Generalized characteristic polynomial of generalized R–vertex corona*

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Abstract

Let G be a graph of order n with adjacency matrix $A(G)$ and diagonal degree matrix $D(G)$. The generalized characteristic polynomial of G is defined to be $f_G(x,t) = \det(xI_n - (A(G) - tD(G)))$. R–graph of G, denoted by $R(G)$, is obtained by adding a new vertex for each edge of G and joining each new vertex to both end vertices of the corresponding edge. The generalized R–vertex corona, denoted by $R(G) \boxdot \wedge_i^n H_i$, is the graph obtained from $R(G)$ and $H_1, \ldots, H_n$ by joining the $i-th$ vertex of $V(G)$ to every vertex of H_i. In this paper, we determine the generalized characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$. As applications, we get infinitely many pairs of generalized cospectral graphs, the number of spanning trees and Kirchhoff index of $R(G) \boxdot \wedge_i^n H_i$.

Keywords: generalized characteristic polynomial, generalized R-vertex corona, cospectral graphs, spanning trees, Kirchhoff index

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1 Introduction

We only consider simple graphs. For a graph G of order n, let $A(G)$ denote the adjacency matrix of G, and $D(G)$ the diagonal degree matrix of G. The Laplacian matrix and signless Lapalcian matrix of G are

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defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. The characteristic polynomial of a $n \times n$ matrix Z is denote by $\phi(Z, \lambda) = \det(\lambda I_n - Z)$, where I_n is the identity matrix of order n. The eigenvalues of $A(G), L(G)$ and $Q(G)$ are called the *A-spectrum*, *L-spectrum* and *Q-spectrum* of G, respectively. The adjacency, Laplacian and signless Laplacian eigenvalues of G are denoted as $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$, $0 = \mu_1(G) \le \mu_2(G) \le \cdots \le \mu_n(G)$ and $\nu_1(G) \le \nu_2(G) \le \cdots \le \nu_n(G)$. The *generalized characteristic polynomial* of G is defined to be $f_G(x,t) = \det(xI_n - (A(G) - tD(G)))$ [1, 4, 13], which generalizes A−spectrum, L−spectrum and Q−spectrum of G. The characteristic polynomials of $A(G)$, $L(G)$ and $Q(G)$ are equal to $f_G(\lambda, 0)$, $(-1)^{|V(G)|} f_G(-\lambda, 1)$ and $f_G(\lambda, -1)$, respectively.

Graphs with the same A-spectrum (respectively, L-spectrum and Q-spectrum) are called *A-cospectral* (respectively, *L-cospectral*, *Q-cospectral*) graphs. For graph G and H, if $f_G(x,t) = f_H(x,t)$, then we call G and H are f−cospectral. Obviously, if G and H are f-cospectral, then they are A-cospectral, L-cospectral and Q-cospectral.

Graph operations, such as the disjoint union, the *corona*, the *edge corona*, the *neighborhood corona* [3, 10–12, 14], are techniques to construct new classes of graphs from old ones. For a graph G, $R(G)$ is a graph obtained from G by adding a vertex u_e and joining u_e to the end vertices of e for each $e \in E(G)$ [5, 9]. Let $I(G)$ be the set of newly added vertices, i.e., $I(G) = V(R(G)) \backslash V(G)$. We define a new graph operation based on R−graph, the generalized R-vertex corona of graph G with n vertices and $H_1, H_2, \ldots, H_n$. We compute the generalized characteristic polynomial of the generalized R-vertex corona. In the rest of this paper, $\mathbf{j}_n$ denotes the column vector of size n consisting entirely of 1's and $\mathbf{0}$ denotes a zero matrix when its size is obvious.

The paper is organized as follows. In Section 2, we give the definition of the generalized R-vertex corona and some useful tools. In Section 3, we compute the generalized characteristic polynomial of the generalized R-vertex corona. Also, we construct many pairs of generalized cospectral graphs. As the applications, *kirchhhoff index* and the number of *spanning trees* of some special R-vertex corona graphs are computed.

2 Preliminaries

Definition 2.1. *Let G be a graph of n vertices with vertex set $V(G)$ and $H_1, H_2, \ldots, H_n$ be n arbitrary graphs. The* generalized $R-$vertex corona *of G and $H_1, H_2, \ldots, H_n$ denoted by $R(G) \boxdot \wedge_i^n H_i$, is the graph obtained from $R(G)$ and $H_1, H_2, \ldots, H_n$ by joining the $i-th$ vertex of $V(G)$ to every vertex of H_i.*

In this paper, we will determine the generalized characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ with the help of the *coronal* of a matrix and the *Kronecker product.* The Z*-coronal* $\Gamma_Z(\lambda)$ of an $n \times n$ matrix Z is defined [3,10–12,14] to be the sum of the entries of the matrix $(\lambda I_n - Z)^{-1}$, that is, $\Gamma_Z(\lambda) = \mathbf{j}_n^T(\lambda I_n - Z)^{-1}\mathbf{j}_n$. It is well known that, if Z is an $n \times n$ matrix with each row sum equals to a constant t, then $\Gamma_Z(\lambda) = \frac{n}{\lambda - t}$.

Let M_1, M_2, M_3 and M_4 be respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible. It is well known that $\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} =$ $\det(M_4)\det\left(M_1 - M_2M_4^{-1}M_3\right) = \det(M_1)\det\left(M_4 - M_3M_1^{-1}M_2\right)$. where $M_1 - M_2M_4^{-1}M_3$ and $M_4 - M_3M_1^{-1}M_2$ are called the *Schur complements* [15] of M_4 and M_1, respectively.

If $A = [a_{ij}]$ is an $m \times n$ matrix and B is an $r \times s$ matrix, then the Kronecker product [7] $A \otimes B$ is defined as the $mr \times ns$ matrix with the block form

$$\begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix}.$$

This is an associative operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist.

Let $t(G)$ denote the number of spanning trees of G. It is well known [5] that if G is a connected graph on n vertices, then $t(G) = \frac{\mu_2(G)\ldots\mu_n(G)}{n}$. The Kirchhoff index of a graph G, denoted by $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices [2,8]. Gutman [6] proved that the Kirchhoff index of a connected graph G with $n (n \geq 2)$ vertices can be expressed as $Kf(G) = n\sum_{i=2}^{n} \frac{1}{\mu_i(G)}$.

3 Generalized characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$

Let G be an arbitrary graph on n vertices and m edges and H_i an arbitrary graph on t_i vertices, for $i = 1, 2, \ldots, n$. Let $N = m+n$ and $M = t_1 + t_2 + \ldots + t_n$.

Label the vertices of G by $1,2,\ldots,n$ and the newly added vertices in $R(G)$ by $n+1,\ldots,n+m$. Label the vertices of H_1 by $n+m+1, n+m+2, \ldots, n+m+t_1$, and the vertices of H_i for $i \geq 2$ by $n+m+\Sigma_{k=1}^{i-1}t_k+1, n+m+\Sigma_{k=1}^{i-1}t_k+2, \ldots, n+m+\Sigma_{k=1}^{i}t_k$.

Theorem 3.1. *Let G be an arbitrary graph with n vertices and m edges, and H_i an arbitrary graph with t_i vertices for $i = 1,2,\ldots,n$. The generalized characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$$f_{R(G)\boxdot\wedge_i^n H_i}(x,t) =$$

$$\det\left(\begin{pmatrix} x-\Gamma_{A(H_1)-tD(H_1)}(x+t)+t\times t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & x-\Gamma_{A(H_n)-tD(H_n)}(x+t)+t\times t_n & 0 \\ 0 & 0 & 0 & xI_m \end{pmatrix} - A(R(G)) + tD(R(G))\right)\prod_i^n f_{H_i}(x+t,t).$$

Proof. The adjacency matrix of $R(G) \boxdot \wedge_i^n H_i$ can be written as:

$$A\left(R(G) \boxdot \wedge_i^n H_i\right) = \begin{pmatrix} A(R(G)) & C \\ C^T & B \end{pmatrix},$$

where $C = \begin{pmatrix} \mathrm{j}_{t_1}^T & 0 & 0 & 0 & 0 \\ 0 & \mathrm{j}_{t_2}^T & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{j}_{t_n}^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{N\times M}$, $B = \begin{pmatrix} A(H_1) & 0 & 0 & 0 \\ 0 & A(H_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A(H_n) \end{pmatrix}$.

Let E be the incidence matrix of G, $A(R(G)) = \begin{pmatrix} A(G) & E \\ E^T & 0 \end{pmatrix}$.

The degree matrix of $R(G) \boxdot \wedge_i^n H_i$ can be written as:

$$D\left(R(G) \boxdot \wedge_i^n H_i\right) = \begin{pmatrix} D(R(G)) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & 0 \\ 0 & D(F)+I_M \end{pmatrix},$$

where $D(R(G)) = \begin{pmatrix} 2D(G) & 0 \\ 0 & 2I_m \end{pmatrix}$, $W = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix}$.

Let F denote $H_1 \cup H_2 \cup \ldots \cup H_n$, $D(F) = \begin{pmatrix} D(H_1) & 0 & 0 & 0 \\ 0 & D(H_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D(H_n) \end{pmatrix}$.

Then, the generalized matrix of $R(G) \boxdot \wedge_i^n H_i$ can be written as:

$$A\left(R(G) \boxdot \wedge_i^n H_i\right) - tD\left(R(G) \boxdot \wedge_i^n H_i\right)$$

$$= \begin{pmatrix} A(R(G)) & C \\ C^T & B \end{pmatrix} - t\begin{pmatrix} D(R(G)) + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & 0 \\ 0 & D(F)+I_M \end{pmatrix}$$

$$= \begin{pmatrix} A(R(G)) - tD(R(G)) - t\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & C \\ C^T & B - tD(F) - tI_M \end{pmatrix}.$$

So, the generalized characteristic polynomial is

$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$

$$= \det \begin{pmatrix} xI_N - A(R(G)) + tD(R(G)) + t\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} & -C \\ -C^T & xI_M - B + tD(F) + tI_M \end{pmatrix}$$

$$= \det((x+t)I_M - B + tD(F))\det(xI_N - A(R(G)) + tD(R(G))$$

$$+ t\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N} - C(xI_M - B + tD(F) + tI_M)^{-1}C^T),$$

where

$$\det((x+t)I_M - B + tD(F))$$

$$= \det \begin{pmatrix} (x+t)I_{t_1} - A(H_1) + tD(H_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (x+t)I_{t_n} - A(H_n) + tD(H_n) \end{pmatrix}$$

$$= \prod_i^n f_{H_i}(x+t, t)$$

and

$$C(xI_M - B + tD(F) + tI_M)^{-1}C^T =$$

$$\begin{pmatrix} J_{t_1}^T\left((x+t)I_{t_1} - A(H_1) + tD(H_1)\right)^{-1}J_{t_1} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & J_{t_n}^T\left((x+t)I_{t_n} - A(H_n) + tD(H_n)\right)^{-1}J_{t_n} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma_{A(H_1)-tD(H_1)}(x+t) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \Gamma_{A(H_n)-tD(H_n)}(x+t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$$

$$= \det(xI_N - A(R(G)) + tD(R(G)) + t\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}_{N\times N}$$

$$- \begin{pmatrix} \Gamma_{A(H_1)-tD(H_1)}(x+t) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \Gamma_{A(H_n)-tD(H_n)}(x+t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}) \prod_i^n f_{H_i}(x+t, t)$$

$$= \det(\begin{pmatrix} x - \Gamma_{A(H_1)-tD(H_1)}(x+t) + t\times t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & x - \Gamma_{A(H_n)-tD(H_n)}(x+t) + t\times t_n & 0 \\ 0 & 0 & 0 & xI_m \end{pmatrix}$$

$$- A(R(G)) + tD(R(G))) \prod_i^n f_{H_i}(x+t, t).$$ □

Theorem 3.2. *Let G be a graph with n vertices and m edges. Let H_i be an arbitrary graph with t_i vertices for $i = 1, 2, \ldots, n$. Then the following hold*
(1) *The adjacency characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$$\phi_{A(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$$

$$= \det\left(\begin{pmatrix} \lambda - \Gamma_{A(H_1)}(\lambda) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda - \Gamma_{A(H_n)}(\lambda) & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} - A(R(G))\right) \prod_i^n \phi_{A(H_i)}(\lambda).$$

(2) *The Laplacian characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$\phi_{L(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$

$$= \det\left(\begin{pmatrix} -\lambda - \Gamma_{-L(H_1)}(-\lambda+1) + t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & -\lambda - \Gamma_{-L(H_n)}(-\lambda+1) + t_n & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} + L(R(G))\right)$$

$$\cdot \prod_i^n \phi_{L(H_i)}(-\lambda+1) \cdot (-1)^{N+M}.$$

(3) *The signless Laplacian characteristic polynomial of* $R(G) \boxdot \wedge_i^n H_i$ *is*

$\phi_{Q(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$

$$= \det\left(\begin{pmatrix} \lambda - \Gamma_{Q(H_1)}(\lambda-1) - t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda - \Gamma_{Q(H_n)}(\lambda-1) - t_n & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} - Q(R(G))\right)$$

$$\cdot \prod_i^n \phi_{Q(H_i)}(\lambda-1).$$

Proof. (1) Since $\phi_{A(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = f_{R(G)\boxdot\wedge_i^n H_i}(\lambda, 0)$, by Theorem 3.1, we have

$\phi_{A(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$

$$= \det\left(\begin{pmatrix} \lambda - \Gamma_{A(H_1)}(\lambda) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda - \Gamma_{A(H_n)}(\lambda) & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} - A(R(G))\right) \prod_i^n \phi_{A(H_i)}(\lambda).$$

(2) Since $\phi_{L(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = (-1)^{N+M} f_{R(G)\boxdot\wedge_i^n H_i}(-\lambda, 1)$, by Theorem 3.1, we have

$\phi_{L(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$

$$= \det\left(\begin{pmatrix} -\lambda - \Gamma_{-L(H_1)}(-\lambda+1) + t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & -\lambda - \Gamma_{-L(H_n)}(-\lambda+1) + t_n & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} + L(R(G))\right)$$

$$\cdot \prod_i^n \phi_{L(H_i)}(-\lambda+1) \cdot (-1)^{N+M}.$$

(3) Since $\phi_{Q(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = f_{R(G)\boxdot\wedge_i^n H_i}(\lambda, -1)$, by Theorem 3.1, we have

$f_{Q(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$

$$= \det\left(\begin{pmatrix} \lambda - \Gamma_{Q(H_1)}(\lambda-1) - t_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda - \Gamma_{Q(H_n)}(\lambda-1) - t_n & 0 \\ 0 & 0 & 0 & \lambda I_m \end{pmatrix} - Q(R(G))\right)$$

$$\cdot \prod_i^n \phi_{Q(H_i)}(\lambda-1).$$ □

Corollary 3.3. *Let* G *be a graph with* n *vertices and* m *edges. Let* H_i *be an arbitrary graph with* t_i *vertices for* $i = 1, 2, \ldots, n$ *and* W *denotes* $diag(t_1, \ldots, t_n)$. *If* $\Gamma_{A(H_i)-tD(H_i)}(x+t) = \Gamma_{A(H)-tD(H)}(x+t)$, *then*

$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$

$= (x+2t)^{m-n} det((x - \Gamma_{A(H)-tD(H)}(x+t))(x+2t)I_n + t(x+2t)W$

$- (x+2t)A(G) + 2t(x+2t)D(G) - A(G) - D(G)) \prod_i^n f_{H_i}(x+t,t).$

Proof. Let

$$\Lambda = \det\begin{pmatrix} x - \Gamma_{A(H_1)-tD(H_1)}(x+t) + t\times t_1 & 0 & 0 & 0 \\ & \ddots & & \\ 0 & & 0 & 0 \\ 0 & 0 & x - \Gamma_{A(H_n)-tD(H_n)}(x+t) + t\times t_n & 0 \\ 0 & 0 & 0 & xI_m \end{pmatrix}$$

$$= \det\begin{pmatrix} (x - \Gamma_{A(H)-tD(H)}(x+t))I_n + tW & 0 \\ 0 & xI_m \end{pmatrix}.$$

Thus,

$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$

$= \det(\Lambda - A(R(G)) + tD(R(G))) \prod_i^n f_{H_i}(x+t,t)$

$= (x+2t)^{m-n} \det((x - \Gamma_{A(H)-tD(H)}(x+t))(x+2t)I_n + t(x+2t)W$

$- (x+2t)A(G) + 2t(x+2t)D(G) - A(G) - D(G)) \prod_i^n f_{H_i}(x+t,t).$ □

From Corollary 3.3, we can get the following 2 Corollaries.

Corollary 3.4. *Let G be a graph with n vertices and m edges. Let H_i be an arbitrary graph with t_i vertices for $i = 1,\ldots,n$ and W denotes $diag(t_1,\ldots,t_n)$. If $\Gamma_{A(H_i)-tD(H_i)}(x+t) = \Gamma_{A(H)-tD(H)}(x+t)$. Then*
(1) *The adjacency characteristic polynomial of $R(G)\boxdot\wedge_i^n H_i$ is*

$$\phi_{A(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = \lambda^{m-n}\det\left(\lambda(\lambda - \Gamma_{A(H)}(\lambda))I_n - (\lambda+1)A(G) - D(G)\right)\prod_i^n \phi_{A(H_i)}(\lambda).$$

(2) *The Laplacian characteristic polynomial of $R(G)\boxdot\wedge_i^n H_i$ is*

$$\phi_{L(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = (-1)^{N+M}(-\lambda+2)^{m-n}\det((-\lambda - \Gamma_{-L(H)}(-\lambda+1))(-\lambda+2)I_n$$
$$+ (-\lambda+2)W - \lambda D(G) + (-\lambda+3)L(G))\prod_i^n \phi_{L(H_i)}(-\lambda+1).$$

(3) *The signless Laplacian characteristic polynomial of $R(G)\boxdot\wedge_i^n H_i$ is*

$$\phi_{Q(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = (\lambda-2)^{m-n}\det((\lambda - \Gamma_{Q(H)}(\lambda-1))(\lambda-2)I_n$$
$$- (\lambda-2)W - (\lambda-1)Q(G) - (\lambda-2)D(G))\prod_i^n \phi_{Q(H_i)}(\lambda-1).$$

Corollary 3.5. *Let G be an arbitrary graph with n vertices and m edges, and $H_1, H_2, \ldots, H_n$ be r-regular graphs on T vertices. Then*

$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$

$$= \det\left(\left(\left(x - \frac{T}{x+t-(r-tr)}\right)(x+2t) + t(x+2t)\times T\right)I_n - (x+2t+1)A(G)\right.$$
$$\left. + (2tx+4t^2-1)D(G)\right)\prod_i^n f_{H_i}(x+t,t)\cdot(x+2t)^{m-n}.$$

Corollary 3.6. *Let G be an arbitrary graph with n vertices and m edges, and $H_1, H_2, \dots, H_n$ be r-regular graphs on T vertices. Then*
(1) *The adjacency characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$$\phi_{A(R(G)\boxdot\wedge_i^n H_i)}(\lambda) = \lambda^{m-n}\det\left(\lambda\left(\lambda - \frac{T}{\lambda - r}\right)I_n - (\lambda+1)A(G) - D(G)\right)\prod_i^n \phi_{A(H_i)}(\lambda).$$

(2) *The Laplacian characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$$\phi_{L(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$$
$$= (-\lambda+2)^{m-n}\det\left(\left(-\lambda - \frac{T}{-\lambda+1} + T\right)(-\lambda+2)I_n + (-\lambda+3)L(G) - \lambda D(G)\right)$$
$$\cdot \phi_{L(H_i)}(-\lambda+1)\cdot(-1)^{N+M}.$$

(3) *The signless Laplacian characteristic polynomial of $R(G) \boxdot \wedge_i^n H_i$ is*

$$\phi_{Q(R(G)\boxdot\wedge_i^n H_i)}(\lambda)$$
$$= (\lambda-2)^{m-n}\det\left(\left(\lambda - \frac{T}{\lambda-1-2r} - T\right)(\lambda-2)I_n - (\lambda-1)Q(G) + (-\lambda+2)D(G)\right)$$
$$\cdot \phi_{Q(H_i)}(\lambda-1).$$

Proof. It is clear from Corollary 3.5. □

Corollary 3.7. *Let G be an r-regular graph with n_1 vertices and m_1 edges, and H_i be arbitrary graphs on n_2 vertices. If $H_i \simeq H$ for $i = 1, 2, \dots, n_1$, then*

$$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$$
$$= \det((x - \Gamma_{A(H)-tD(H)}(x+t))(x+2t) + t(x+2t)(n_2))I_{n_1} - (x+2t)A(G)$$
$$- 2t(x+2t)D(G) - A(G) - D(G))\,(f_H(x+t,t))^{n_1}\,(x+2t)^{m_1-n_1}.$$

Proof. By Corollary 3.3, we have

$$f_{R(G)\boxdot\wedge_i^n H_i}(x,t)$$
$$= \det((x - \Gamma_{A(H)-tD(H)}(x+t))(x+2t)I_{n_1} + t(x+2t)W - (x+2t)A(G)$$
$$- 2t(x+2t)D(G) - A(G) - D(G))\,(f_H(x+t,t))^{n_1}\,(x+2t)^{m_1-n_1}$$
$$= \det((x - \Gamma_{A(H)-tD(H)}(x+t))(x+2t) + t(x+2t)(n_2))I_{n_1} - (x+2t)A(G)$$
$$- 2t(x+2t)D(G) - A(G) - D(G))\,(f_H(x+t,t))^{n_1}\,(x+2t)^{m_1-n_1}.$$

Following conclusions are consistant with the result in literature [9].

Corollary 3.8. *Let G be an r-regular graph with n_1 vertices and m_1 edges, and H_i be arbitrary graphs on n_2 vertices. If $H_i \simeq H$ for $i = 1, 2, \dots, n_1$, then*
(1) *The adjacency characteristic polynomial of $R(G) \boxdot \wedge_i^{n_1} H_i$ is*

$$\phi_{A(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= \lambda^{m_1-n_1}\prod_{i=1}^{n_1}\left(\lambda^2 - (\Gamma_{A(H)}(\lambda) - \lambda_i(G))\lambda - r_1 - \lambda_i(G)\right)\prod_{i=1}^{n_2}(\lambda - \lambda_i(G_2))^{n_1},$$

where λ_i denotes the ith adjacency eigenvalue of G.
(2) *The Laplacian characteristic polynomial of $R(G) \boxdot \wedge_i^{n_1} H_i$ is*

$$\phi_{L(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= (\lambda-2)^{m_1-n_1}\prod_{i=2}^{n_2}(\lambda - 1 - \mu_i(H))^{n_1}\prod_{i=1}^{n_1}(\lambda^3 - (r_1 + n_2 + 3 + \mu_i(G))\lambda^2$$
$$+ (4\mu_i(G) + 2n_2 + r_1 + 2)\lambda - 3\mu_i(G)),$$

where μ_i denotes the ith Laplacian eigenvalue of G.
(3) The signless Laplacian characteristic polynomial of $R(G) \boxdot \wedge_i^{n_1} H_i$ is
$$\phi_{Q(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= (\lambda - 2)^{m_1 - n_1} \prod_{i=1}^{n_2}(\lambda - 1 - \nu_i(H))^{n_1} \prod_{i=1}^{n_1}(\lambda^2 - (\Gamma_{Q(H)}(\lambda - 1)$$
$$+ \nu_i(G) + r_1 + n_2 + 2)\lambda + 2r_1 + 2n_2 + 2\Gamma_{Q(H)}(\lambda - 1) + \nu_i(G)),$$
where ν_i denotes the ith signless Laplacian eigenvalue of G.

Proof. (1) Since $\phi_{A(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda) = f_{R(G)\boxdot\wedge_i^{n_1} H_i}(\lambda, 0)$, by Corollary 3.7, we have
$$\phi_{A(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= \lambda^{m_1 - n_1} det\left(\lambda(\lambda - \Gamma_{A(H)}(\lambda))I_n - (\lambda + 1)A(G) - D(G)\right)(f_{A(H)}(\lambda))^{n_1}$$
$$= \lambda^{m_1 - n_1} \prod_{i=1}^{n_1}\left(\lambda^2 - (\Gamma_{A(H)}(\lambda) - \lambda_i(G))\lambda - r_1 - \lambda_i(G)\right) \prod_{i=1}^{n_2}(\lambda - \lambda_i(G_2))^{n_1}.$$
(2) Since $\phi_{L(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda) = (-1)^{n_1 + m_1 + n_1 n_2} f_{R(G)\boxdot\wedge_i^{n_1} H_i}(-\lambda, 1)$, by Corollary 3.7, we have
$$\phi_{L(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= det((-\lambda - \Gamma_{-L(H)}(-\lambda + 1))(-\lambda + 2)I_{n_1} + (-\lambda + 2)n_2 I_{n_1} - (-\lambda + 2)A(G) - A(G)$$
$$+ 2(-\lambda + 2)D(G) - D(G))(-1)^{n_1 + m_1 + n_1 n_2}(-\lambda + 2)^{m_1 - n_1}(f_{L(H)}(-\lambda + 1))^{n_1}$$
$$= (-1)^{n_1 + m_1 + n_1 n_2} \prod_{i=1}^{n_1}\left((\lambda - 2)\left(\lambda - 2r_1 - n_2 - \frac{n_2}{\lambda - 1}\right) - r_1 - u_i(G)\right)$$
$$\cdot \prod_{i=1}^{n_2}(-\lambda + 1 + \mu_i(H))^{n_1} \cdot (-\lambda + 2)^{m_1 - n_1}.$$
Note that $\mu_i(G) = 0$. Now the result follows easily.
(3) Since $\phi_{Q(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda) = f_{R(G)\boxdot\wedge_i^{n_1} H_i}(\lambda, -1)$, by Corollary 3.7, we have
$$\phi_{Q(R(G)\boxdot\wedge_i^{n_1} H_i)}(\lambda)$$
$$= (\lambda - 2)^{m_1 - n_1} det((\lambda - \Gamma_{Q(H)}(\lambda - 1))(\lambda - 2)I_{n_1} - (\lambda - 2)n_2 I_{n_1} - (\lambda - 2)A(G)$$
$$- 2(\lambda - 2)D(G) - Q(G))(f_{Q(H)}(\lambda - 1))^{n_1}$$
$$= \prod_{i=1}^{n_1}(\lambda^2 - (\Gamma_{Q(H)}(\lambda - 1) + \nu_i(G) + r_1 + n_2 + 2)\lambda + 2r_1 + 2n_2 + 2\Gamma_{Q(H)}(\lambda - 1)$$
$$+ \nu_i(G))(\lambda - 2)^{m_1 - n_1} \prod_{i=1}^{n_2}(\lambda - 1 - \nu_i(H))^{n_1}.$$
□

Corollary 3.9. *If G_1 and G_2 are generalized cospectral regular graphs, and H_i is an arbitary graph for $i = 1, 2, \ldots, n$, then $R(G_1) \boxdot \wedge_i^n H_i$ and $R(G_2) \boxdot \wedge_i^n H_i$ are generalized cospectral.*

Proof. It is clear from Corollary 3.3. □

Corollary 3.10. *Let G be an r-regular graph with n_1 vertices and m_1 edges,* *and H_i be an arbitrary graph with n_2 vertices. If $H_i \simeq H$ for $i = 1, 2, \ldots, n_1$, then the $L -$ spectrum of $R(G) \boxdot \wedge_i^{n_1} H_i$ is*

(a) 2, repeated $m_1 - n_1$ times;

(b) $\mu_i(H) + 1$, repeated n_1 times for $i = 2, \ldots, n_2$;

(c) three roots of the equation $x^3 - (r_1 + n_2 + 3 + \mu_i(G))x^2 + (4\mu_i(G) + 2n_2 + r_1 + 2)x - 3\mu_i(G) = 0$, for $i = 1, 2, \ldots, n_1$.

Proof. It is clear from Corollary 3.8. □

Corollary 3.11. *Let G be an r-regular graph with n_1 vertices and m_1 edges, and H_i be an arbitrary graph with n_2 vertices. If $H_i \simeq H$ for $i = 1, 2, \ldots, n_1$ that the Kirchhoff index and the number of Spannning trees of $R(G) \boxdot \wedge_i^{n_1} H_i$ are*

$$Kf(R(G) \boxdot \wedge_i^{n_1} H_i) = \left(\sum_{i=2}^{n_2} \frac{n_1}{\mu_i(H) + 1} + \frac{m_1 - n_1}{2} + \frac{r_1 + n_2 + 3}{2n_2 + r_1 + 2} + \sum_{i=2}^{n_1} \frac{4\mu_i(G) + 2n_2 + r_1 + 2}{3\mu_i(G)} \right) \cdot (m_1 + n_1 + n_1 n_2),$$

$$t(R(G) \boxdot \wedge_i^{n_1} H_i) = \frac{2^{m_1 - n_1} \prod_{i=1}^{n_2} (\mu_i(G_2) + 1)^{n_1} (2n_2 + r_1 + 2) \prod_{i=2}^{n_1} (3\mu_i(G))}{m_1 + n_1 + n_1 n_2}.$$

Proof. It is clear from Corollary 3.10. □

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