + f_{n+3} b, \ldots \end{aligned}$$

Using the above, the sequence becomes:

$f_{-1}a + f_0 b, f_0 a + f_1 b, f_1 a + f_2 b, f_2 a + f_3 b, \ldots, f_{n-1} a + f_n b,$
$f_n a + f_{n+1} b, f_{n+1} a + f_{n+2} b, f_{n+2} a + f_{n+3} b, \ldots$

where f_n denote the n th term of the ordinary 2-step Fibonacci sequence. That is, two consecutive terms of this sequences are $f_{n-1}a + f_n b$ and $f_n a + f_{n+1} b$. From Consequence 1.1., we have $f_{n-1}a + f_n b \equiv a$ and $f_n a + f_{n+1} b \equiv b$ where $f_{n-1} \equiv 1$ and $f_{n+1} \equiv 1$. Thus, the sequence $F^F_{(a,b)}(x)$ is periodic and $PF^F_{(a,b)}(x) = k(p)$.

There is a similar proof for the sequence $F^F_{(b,a)}(x)$.

Proposition 4.7. For any prime p, let

$$G = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a \rangle$$

with generators a and b. The period of the Fibonacci Polynomial-type orbit $F^G_{(a,b)}(x)$ is $k(p)$ and $F^G_{(b,a)}(x)$ is $2p$.

Proof. It can clearly be seen that sequence created by Fibonacci Polynomial-type orbit $F^G_{(a,b)}(x)$ is similar to sequence created by Fibonacci Polynomial-type orbit $F^F_{(a,b)}(x)$. Thus, sequence of Fibonacci Polynomial-type orbit $F^G_{(a,b)}(x)$ is periodic and $PF^G_{(a,b)}(x) = k(p)$.

Let us consider the Fibonacci Polynomial-type orbit $F^G_{(b,a)}(x)$. The sequence $F^G_{(b,a)}(x)$ is as follows:

$x_0 = b,\ \ x_1 = a,\ \ x_2 = a^2 + b,\ \ x_3 = ab + a = 2a,\ \ x_4 = 2a^2 + b = b,$
$x_5 = ab + 2a = 3a,\ x_6 = 3a^2 + b = b,\ x_7 = ab + 3a = 4a,\ x_8 = 4a^2 + b = b, \dots, x_{2n} = b, x_{2n+1} = (n+1)a,$
$x_{2n+2} = ax_{2n+1} + x_{2n} = a[(n+1)a] + b = (n+1)a^2 + b = b,$
$x_{2n+3} = ax_{2n+2} + x_{2n+1} = ab + (n+1)a = a + (n+1)a = (n+2)a,$
$\dots$.
Using the above, the sequence becomes:

$$b, a, b, 2a, b, 3a, b, 4a, b, 5a, \dots, b, (n+1)a, b, (n+2)a, \dots$$

Notice that each element x_n of this sequence has the form

$$x_n = \begin{cases} b & n = 2m \\ (m+1)a & n = 2m+1 \end{cases}$$

It can be seen that the period of the sequence is determined by prime number p. The residue class has p elements according to modulo $p \geq 2$ and there are b of term p times. Thus, the sequence $F^G_{(b,a)}(x)$ is periodic and $PF^G_{(b,a)}(x) = 2p$.

Proposition 4.8. For any prime $p \neq 2$,

$$H = \langle a, b \mid pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle \cong \mathbb{Z}_p + C_p(0)$$

with generators a and b.The period of the Fibonacci Polynomial-type orbit $F^H_{(a,b)}(x)$ is $k(p)$.

Proof. It can clearly be seen that sequence created by Fibonacci Polynomial-type orbit $F^H_{(a,b)}(x)$ is similar to sequence created by Fibonacci Polynomial-type orbit $F^D_{(a,b)}(x)$. Thus, sequence of Fibonacci Polynomial-type orbit $F^H_{(a,b)}(x)$ is periodic and $PF^H_{(a,b)}(x) = k(p)$.

Proposition 4.9. For any prime $p \neq 2$,

$$I = \langle a, b \mid pa = pb = 0, a^2 = b, ab = 0 \rangle$$

with generators a and b. The period of the Fibonacci Polynomial-type orbit $F^I_{(b,a)}(x)$ is $2p$.

Proof. It can clearly be seen that sequence created by Fibonacci Polynomial-type orbit $F^I_{(b,a)}(x)$ is similar to sequence created by Fibonacci Polynomial-type orbit $F^G_{(b,a)}(x)$. Thus, sequence of Fibonacci Polynomial-type orbit $F^I_{(b,a)}(x)$ is periodic and $PF^I_{(b,a)}(x) = 2p$.

Proposition 4.10. For any prime $p \neq 2$,

$$J = \langle a, b \mid pa = pb = 0, a^2 = b^2 = 0 \rangle \cong C_p \times C_p(0)$$

with generators a and b. The periods of the Fibonacci Polynomial-type orbits $F^J_{(a,b)}(x)$ and $F^J_{(b,a)}(x)$ are $2p$.

Proof. It can clearly be seen that sequences created by Fibonacci Polynomial-type orbit $F^J_{(a,b)}(x)$ and $F^J_{(b,a)}(x)$ are similar to sequences created by Fibonacci Polynomial-type orbit $F^G_{(b,a)}(x)$. Thus, sequences of Fibonacci Polynomial-type orbits $F^J_{(a,b)}(x)$ and $F^J_{(b,a)}(x)$ are periodic and $PF^J_{(a,b)}(x) = PF^J_{(b,a)}(x) = 2p$.

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