A note on non-existence of cubic semisymmetric graphs of order $8p$ or $8p^2$

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Abstract

The aim of this note is to present a short proof of a result of Alaeiyan et al. [Bull. Austral. Math. Soc. 77 (2008) 315–323; Proc. Indian Acad. Sci., Math. Sci. 119 (2009) 647–653] concerning the non-existence of cubic semisymmetric graphs of order $8p$ or $8p^2$, where p is a prime. In those two papers the authors choose the heavy weaponry of covering techniques. Our proof relies on the analysis of the subgroup structure of the full automorphism group of the graph and the normal quotient graph theory.

Keywords Edge-transitive graph, semisymmetric graph, vertex-transitive graph.

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1 Introduction

Throughout this paper a *graph* means a finite, connected, simple and undirected graph. For a graph X, denote by $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,(X)$ the vertex set, edge set, arc-set and the full automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if $\mathrm{Aut}\,(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. If the graph X has regular valency and is edge- but not vertex-transitive, then it is called *semisymmetric*.

The study of semisymmetric graphs was initiated by Folkman [7] who gave constructions of several infinite families of such graphs, and posed a number of open problems which spurred the interest in this topic (see for example [3, 4, 5, 7, 8, 12, 13, 14, 15, 18, 19, 20, 22, 23, 24, 25, 26]).

Let p be a prime. Folkman [7] proved that there is no cubic semisymmetric graph of order $2p$ or $2p^2$. Following this, some authors considered the classification of cubic semisymmetric graphs with given orders. For example, Malnič et al. [21] classified cubic semisymmetric graphs of order $2p^3$. By Du [6], there is no connected cubic semisymmetric graph of order $6p$, and Lu et al. [17] classified connected cubic semisymmetric graphs of order $6p^2$. Recently, Hua and Feng [10] classified cubic semisymmetric graphs of order $8p^3$, and by using covering techniques, Alaeiyan et al. [1, 2], proved the following result.

Theorem 1.1 *Every cubic edge-transitive graph of order $8p$ or $8p^2$ is vertex-transitive for each prime p.*

In this paper, we shall present a short proof of this theorem by using group theory and quotient graph theory.

2 Main Results

We start by stating some preliminary results.

Proposition 2.1 [21, Corollary 2.3] *Let X be a connected bipartite graph admitting an abelian subgroup $G \leq \mathrm{Aut}\,(X)$ acting regularly on each of the* *bipartition sets. Then X is vertex-transitive.*

Let X be a cubic graph and let $G \leq \mathrm{Aut}(X)$ act transitively on the edges of X. Let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. Below we introduce two propositions, of which the first is a special case of [16, Theorem 9].

Proposition 2.2 *Let G be transitive on $V(X)$. Then G is an s-regular subgroup of* $\mathrm{Aut}(X)$ *for some integer s. If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a cubic symmetric graph with G/N as an s-regular group of automorphisms, and X is a regular N-cover of X_N.*

The next proposition is a special case of [17, Lemma 3.2].

Proposition 2.3 *Let G be intransitive on $V(X)$. Then X is a bipartite graph with two partition sets, say V_0 and V_1. If N is intransitive on the bipartition sets, then N is semiregular on both V_0 and V_1, X_N is a cubic graph with G/N as an edge- but not vertex-transitive group of automorphisms and X is a regular N-cover of X_N.*

Before giving a proof of Theorem 1.1, we need to show the following two lemmas.

Lemma 2.4 *Let X be a cubic edge-transitive graph order $8p^n$, where $p > 7$ is a prime and n is an integer. Then every Sylow p-subgroup of* $\mathrm{Aut}(X)$ *is normal.*

Proof Set $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. By [21, Proposition 2.4], the stabilizer A_v of $v \in V(X)$ has order dividing $2^r \cdot 3$ for some integer r. Without loss of the generality, assume $|A_v| = 2^\ell \cdot 3$ for some integer ℓ. Then, $|A| \mid 2^{3+\ell} \cdot 3 \cdot p^n$. If A is non-solvable, then A has a non-abelian simple chief factor T_1/T_2. Noting that $|T_1/T_2| \mid 2^{3+\ell} \cdot 3 \cdot p^n$, by [9, pp.12-14], $T_1/T_2 \cong T = A_5$ or PSL$(2,7)$, contrary to the assumption that $p > 7$. Thus, A is solvable.

For each divisor r of A, denote by $O_r(A)$ the maximal normal r subgroup of A. We have the following claim.

Claim: $O_p(A) > 1$.

Suppose to the contrary that $O_p(A) = 1$. By the solvability of A, we know that either $O_2(A) > 1$ or $O_2(A) = 1$. Let $M = O_2(A)$. It is easy to check that M has more than two orbits on $V(X)$, and moreover, if X is not vertex-transitive, then M is intransitive on each partition set of X. By Propositions 2.2 and 2.3, M is semi-regular, and so then $|M| \mid 4$. Let H/M be a minimal normal subgroup of A/M. Then H/M must be an elementary abelian p-group. Let Q be the Sylow p-subgroup of H. Since $p > 7$, Sylow theorem implies that $Q \trianglelefteq H$. Therefore, Q is characteristic in H, and hence it is normal in A because $H \trianglelefteq A$. This implies that $O_p(A) > 1$, a contradiction.

Now we are ready to finish the proof. Set $N = O_p(A)$. Suppose $N < P$. Applying Propositions 2.2 and 2.3, the quotient graph X_N of X relative to N is a cubic graph of order $8p^t$ with $t = |P|/|N| > 1$. Furthermore, A/N is an edge-transitive automorphism group of X_N. By Claim, Aut(X_N) has a normal p-subgroup, say T/N. It is easy to see that P/N is also a Sylow p-subgroup of Aut(X_N) and so $T/N \leq P/N$. Consequently, $T/N \trianglelefteq A/N$. This is contrary to the fact that $N = O_p(A)$. Thus, $N = P$, namely, $P \trianglelefteq A$.
□

Lemma 2.5 *Let $G =$ GL$(2,p)$ where p is an odd prime. Then G has no subgroup isomorphic to A_4.*

Proof Suppose to the contrary that L is a subgroup of G isomorphic to A_4. By elementary group theory, $H =$ SL$(2,p)$ is a subgroup of $G =$ GL$(2,p)$ of index 2, and SL$(2,p)$ has a unique involution. As $L \cong A_4$ has three involutions, L is not contained in H. It follows that $LH = G$, and hence $L \cap H$ is a subgroup of L of index 2. However, by elementary group theory, A_4 has no subgroups of index 2, a contradiction. □

Proof of Theorem 1.1 Suppose to the contrary that X is a cubic semisymmetric graph of order $8p^t$ with $t = 1$ or 2. By [5, pp.282-290], there exists no cubic semisymmetric graph of order $8p^t$ with $p \leq 7$. In what follows, assume that $p > 7$. Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. By Lemma 2.4, P is normal in A. By Proposition 2.3, P is semi-regular, and the quotient graph X_P of X relative to P is a cubic graph with A/P as an edge-transitive group of automorphisms. It follows that $X_P \cong Q_3$. Note that $\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$, and the subgroup of $\mathrm{Aut}(Q_3)$ fixing each partition set of Q_3 setwise is isomorphic to S_4. Hence, $A/P \leq S_4$. Since A/P is edge-transitive, we may take $B/P \leq A/P$ such that $B/P \cong A_4$.

Let $C = C_B(P)$. Then $P \leq C$. If $P = C$, then $A_4 \cong B/P = B/C \leq \mathrm{Aut}(P)$. As $|P| = p^t$ with $t = 1$ or 2, one has $P \cong \mathbb{Z}_{p^t}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. For the former, we have $\mathrm{Aut}(P) \cong \mathbb{Z}_{p^{t-1}(p-1)}$. This is clearly impossible because $A_4 \leq \mathrm{Aut}(P)$. For the latter, $\mathrm{Aut}(P) \cong \mathrm{GL}(2,p)$. This is also impossible by Lemma 2.5. Therefore, we must have $C > P$. Take a minimal normal subgroup of B/P contained in C/P, say M/P. Since $B/P \cong A_4$, one has $M/P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $M \leq C$, one has $M = P \times Q$, where $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Sylow 2-subgroup of M. Clearly, M is abelian. Furthermore, M acts regularly on each partition set of X. By Proposition 2.1, X is vertex-transitive, a contradiction. □

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