ON LOWER ORIENTABLE STRONG DIAMETER AND STRONG RADIUS OF SOME GRAPHS

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Abstract

For two vertices u and v in a strong digraph D, the strong distance between u and v is the minimum number of arcs of a strong subdigraph of D containing u and v. The strong eccentricity of a vertex v of D is the strong distance between v and a vertex farthest from v. The strong diameter (strong radius) of D is the maximum (minimum) strong eccentricity among all vertices of D. The lower orientable strong diameter (lower orientable strong radius), $sdiam(G)$ ($srad(G)$), of a 2-edge-connected graph G is the minimum strong diameter (minimum strong radius) over all strong orientations of G. In this paper, a conjecture of Chen and Guo is disproved by proving $sdiam(K_3 \square K_3) = sdiam(K_3 \square K_4) = 5$, $sdiam(K_m \square P_n)$ is determined, $sdiam(G)$ and $srad(G)$ for cycle vertex multiplications are computed, and some results concerning $sdiam(G)$ are described.

Keywords: Lower orientable strong diameter, Lower orientable strong radius, Cartesian product, Cycle vertex multiplication.

1 Introduction

Let G be a finite undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity of* v is $e_G(v)$ = max

$\{d_G(v,x) \mid x \in V(G)\}$, where $d_G(v,x)$ denotes the length of a shortest (v,x)-path in G. The *diameter* of G is $d(G) = \max\{e_G(v) \mid v \in V(G)\}$ and the *radius* of G is $r(G) = \min\{e_G(v) \mid v \in V(G)\}$.

Let D be a directed graph (digraph) with vertex set $V(D)$ and arc set $A(D)$ which has no loops and no two of its arcs have same tail and same head. The *distance* $d_D(u,v)$ from a vertex u to a vertex v in D is the length of a shortest directed (u,v)-path in D. Since the distance d_D does not satisfy the *symmetric property*, the distance d_D is not a *metric*. For $v \in V(D)$, the notions $e_D(v)$, $d(D)$ and $r(D)$ are defined as in the undirected graph. The *underlying* graph $G(D)$ of a digraph D is arising when directions of arcs are ignored.

A vertex v is *reachable* from a vertex u of D if there is a directed path in D from u to v. A digraph D is *strongly connected* or *strong* if any pair of vertices in D are mutually reachable in D. The underlying graph $G(D)$ of a strong digraph D is necessarily 2-edge-connected.

As the *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest (u,v)-path in G, the distance $d_G(u,v)$ is the minimum number of edges in a connected subgraph of G containing u and v. This equivalent formulation of the distance d_G was extended by Chartrand, Erwin, Raines, and Zhang [2] to strongly connected digraphs, in particular to strong oriented graphs.

The *strong distance*, $sd_D(u,v)$, between u and v is defined, in [2], as the minimum number of arcs of a strong subdigraph of D containing u and v. The strong distance sd_D is a metric on $V(D)$.

The *strong eccentricity* of a vertex v in D is $se_D(v) = \max\{sd_D(v,x) \mid x \in V(D)\}$. The *strong diameter* of D is $sdiam(D) = \max\{se_D(v) \mid v \in V(D)\}$ and the *strong radius* of D is $srad(D) = \min\{se_D(v) \mid v \in V(D)\}$.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edges. For a 2-edge-connected graph G, let $\mathscr{D}(G)$ denote the set of all strong orientations of G; the *lower orientable strong diameter* of G is $sdiam(G) = \min\{sdiam(D) : D \in \mathscr{D}(G)\}$; the *lower orientable strong radius* of G is $srad(G) = \min\{srad(D) : D \in \mathscr{D}(G)\}$ ([13]); and the *orientation number* of G is $\vec{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$ ([8]).

Notations and terminology not described here can be seen in [1].

For $X \subseteq V(D)$, the subdigraph of D induced by X is denoted by $D[X]$. The *size* of D is the number of arcs in D, and $u \to v$ means (u,v) is an arc in D.

In this paper, we concentrate on $sdiam(G)$ and $srad(G)$.

2 Lower orientable strong diameter of cartesian product of graphs

In this section, we consider lower orientable strong diameter of cartesian product of graphs.

Let G be a 2-edge-connected graph. Juan, Huang and Sun [5] proved that $sdiam(G) \geq 2d(G)$, and Chen, Guo and Zhai [4] proved that $sdiam(G) \leq 2\vec{d}(G)$. Consequently, if $\vec{d}(G) = d(G)$, then $sdiam(G) = 2d(G)$. Hence to compute $sdiam(G)$, it is enough to consider G with $\vec{d}(G) > d(G)$.

The *cartesian product* $G \square H$ of two graphs G and H has $V(G \square H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \square H$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

For the n-dimensional hypercube $Q_n = Q_{n-1} \square K_2$, McCanna [6] evaluated $\vec{d}(Q_n)$ as follows: $\vec{d}(Q_2) = 3$, $\vec{d}(Q_3) = 5$ and $\vec{d}(Q_n) = n$ for $n \geq 4$. As $d(Q_n) = n$, $sdiam(Q_n) = 2n$ for $n \geq 4$ ([12], see Theorem 3). Juan, Huang and Sun [5] proved that $srad(G) \geq 2r(G)$. Hence, $2r(G) \leq srad(G) \leq sdiam(G) \leq 2\vec{d}(G)$. Since $r(Q_n) = n$, $srad(Q_n) = 2n$ for $n \geq 4$ ([12], see Theorem 1).

For the complete graph K_ν, let $V(K_\nu) = \{1, 2, \ldots, \nu\}$; for the path P_ν on ν vertices, let $V(P_\nu) = \{1, 2, \ldots, \nu\}$ and $E(P_\nu) = \{\{i, i+1\} : i \in \{1, 2, \ldots, \nu - 1\}\}$; and for the cycle C_ν on ν vertices, let $V(C_\nu) = V(P_\nu)$ and $E(C_\nu) = E(P_\nu) \cup \{\{\nu, 1\}\}$.

Theorem 2.1 *Let G be a graph with vertices u and v such that $d_G(u, v) = d(G)$ and between u and v there is a unique path in G. Then, for any integer $m \geq 3$, $sdiam(K_m \square G) \geq 2d(G) + 3$.*

Proof. Let $P : x_1 x_2 \ldots x_k$ be the unique (u, v)-path in G, where $u = x_1$, $v = x_k$ and $k = d(G) + 1$. As $d(K_m \square G) = d(G) + 1$, $sdiam(K_m \square G) \geq 2(d(G) + 1)$.

Suppose $sdiam(K_m \square G) = 2(d(G) + 1)$, then there exist an orientation D of $K_m \square G$ with $sdiam(D) = 2(d(G) + 1)$. Let $i, j \in \{1, 2, \ldots, m\}$ and $i \neq j$. Then $d_{K_m \square G}((i, u), (j, v)) = d(G) + 1$ and there is a unique 2-edge-connected subgraph with $2(d(G) + 1)$ edges containing (i, u) and (j, v) in $K_m \square G$, namely, the cycle $C_{i,j} : (i, x_1)\ (i, x_2)\ \ldots\ (i, x_{k-1})\ (i, x_k)$

(j, x_k) (j, x_{k-1}) $\cdots$ (j, x_2) (j, x_1) (i, x_1). $C_{i,j}$ must be a directed cycle in D. Without loss of generality assume that the orientation of $C_{1,2}$ in D is $(1, x_1) \to (1, x_2) \to \cdots \to (1, x_{k-1}) \to (1, x_k) \to (2, x_k) \to (2, x_{k-1}) \to \cdots \to (2, x_2) \to (2, x_1) \to (1, x_1)$. Consequently, the orientation of $C_{1,3}$ in D is $(1, x_1) \to (1, x_2) \to \cdots \to (1, x_{k-1}) \to (1, x_k) \to (3, x_k) \to (3, x_{k-1}) \to \cdots \to (3, x_2) \to (3, x_1) \to (1, x_1)$. Now the orientation of $C_{2,3}$ in D is not a directed cycle, a contradiction. Hence, $sdiam(K_m \square G) > 2(d(G)+1)$. ■

In [7], Koh and Tay proved that for $m \geq 2$ and $n \geq 2$,

$$\vec{d}(K_m \square P_n) = \begin{cases} n+2 & \text{if } (m,n) \in \{(2,3),(2,5),(3,2)\}, \\ n+1 & \text{otherwise.} \end{cases}$$

Clearly, $sdiam(K_2 \square P_2) = 4$. In [3], Chen and Guo proved that, for $m \geq 3$, $sdiam(K_m \square P_2) = 5$. We have the following.

Theorem 2.2 *For positive integers $m \geq 3$ and $n \geq 3$, $sdiam(K_m \square P_n) = 2n+1$.*

Proof. By Theorem 2.1, $sdiam(K_m \square P_n) \geq 2d(P_n) + 3 = 2n+1$. To complete the proof, it suffices to provide an orientation D of $K_m \square P_n$ with $sdiam(D) \leq 2n+1$. We consider two cases.

Case 1. $m = 3$.

Define an orientation D of $K_3 \square P_n$ as follows:
$(1,1) \to (1,2) \to (1,3) \to \cdots \to (1,n)$,
$(2,1) \to (2,2) \to (2,3) \to \cdots \to (2,n)$,
$(3,1) \leftarrow (3,2) \leftarrow (3,3) \leftarrow \cdots \leftarrow (3,n)$,
$(3,1) \to (1,1) \to (2,1)$ and $(3,1) \to (2,1)$,
for $2 \leq j \leq n-1$, $(1,j) \to (2,j) \to (3,j) \to (1,j)$, (∗)
$(3,n) \leftarrow (1,n) \leftarrow (2,n)$ and $(3,n) \leftarrow (2,n)$.

Claim 1. For $i_1, i_2 \in \{1,2,3\}$ and $j_1, j_2 \in \{2,3,4,\ldots,n-1\}$ with $j_1 < j_2$, $sd_D((i_1,j_1),(i_2,j_2)) \leq 2n$.

Claim 1 follows from the strong subdigraph $[(1,j_1) \to (2,j_1) \to (3,j_1) \to (1,j_1)] \cup [(1,j_2) \to (2,j_2) \to (3,j_2) \to (1,j_2)] \cup [(2,j_1) \to (2,j_1+1) \to \cdots \to (2,j_2)] \cup [(3,j_2) \to (3,j_2-1) \to \cdots \to (3,j_1)]$ in D.

Claim 2. For $j, j_1, j_2 \in \{1,2,\ldots,n\}$, $se_D((3,j)) \leq 2n$, $sd_D((1,j_1),(1,j_2)) \leq 2n$ and $sd_D((2,j_1),(2,j_2)) \leq 2n$.

Claim 2 follows from the directed $2n$-cycles $(1,1) \to (1,2) \to \cdots \to (1,n) \to (3,n) \to (3,n-1) \to (3,n-2) \to \cdots \to (3,1) \to (1,1)$ and $(2,1) \to (2,2) \to \cdots \to (2,n) \to (3,n) \to (3,n-1) \to (3,n-2) \to \cdots \to (3,1) \to (2,1)$ in D.

Claim 3. $se_D((1,1)) \leq 2n+1$.

Claim 3 follows from the directed $(2n+1)$-cycle $(1,1) \to (2,1) \to (2,2) \to \cdots \to (2,n) \to (3,n) \to (3,n-1) \to \cdots \to (3,1) \to (1,1)$ in D and from $sd_D((1,j_1),(1,j_2)) \leq 2n$ for $j_1, j_2 \in \{1,2,\ldots,n\}$.

Claim 4. $se_D((1,n)) \leq 2n+1$.

Claim 4 follows from the directed $(2n+1)$-cycle $(2,1) \to (2,2) \to (2,3) \to \cdots \to (2,n) \to (1,n) \to (3,n) \to (3,n-1) \to (3,n-2) \to \cdots \to (3,1) \to (2,1)$ in D and from $sd_D((1,j_1),(1,j_2)) \leq 2n$ for $j_1, j_2 \in \{1,2,\ldots,n\}$.

Claim 5. For $j \in \{2,3,\ldots,n-1\}$, $sd_D((2,1),(1,j)) \leq 2n$.

Claim 5 follows from the strong subdigraph $[(3,j) \to (3,j-1) \to \cdots \to (3,1) \to (2,1) \to (2,2) \to \cdots \to (2,j)] \cup [(1,j) \to (2,j) \to (3,j) \to (1,j)]$ in D.

Claim 6. For $j \in \{2,3,\ldots,n-1\}$, $sd_D((2,n),(1,j)) \leq 2n$.

Claim 6 follows from the strong subdigraph $[(2,j) \to (2,j+1) \to \cdots \to (2,n) \to (3,n) \to (3,n-1) \to \cdots \to (3,j)] \cup [(1,j) \to (2,j) \to (3,j) \to (1,j)]$ in D.

By Claims 1-6 and by $(*)$, $sdiam(D) \leq 2n+1$.

Case 2. $m \geq 4$.

In what follows, we consider the orientation D of $K_m \square P_n$ obtained by Koh and Tay in Proposition 1 of [7]. It is known that $\vec{d}(K_\nu)$ is 2 if $\nu \neq 4$ and it is 3 if $\nu = 4$. If $\nu = 4$, consider the orientation $1 \to 2 \to 3 \to 4 \to 1$, $3 \to 1$ and $2 \to 4$ of K_4. Observe that in this orientation of K_4, $d(i,j) \leq 2$ if $(i,j) \neq (4,3)$; and upto isomorphism, K_4 has a unique strong orientation. Let $A \in \mathscr{D}(K_{m-1})$ such that $d(A)$ is 2 if $m \neq 5$ and it is 3 if $m = 5$; and let $B \in \mathscr{D}(K_m)$ such that $d(B)$ is 2 if $m \neq 4$ and it is 3 if $m = 4$. Define D as follows:

$D_1 = D[\{1,2,\ldots,m-1\} \times \{1\}] \equiv A$;

for $i \in \{1,2,\ldots,m-1\}$, $(m,1) \to (i,1)$;

for $j \in \{2,3,\ldots,n-1\}$, $D_j = D[\{1,2,\ldots,m\} \times \{j\}] \equiv B$;

$D_n = D[\{1,2,\ldots,m-1\} \times \{n\}] \equiv \tilde{A}$, the converse digraph of A;

for $i \in \{1,2,\ldots,m-1\}$, $(i,n) \to (m,n)$;

for $i \in \{1,2,\ldots,m-1\}$, $(i,1) \to (i,2) \to \cdots \to (i,n)$.

$(m,n) \to (m,n-1) \to \cdots \to (m,1)$.

Claim 1. For $j \in \{1,2,\ldots,n\}$, $se_D((m,j)) \leq 2n$.

Claim 2. For $i \in \{1,2,\ldots,m-1\}$ and $j_1, j_2 \in \{1,2,\ldots,n\}$ with $j_1 \neq j_2$, $sd_D((i,j_1),(i,j_2)) \leq 2n$.

Claims 1 and 2 follow from the set $\{(i,1) \to (i,2) \to \cdots \to (i,n) \to (m,n) \to (m,n-1) \to \cdots \to (m,1) \to (i,1) : i \in \{1,2,\ldots,m-1\}\}$ of directed $2n$-cycles in D.

Claim 3. For $i_1, i_2 \in \{1,2,\ldots,m-1\}$ with $i_1 \neq i_2$ and $j \in \{1,2,\ldots,n\}$, $sd_D((i_1,j),(i_2,j)) \leq 4$.

If neither $m = 5$ and $j \in \{1,n\}$ nor $m = 4$ and $j \in \{2,3,\ldots,n-1\}$, then Claim 3 follows, since the vertices (i_1,j) and (i_2,j) belong to a directed cycle of length at most 4 in D_j. If either $m=5$ and $j\in\{1,n\}$ or $m=4$ and $j\in\{2,3,\dots,n-1\}$, then since the strong orientation of K_4 contains a directed 4-cycle, the vertices (i_1,j) and (i_2,j) belong to a directed cycle of length at most 4 in D_j.

Claim 4. For $i_1,i_2\in\{1,2,\dots,m-1\}$ with $i_1\neq i_2$ and $j_1,j_2\in\{2,3,\dots,n-1\}$ with $j_1\neq j_2$, $sd_D((i_1,j_1),(i_2,j_2))\leq 2n$.

Without loss of generality assume that $j_1<j_2$. For $m\neq 4$, Claim 4 follows from the closed directed trail $(i_1,j_1)\overrightarrow{P}(i_2,j_1)\to(i_2,j_1+1)\to\cdots\to(i_2,j_2)\overrightarrow{Q}(m,j_2)\to(m,j_2-1)\to\cdots\to(m,j_1)\overrightarrow{R}(i_1,j_1)$ in D of length at most $2+(n-3)+2+(n-3)+2=2n$, where $\overrightarrow{P}$, $\overrightarrow{Q}$ and $\overrightarrow{R}$ are, respectively, directed $((i_1,j_1),(i_2,j_1))$, $((i_2,j_2),(m,j_2))$ and $((m,j_1),(i_1,j_1))$ paths of length at most 2 in D_{j_1}, D_{j_2} and D_{j_1}. Hence, assume that $m=4$. In the above closed directed trail, if $(i_1,i_2)=(1,2)$, take $\overrightarrow{P}:(1,j_1)\to(2,j_1)$, $\overrightarrow{Q}:(2,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)$; if $(i_1,i_2)=(1,3)$, take $\overrightarrow{P}:(1,j_1)\to(2,j_1)\to(3,j_1)$, $\overrightarrow{Q}:(3,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)$; if $(i_1,i_2)=(2,1)$, take $\overrightarrow{P}:(2,j_1)\to(3,j_1)\to(1,j_1)$, $\overrightarrow{Q}:(1,j_2)\to(2,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)\to(2,j_1)$; if $(i_1,i_2)=(2,3)$, take $\overrightarrow{P}:(2,j_1)\to(3,j_1)$, $\overrightarrow{Q}:(3,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)\to(2,j_1)$; if $(i_1,i_2)=(3,1)$, take $\overrightarrow{P}:(3,j_1)\to(1,j_1)$, $\overrightarrow{Q}:(1,j_2)\to(2,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)\to(2,j_1)\to(3,j_1)$; if $(i_1,i_2)=(3,2)$, take $\overrightarrow{P}:(3,j_1)\to(1,j_1)\to(2,j_1)$, $\overrightarrow{Q}:(2,j_2)\to(4,j_2)$ and $\overrightarrow{R}:(4,j_1)\to(1,j_1)\to(2,j_1)\to(3,j_1)$.

Claim 5. For $i_1,i_2\in\{1,2,\dots,m-1\}$ with $i_1\neq i_2$ and $j\in\{2,3,\dots,n-1\}$, $sd_D((i_1,1),(i_2,j))\leq 2n+1$.

For $m\neq 5$, Claim 5 follows from the closed directed trail $(i_1,1)\overrightarrow{P}(i_2,1)\to(i_2,2)\to\cdots\to(i_2,j)\overrightarrow{Q}(m,j)\to(m,j-1)\to\cdots\to(m,1)\to(i_1,1)$ in D of length at most $2+(n-2)+2+(n-2)+1=2n+1$, where $\overrightarrow{P}$ and $\overrightarrow{Q}$ are, respectively, directed $((i_1,1),(i_2,1))$ and $((i_2,j),(m,j))$ paths of length at most 2 in D_1 and D_j. Note that for $m=4$, $\overrightarrow{Q}$ is a directed $((i_2,j),(4,j))$-path of length at most 2 in D_j. Hence, assume that $m=5$. If $(i_1,i_2)\neq(4,3)$, then $\overrightarrow{P}$ is a directed $((i_1,1),(i_2,1))$-path of length at most 2 in D_1. So assume that $(i_1,i_2)=(4,3)$. Consider the closed directed trail $(4,1)\to(4,2)\to\cdots\to(4,j)\overrightarrow{P}(3,j)\overrightarrow{Q}(5,j)\to(5,j-1)\to\cdots\to(5,1)\to(4,1)$ in D of length $(n-2)+2+2+(n-2)+1=2n+1$, where $\overrightarrow{P}$ and $\overrightarrow{Q}$ are, respectively, directed $((4,j),(3,j))$ and $((3,j),(5,j))$ paths of length at most 2 in D_j.

Claim 6. For $i_1,i_2\in\{1,2,\dots,m-1\}$ with $i_1\neq i_2$ and $j\in\{2,3,\dots,n-1\}$, $sd_D((i_1,j),(i_2,n))\leq 2n+1$.

For $m \neq 4$, Claim 6 follows from the closed directed trail $(i_1,j)\overrightarrow{P}(i_2,j) \rightarrow (i_2,j+1) \rightarrow \cdots \rightarrow (i_2,n) \rightarrow (m,n) \rightarrow (m,n-1) \rightarrow \cdots \rightarrow (m,j)\overrightarrow{Q}(i_1,j)$ in D of length at most $2+(n-2)+1+(n-2)+2 = 2n+1$, where $\overrightarrow{P}$ and $\overrightarrow{Q}$ are, respectively, directed $((i_1,j),(i_2,j))$ and $((m,j),(i_1,j))$ paths of length at most 2 in D_j. Hence, assume that $m = 4$. If $i_1 \neq 3$, then $\overrightarrow{P}$ and $\overrightarrow{Q}$ are, respectively, directed $((i_1,j),(i_2,j))$ and $((4,j),(i_1,j))$ paths of length at most 2 in D_j. So, assume that $i_1 = 3$. If $i_2 = 1$, then $\overrightarrow{P}$ and $\overrightarrow{Q}$ are, respectively, directed $((3,j),(1,j))$ and $((4,j),(3,j))$ paths of length at most 1 and 3 in D_j. So, assume that $i_2 = 2$. Now, consider the strong subdigraph $[(3,j) \rightarrow (1,j) \rightarrow (2,j) \rightarrow (3,j)] \cup [(2,j) \rightarrow (2,j+1) \rightarrow \cdots \rightarrow (2,n) \rightarrow (4,n) \rightarrow (4,n-1) \rightarrow \cdots \rightarrow (4,j) \rightarrow (1,j)]$ in D of size $3+(n-2)+1+(n-2)+1 = 2n+1$.

Claim 7. For $i_1, i_2 \in \{1,2,\ldots,m-1\}$ with $i_1 \neq i_2$, $sd_D((i_1,1),(i_2,n)) \leq 2n+1$.

If $(i_1,1) \rightarrow (i_2,1)$ is in D, then consider the directed $(2n+1)$-cycle $(i_1,1) \rightarrow (i_2,1) \rightarrow (i_2,2) \rightarrow \cdots \rightarrow (i_2,n) \rightarrow (m,n) \rightarrow (m,n-1) \rightarrow \cdots \rightarrow (m,1) \rightarrow (i_1,1)$ in D; otherwise $(i_1,1) \leftarrow (i_2,1)$ is in D, and hence $(i_1,n) \rightarrow (i_2,n)$ is in D, now consider the directed $(2n+1)$-cycle $(i_1,1) \rightarrow (i_1,2) \rightarrow \cdots \rightarrow (i_1,n) \rightarrow (i_2,n) \rightarrow (m,n) \rightarrow (m,n-1) \rightarrow \cdots \rightarrow (m,1) \rightarrow (i_1,1)$ in D.

By Claims 1-7, $sdiam(D) \leq 2n+1$.

This completes the proof. ■

In [7], Koh and Tay proved that for $m \geq 4$ and $k \geq 1$, $\vec{d}(K_m \Box C_{2k+1}) = k+2$. We have:

Corollary 2.1 *For $m \geq 3$ and $k \geq 1$, $2k+3 \leq sdiam(K_m \Box C_{2k+1}) \leq 2k+4$.*

Proof. By Thoerem 2.1, $sdiam(K_m \Box C_{2k+1}) \geq 2k+3$. Upper bound follows from Propositions 7 and 8 in [7]. ■

In [3], Chen and Guo proved that $sdiam(K_2 \Box K_n) = 5$ for $n \geq 3$; $5 \leq sdiam(K_m \Box K_n) \leq 6$ for $3 \leq m \leq n$, and conjectured that $sdiam(K_m \Box K_n) = 6$ for $3 \leq m \leq n$. We disprove this conjecture for the two pairs $(m,n) = (3,3)$ and $(3,4)$. The digraph D_1 in Figure 1 is an orientation of $K_3 \Box K_3$; the directed 5-cycles

$(1,1) \rightarrow (1,2) \rightarrow (2,2) \rightarrow (3,2) \rightarrow (3,1) \rightarrow (1,1)$,
$(1,2) \rightarrow (2,2) \rightarrow (3,2) \rightarrow (3,3) \rightarrow (1,3) \rightarrow (1,2)$,
$(1,2) \rightarrow (2,2) \rightarrow (2,1) \rightarrow (2,3) \rightarrow (1,3) \rightarrow (1,2)$,
$(1,2) \rightarrow (3,2) \rightarrow (3,1) \rightarrow (3,3) \rightarrow (1,3) \rightarrow (1,2)$,
$(2,1) \rightarrow (2,3) \rightarrow (2,2) \rightarrow (3,2) \rightarrow (3,1) \rightarrow (2,1)$, and

$$(2,1) \to (2,3) \to (3,3) \to (1,3) \to (1,1) \to (2,1)$$

in D_1 shows that $sdiam(D_1) \le 5$, and hence $sdiam(K_3 \square K_3) = 5$. The digraph D_2 in Figure 2 is an orientation of $K_3 \square K_4$; the directed 5-cycles

$(1,1) \to (1,2) \to (2,2) \to (2,4) \to (1,4) \to (1,1)$,
$(1,1) \to (1,2) \to (3,2) \to (3,3) \to (1,3) \to (1,1)$,
$(1,2) \to (3,2) \to (3,1) \to (3,3) \to (1,3) \to (1,2)$,
$(1,2) \to (2,2) \to (2,1) \to (2,3) \to (1,3) \to (1,2)$,
$(1,1) \to (2,1) \to (2,3) \to (2,4) \to (1,4) \to (1,1)$,
$(1,1) \to (3,1) \to (3,4) \to (2,4) \to (1,4) \to (1,1)$,
$(1,2) \to (2,2) \to (2,4) \to (1,4) \to (1,3) \to (1,2)$,
$(1,3) \to (1,1) \to (3,1) \to (3,4) \to (1,4) \to (1,3)$,
$(1,4) \to (1,2) \to (3,2) \to (3,3) \to (3,4) \to (1,4)$,
$(2,1) \to (2,3) \to (3,3) \to (3,4) \to (2,4) \to (2,1)$,
$(2,1) \to (2,3) \to (2,2) \to (3,2) \to (3,1) \to (2,1)$,
$(2,2) \to (3,2) \to (3,4) \to (1,4) \to (1,2) \to (2,2)$,
$(2,2) \to (3,2) \to (3,3) \to (1,3) \to (1,2) \to (2,2)$, and
$(2,4) \to (1,4) \to (1,2) \to (3,2) \to (3,4) \to (2,4)$,

in D_2 shows that $sdiam(D_2) \le 5$, and hence $sdiam(K_3 \square K_4) = 5$.

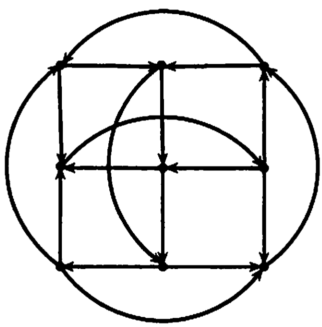

Figure. 1. An orientation D_1 of $K_3 \square K_3$.

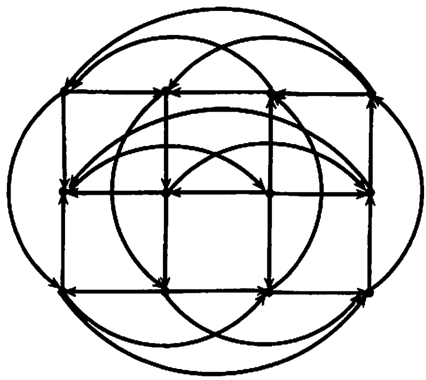

Figure. 2. An orientation D_2 of $K_3 \square K_4$.

Theorem 2.3 $sdiam(K_3 \square K_3) = 5 = sdiam(K_3 \square K_4)$.

3 Lower orientable strong diameter and radius for cycle vertex multiplications

Let G be a given connected graph of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For any sequence of n positive integers s_i, let $G(s_1, s_2, \dots, s_n)$ denote the graph with vertex set V^* and edge set E^* such that $V^* = \bigcup_{i=1}^{n} V_i$, where V_i 's are pairwise disjoint sets with $|V_i| = s_i$, $i \in \{1, 2, \dots, n\}$, and for any two distinct vertices x, y in V^*, $xy \in E^*$ if and only if $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. Call the graph $G(s_1, s_2, \dots, s_n)$ a *G-vertex multiplication*. For $s = 1, 2, \dots,$ denote $G(s, s, \dots, s)$ by $G^{(s)}$.

In this section, we consider the lower orientable strong diameter and the lower orientable strong radius for cycle vertex multiplications.

For the cycle C_n on n vertices, let $V(C_n) = \{1, 2, \dots, n\}$ and $E(C_n) = \{\{i, i+1\} : i \in \{1, 2, \dots, n-1\}\} \cup \{n, 1\}$. Write, for $i \in \{1, 2, \dots, n\}$, $V_i = \{(p, i) \mid p \in \{1, 2, \dots, s_i\}\}$ and call (p, i) the p-th vertex in V_i.

In [9], Koh and Tay proved that for $n \geq 5$, $\vec{d}(C_n^{(2)}) = d(C_n^{(2)}) + 1$. In [10], Ng and Koh proved that:

- for $6 \leq n \leq 9$, $\vec{d}(C_n^{(3)}) = d(C_n^{(3)}) + 1$,
- for $n \geq 10$ and $s_i \geq 3$, $\vec{d}(C_n^{(s_i)}) = d(C_n^{(s_i)})$,
- for $n \geq 6$, $\vec{d}(C_n^{(4)}) = d(C_n^{(4)})$.

Consequently, for $n \geq 10$ and $s_i \geq 3$, $sdiam(C_n^{(s_i)}) = 2d(C_n^{(s_i)})$ and for $n \geq 6$, $sdiam(C_n^{(4)}) = 2d(C_n^{(4)})$. It is known from [11] that $\vec{d}(C_5^{(3)}) = \vec{d}(C_5^{(4)}) = 3$.

Theorem 3.1 *$sdiam(C_6^{(3)}) = 6$, $sdiam(C_8^{(3)}) = 8$ and $sdiam(C_9^{(3)}) = 8$.*

Proof. Clearly, $sdiam(C_6^{(3)}) \geq 2d(C_6^{(3)}) = 6$. To prove $sdiam(C_6^{(3)}) \leq 6$, we have to obtain an orientation D of $C_6^{(3)}$ such that $sdiam(D) = 6$. Orient the edges of $C_6^{(3)}$ as follows:
- $(i,j) \to (i,j+1)$ if $i \in \{1,2,3\}$ and $j \in \{1,2,\dots,6\}$;
- $(k,j+1) \to (i,j)$ if $i,k \in \{1,2,3\}$, $k \neq i$ and $j \in \{1,2,\dots,6\}$.

Let D_6 be the resulting digraph. As D_6 is vertex-transitive, we only check that $se_{D_6}((1,1)) \leq 6$. The existence of the directed 6-cycles $(1,1) \to (1,2) \to (1,3) \to (1,4) \to (1,5) \to (1,6) \to (1,1)$, $(1,1) \to (3,6) \to (2,5) \to (3,4) \to (2,3) \to (3,2) \to (1,1)$, $(1,1) \to (2,6) \to (3,5) \to (2,4) \to (3,3) \to (2,2) \to (1,1)$ and $(1,1) \to (1,2) \to (2,1) \to (2,2) \to (3,1) \to (3,2) \to (1,1)$, in D_6, shows that $se_{D_6}((1,1)) \leq 6$.

Clearly, $sdiam(C_8^{(3)}) \geq 2d(C_8^{(3)}) = 8$. To show $sdiam(C_8^{(3)}) \leq 8$, we have to find an orientation D of $C_8^{(3)}$ such that $sdiam(D) = 8$. Orient the edges of $C_8^{(3)}$ as follows:
- $(i,j) \to (i,j+1)$ if $i \in \{1,2,3\}$ and $j \in \{1,2,\dots,8\}$;
- $(k,j+1) \to (i,j)$ if $i,k \in \{1,2,3\}$, $k \neq i$ and $j \in \{1,2,\dots,8\}$.

Let D_8 be the resulting digraph. As D_8 is vertex-transitive, we only check that $se_{D_8}((1,1)) \leq 8$. The existence of the directed 8-cycles $(1,1) \to (1,2) \to (1,3) \to (1,4) \to (1,5) \to (1,6) \to (1,7) \to (1,8) \to (1,1)$, $(1,1) \to (3,8) \to (2,7) \to (3,6) \to (2,5) \to (3,4) \to (2,3) \to (3,2) \to (1,1)$, $(1,1) \to (2,8) \to (3,7) \to (2,6) \to (3,5) \to (2,4) \to (3,3) \to (2,2) \to (1,1)$ and the directed 6-cycle $(1,1) \to (1,2) \to (2,1) \to (2,2) \to (3,1) \to (3,2) \to (1,1)$, in D_8, shows that $se_{D_8}((1,1)) \leq 8$.

Clearly, $sdiam(C_9^{(3)}) \geq 2d(C_9^{(3)}) = 8$. For $sdiam(C_9^{(3)}) \leq 8$, we exhibit below an orientation D of $C_9^{(3)}$ such that $sdiam(D) = 8$. Orient the edges of $C_9^{(3)}$ as follows:

(*i*) for $i \in \{1,3,5,7\}$,

$$\{(1,i+1),(3,i+1)\} \to (1,i) \to (2,i+1),$$
$$(2,i+1) \to (2,i) \to \{(1,i+1),(3,i+1)\},$$
$$\{(1,i+1),(3,i+1)\} \to (3,i) \to (2,i+1);$$

(*ii*) for $i \in \{2,4,6\}$,

$$(3,i+1) \to (1,i) \to \{(1,i+1),(2,i+1)\},$$
$$(1,i+1) \to (2,i) \to \{(2,i+1),(3,i+1)\},$$
$$\{(1,i+1),(2,i+1)\} \to (3,i) \to (3,i+1);$$

(*iii*) $\{(1,1),(2,1)\} \to (1,9) \to (3,1),$
$(1,1) \to (2,9) \to \{(2,1),(3,1)\},$
$(3,1) \to (3,9) \to \{(1,1),(2,1)\};$

(*iv*) $(3,8) \to (1,9) \to \{(1,8),(2,8)\},$
$\{(2,8),(3,8)\} \to (2,9) \to (1,8),$
$\{(1,8),(2,8)\} \to (3,9) \to (3,8).$

Let D_9 be the resulting digraph.

For $i \in \{1,3,5,7\}$, the strong subdigraph $(2,i+1) \to (2,i) \oplus (2,i) \to (1,i+1) \to (1,i) \to (2,i+1) \oplus (2,i) \to (3,i+1) \to (3,i) \to (2,i+1)$ with 7 arcs, for $i \in \{2,4,6\}$, the directed 6-cycle $(2,i+1) \to (3,i) \to (3,i+1) \to (1,i) \to (1,i+1) \to (2,i) \to (2,i+1)$, the directed 6-cycle $(1,1) \to (2,9) \to (2,1) \to (1,9) \to (3,1) \to (3,9) \to (1,1)$, and the directed 6-cycle $(2,8) \to (2,9) \to (1,8) \to (3,9) \to (3,8) \to (1,9) \to (2,8)$ in D_9 shows that $sd_{D_9}(u,v) \le 7$ for $u,v \in V_i \cup V_{i+1}$, where $i \in \{1,2,\ldots,9\}$ and $V_{9+1} = V_1$.

The existence of the following strong subdigraphs, each with at most 8 arcs, in D_9 :

for $i \in \{1,3\}$,

$(1,i) \to (2,i+1) \to (2,i+2) \to (1,i+3) \to (1,i+4) \to (3,i+3) \to (1,i+2) \to (3,i+1) \to (1,i)$, $(1,i) \to (2,i+1) \to (2,i+2) \to (1,i+3) \to (2,i+4) \to (3,i+3) \to (1,i+2) \to (3,i+1) \to (1,i)$,

$(1,i) \to (2,i+1) \to (3,i+2) \to (1,i+1) \to (1,i) \oplus (3,i+2) \to (2,i+3) \to (3,i+4) \to (1,i+3) \to (3,i+2)$,

$(2,i) \to (1,i+1) \to (2,i+2) \to (1,i+3) \to (1,i+4) \to (3,i+3) \to (1,i+2) \to (2,i+1) \to (2,i)$, $(2,i) \to (3,i+1) \to (3,i+2) \to (2,i+3) \to (2,i+4) \to (3,i+3) \to (1,i+2) \to (2,i+1) \to (2,i)$,

$(2,i) \to (3,i+1) \to (3,i+2) \to (2,i+3) \to (3,i+4) \to (1,i+3) \to (1,i+2) \to (2,i+1) \to (2,i)$,

$(3,i) \to (2,i+1) \to (2,i+2) \to (3,i+1) \to (3,i) \oplus (2,i+2) \to (1,i+3) \to (1,i+4) \to (2,i+3) \to (2,i+2)$, $(3,i) \to (2,i+1) \to (3,i+2) \to (2,i+3) \to (2,i+4) \to (3,i+3) \to (1,i+2) \to (3,i+1) \to (3,i)$,

$(3,i) \to (2,i+1) \to (3,i+2) \to (1,i+1) \to (3,i) \oplus (3,i+2) \to (2,i+3) \to (3,i+4) \to (1,i+3) \to (3,i+2)$;

for $i \in \{2,4\}$,

$(1,i) \to (2,i+1) \to (1,i+2) \to (3,i+1) \to (1,i) \oplus (1,i+2) \to (2,i+3) \to (1,i+4) \to (3,i+3) \to (1,i+2)$, $(1,i) \to (1,i+1) \to (2,i+2) \to (3,i+3) \to (2,i+4) \to (2,i+3) \to (3,i+2) \to (3,i+1) \to (1,i)$,

$(1,i) \to (1,i+1) \to (2,i+2) \to (2,i+3) \to (3,i+4) \to (1,i+3) \to (3,i+2) \to (3,i+1) \to (1,i)$,

$(2,i) \to (2,i+1) \to (1,i+2) \to (2,i+3) \to (1,i+4) \to (1,i+3) \to (3,i+2) \to (1,i+1) \to (2,i)$, $(2,i) \to (3,i+1) \to (2,i+2) \to (3,i+3) \to (2,i+4) \to (2,i+3) \to (3,i+2) \to (1,i+1) \to (2,i)$,

$(2,i) \to (3,i+1) \to (2,i+2) \to (2,i+3) \to (3,i+4) \to (3,i+3) \to (1,i+2) \to (1,i+1) \to (2,i)$,

$(3,i) \to (3,i+1) \to (2,i+2) \to (2,i+1) \to (3,i) \oplus (2,i+2) \to (2,i+3) \to (1,i+4) \to (1,i+3) \to (2,i+2)$, $(3,i) \to (3,i+1) \to (2,i+2) \to (3,i+3) \to (2,i+4) \to (2,i+3) \to (3,i+2) \to (1,i+1) \to (3,i)$,

$(3,i) \to (3,i+1) \to (2,i+2) \to (2,i+3) \to (3,i+4) \to (3,i+3) \to (1,i+2) \to (1,i+1) \to (3,i)$;

for $i = 5$,

$(1,5) \to (2,6) \to (2,7) \to (3,8) \to (1,9) \to (1,8) \to (3,7) \to (1,6) \to (1,5)$, $(1,5) \to (2,6) \to (3,7) \to (2,8) \to (2,9) \to (1,8) \to (1,7) \to (3,6) \to (1,5)$, $(1,5) \to (2,6) \to (2,7) \to (1,8) \to (3,9) \to (3,8) \to (1,7) \to (3,6) \to (1,5)$,

$(2,5) \to (1,6) \to (2,7) \to (3,8) \to (1,9) \to (1,8) \to (1,7) \to (2,6) \to (2,5)$, $(2,5) \to (3,6) \to (3,7) \to (2,8) \to (2,9) \to (1,8) \to (1,7) \to (2,6) \to (2,5)$, $(2,5) \to (3,6) \to (3,7) \to (2,8) \to (3,9) \to (3,8) \to (1,7) \to (2,6) \to (2,5)$,

$(3,5) \to (2,6) \to (2,7) \to (3,8) \to (1,9) \to (1,8) \to (1,7) \to (3,6) \to (3,5)$, $(3,5) \to (2,6) \to (2,7) \to (3,8) \to (2,9) \to (1,8) \to (3,7) \to (1,6) \to (3,5)$, $(3,5) \to (2,6) \to (3,7) \to (2,8) \to (3,9) \to (3,8) \to (1,7) \to (3,6) \to (3,5)$;

for $i = 6$,

$(1,6) \to (1,7) \to (2,8) \to (3,9) \to (1,1) \to (1,9) \to (1,8) \to (3,7) \to (1,6)$, $(1,6) \to (2,7) \to (3,8) \to (2,9) \to (2,1) \to (1,9) \to (1,8) \to (3,7) \to (1,6)$, $(1,6) \to (1,7) \to (2,8) \to (2,9) \to (3,1) \to (3,9) \to (3,8) \to (3,7) \to (1,6)$,

$(2,6) \to (3,7) \to (2,8) \to (3,9) \to (1,1) \to (1,9) \to (1,8) \to (1,7) \to (2,6)$, $(2,6) \to (2,7) \to (3,8) \to (2,9) \to (2,1) \to (1,9) \to (1,8) \to (1,7) \to (2,6)$, $(2,6) \to (2,7) \to (3,8) \to (2,9) \to (3,1) \to (3,9) \to (3,8) \to (1,7) \to (2,6)$,

$(3,6) \to (3,7) \to (2,8) \to (3,9) \to (1,1) \to (1,9) \to (1,8) \to (1,7) \to (3,6)$, $(3,6) \to (3,7) \to (2,8) \to (2,7) \to (3,6) \oplus (2,8) \to (2,9) \to (2,1) \to (1,9) \to (2,8)$, $(3,6) \to (3,7) \to (2,8) \to (2,9) \to (3,1) \to (3,9) \to (3,8) \to (1,7) \to (3,6)$;

for $i = 7$,

$(1,7) \to (2,8) \to (2,9) \to (2,1) \to (1,2) \to (1,1) \to (1,9) \to (1,8) \to (1,7)$, $(1,7) \to (2,8) \to (3,9) \to (1,1) \to (2,2) \to (2,1) \to (1,9) \to (1,8) \to (1,7)$, $(1,7) \to (2,8) \to (2,9) \to (2,1) \to (3,2) \to (3,1) \to (3,9) \to (3,8) \to (1,7)$,

$(2,7) \to (1,8) \to (3,9) \to (2,1) \to (1,2) \to (1,1) \to (1,9) \to (2,8) \to (2,7)$, $(2,7) \to (3,8) \to (2,9) \to (3,1) \to (2,2) \to (2,1) \to (1,9) \to (2,8) \to (2,7)$, $(2,7) \to (1,8) \to (3,9) \to (2,1) \to (3,2) \to (1,1) \to (1,9) \to (2,8) \to (2,7)$,

$(3,7) \to (2,8) \to (2,9) \to (2,1) \to (1,2) \to (3,1) \to (3,9) \to (3,8) \to$

$(3,7)$, $(3,7) \to (2,8) \to (3,9) \to (1,1) \to (2,2) \to (2,1) \to (1,9) \to (1,8) \to (3,7)$, $(3,7) \to (2,8) \to (2,9) \to (2,1) \to (3,2) \to (3,1) \to (3,9) \to (3,8) \to (3,7)$;
for $i = 8$,
$(1,8) \to (3,9) \to (2,1) \to (1,2) \to (1,3) \to (3,2) \to (1,1) \to (1,9) \to (1,8)$, $(1,8) \to (3,9) \to (2,1) \to (1,2) \to (2,3) \to (3,2) \to (1,1) \to (2,9) \to (1,8)$, $(1,8) \to (3,9) \to (2,1) \to (3,2) \to (3,3) \to (1,2) \to (1,1) \to (1,9) \to (1,8)$, $(1,8) \to (3,9) \to (1,1) \to (2,2) \to (2,1) \to (1,9) \to (1,8)$, $(1,8) \to (3,9) \to (1,1) \to (1,9) \to (1,8) \oplus (3,9) \to (2,1) \to (1,9) \to (3,1) \to (3,9)$,
$(2,8) \to (3,9) \to (2,1) \to (1,2) \to (1,3) \to (3,2) \to (1,1) \to (1,9) \to (2,8)$, $(2,8) \to (3,9) \to (2,1) \to (1,2) \to (2,3) \to (3,2) \to (1,1) \to (1,9) \to (2,8)$, $(2,8) \to (2,9) \to (3,1) \to (2,2) \to (3,3) \to (1,2) \to (1,1) \to (1,9) \to (2,8)$,
$(3,8) \to (2,9) \to (2,1) \to (1,2) \to (1,3) \to (3,2) \to (3,1) \to (3,9) \to (3,8)$, $(3,8) \to (2,9) \to (3,1) \to (3,9) \to (3,8) \oplus (3,1) \to (2,2) \to (2,3) \to (3,2) \to (3,1)$, $(3,8) \to (1,9) \to (3,1) \to (3,9) \to (3,8) \oplus (3,1) \to (2,2) \to (3,3) \to (1,2) \to (3,1)$, $(3,8) \to (2,9) \to (2,1) \to (1,9) \to (3,1) \to (3,9) \to (3,8) \oplus (3,9) \to (1,1) \to (1,9)$;
for $i = 9$,
$(1,9) \to (3,1) \to (2,2) \to (2,3) \to (1,4) \to (3,3) \to (1,2) \to (1,1) \to (1,9)$, $(1,9) \to (3,1) \to (2,2) \to (3,3) \to (2,4) \to (2,3) \to (3,2) \to (1,1) \to (1,9)$, $(1,9) \to (3,1) \to (2,2) \to (2,1) \to (1,9) \oplus (2,2) \to (2,3) \to (3,4) \to (1,3) \to (2,2)$,
$(2,9) \to (3,1) \to (2,2) \to (2,3) \to (1,4) \to (1,3) \to (3,2) \to (1,1) \to (2,9)$, $(2,9) \to (2,1) \to (3,2) \to (1,1) \to (2,9) \oplus (3,2) \to (3,3) \to (2,4) \to (2,3) \to (3,2)$, $(2,9) \to (3,1) \to (2,2) \to (2,3) \to (3,4) \to (3,3) \to (1,2) \to (1,1) \to (2,9)$,
$(3,9) \to (1,1) \to (2,2) \to (2,3) \to (1,4) \to (1,3) \to (3,2) \to (3,1) \to (3,9)$, $(3,9) \to (2,1) \to (3,2) \to (3,1) \to (3,9) \oplus (3,2) \to (3,3) \to (2,4) \to (2,3) \to (3,2)$ and $(3,9) \to (1,1) \to (2,2) \to (2,3) \to (3,4) \to (3,3) \to (1,2) \to (3,1) \to (3,9)$
shows that $sd_{D_9}(u,v) \leq 8$ for $u \in V_i$, $v \in V_{i+2} \cup V_{i+3} \cup V_{i+4}$, where $i \in \{1,2,\ldots,9\}$, and in suffix $8+2 = 7+3 = 6+4 = 1$, $9+2 = 8+3 = 7+4 = 2$, $9+3 = 8+4 = 3$, and $9+4 = 4$. Thus, for $u,v \in V(D_9)$, $sd_{D_9}(u,v) \leq 8$ and hence $sdiam(D_9) \leq 8$.

This completes the proof. ■

Theorem 3.2 *For* $n \geq 2$, $sdiam(C^{(2)}_{2n+1}) = 2n+1$.

Proof. Suppose there is an orientation D of $C^{(2)}_{2n+1}$ such that $sdiam(D) \leq 2n$. If, in D, $(1,1) \to \{(1,2),(2,2)\}$, then as, $d_D((1,n+1),(1,1)) > n$, we have $sd_D((1,1),(1,n+1)) > 2n$, a contradiction. Also, if, in D, $\{(1,2),(2,2)\} \to (1,1)$, then as, $d_D((1,1),(1,n+1)) > n$, we have $sd_D((1,1),(1,n+1)) > 2n$, a contradiction. Hence, by symmetry, assume that, in D, $(1,i) \to (1,i+1) \to (2,i) \to (2,i+1) \to (1,i)$ for all $i \in \{1,2,\ldots,2n\}$ and either $(1,2n+1) \to (1,1) \to (2,2n+1) \to (2,1) \to (1,2n+1)$ or $(1,2n+1) \to (2,1) \to (2,2n+1) \to (1,1) \to (1,2n+1)$. Then, as $d_D((1,1),(2,n+1)) > n$, we have $sd_D((1,1),(2,n+1)) > 2n$, a contradiction. Thus, $sdiam(C_{2n+1}^{(2)}) \geq 2n+1$.

Now, orient the edges of $C_{2n+1}^{(2)}$ as follows:
(*i*) $(1,1) \to (1,2) \to (1,3) \to \cdots \to (1,2n+1) \to (1,1)$;
(*ii*) $(2,1) \leftarrow (2,2) \leftarrow (2,3) \leftarrow \cdots \leftarrow (2,2n+1) \leftarrow (2,1)$; and
(*iii*) $(1,i) \to (2,i+1)$ and $(2,i) \to (1,i+1)$, where $i \in \{1,2,\ldots,2n+1\}$ and $(2n+1)+1 = 1$.
Let D be the resulting digraph.

By the nature of orientation, we compute strong eccentricity only for the vertices $(1,1)$ and $(2,1)$ in $V(D)$. The existence of the directed $(2n+1)$-cycles

$(1,1) \to (1,2) \to (1,3) \to \cdots \to (1,2n+1) \to (1,1)$,
$(1,1) \to (2,2) \to (1,3) \to (1,4) \to \cdots \to (1,2n+1) \to (1,1)$,
$(1,1) \to (1,2) \to \cdots \to (1,i-1) \to (2,i) \to (1,i+1) \to (1,i+2) \to \cdots \to (1,2n+1) \to (1,1)$, $i \in \{3,4,\ldots,2n-1\}$,
$(1,1) \to (1,2) \to \cdots \to (1,2n-1) \to (2,2n) \to (1,2n+1) \to (1,1)$,
$(1,1) \to (1,2) \to \cdots \to (1,2n) \to (2,2n+1) \to (1,1)$,

and the directed 4-cycle

$(1,1) \to (2,2) \to (2,1) \to (2,2n+1) \to (1,1)$,

in D, shows that $se_D((1,1)) \leq 2n+1$.

Now the existence of the directed $(2n+1)$-cycles

$(2,1) \leftarrow (2,2) \leftarrow (2,3) \leftarrow \cdots \leftarrow (2,2n+1) \leftarrow (2,1)$,
$(2,1) \to (1,2) \to (1,3) \to (1,4) \to \cdots \to (1,2n+1) \to (2,1)$,

and the directed 4-cycle

$(2,1) \to (2,2n+1) \to (1,1) \to (2,2) \to (2,1)$,

in D, shows that $se_D((2,1)) \leq 2n+1$.

Hence $sdiam(D) \leq 2n+1$. Consequently, $sdiam(C_{2n+1}^{(2)}) \leq 2n+1$.

This completes the proof. ∎

Recall that: Juan, Huang and Sun [5] proved that $2rad(G) \leq srad(G)$. As $rad(Q_n) = n$, $srad(Q_n) = 2n$ for $n \geq 4$ ([12], see Theorem 1).

Theorem 3.3 *For each* $s_i \geq 2$ *and* $n \geq 4$,

$$srad(C_n(s_1,s_2,\ldots,s_n)) = \begin{cases} n & if \quad n \quad is \quad even, \\ n-1 & if \quad n \quad is \quad odd. \end{cases}$$

Proof. Clearly, $srad(C_n(s_1, s_2, \ldots, s_n)) \geq 2\,rad(C_n(s_1, s_2, \ldots, s_n)) = 2\lfloor \frac{n}{2} \rfloor$. To prove $srad(C_n(s_1, s_2, \ldots, s_n)) \leq 2\lfloor \frac{n}{2} \rfloor$, we only need to give an orientation D of $C_n(s_1, s_2, \ldots, s_n)$ such that $srad(D) = 2\lfloor \frac{n}{2} \rfloor$. Orient the edges of $C_n(s_1, s_2, \ldots, s_n)$ as follows:

(*i*) $(1,1) \to (1,2) \to (i,1) \to (j,2) \to (1,1)$ if $i \in \{2,3,\ldots,s_1\}$ and $j \in \{2,3,\ldots,s_2\}$;

(*ii*) for each $i \in \{2,3,\ldots,\lfloor \frac{n}{2} \rfloor\}$, $(1,i) \to (j,i+1) \to (2,i)$ if $j \in \{1,2,\ldots,s_{i+1}\}$;

(*iii*) for each $i \in \{\lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 3, \ldots, n-2\}$, $(1,i+1) \to (j,i) \to (2,i+1)$ if $j \in \{1,2,\ldots,s_i\}$;

(*iv*) $(1,n) \to (j,n-1)$ if $j \in \{1,2,\ldots,s_i\}$;

(*v*) $(i,n-1) \to (j,n)$ if $i \in \{2,3,\ldots,s_{n-1}\}$ and $j \in \{2,3,\ldots,s_n\}$;

(*vi*) $(j,n) \to (1,1) \to (1,n)$ if $j \in \{2,3,\ldots,s_n\}$;

(*vii*) orient the remaining edges of $C_n(s_1, s_2, \ldots, s_n)$ arbitrarily.

Let D be the resulting digraph.

Claim. $se_D((1,1)) = 2\lfloor \frac{n}{2} \rfloor$.

The existence of

- the set $\{(1,1) \to (1,2) \to (i,1) \to (j,2) \to (1,1) : i \in \{2,3,\ldots,s_1\}$ and $j \in \{2,3,\ldots,s_2\}\}$ of directed 4-cycles,
- for $i \in \{2,3,\ldots,\lfloor \frac{n}{2} \rfloor\}$, the set $\{(1,1) \to (1,2) \to (1,3) \to \cdots \to (1,i) \to (j,i+1) \to (2,i) \to (2,i-1) \to \cdots \to (2,2) \to (1,1) : j \in \{1,2,\ldots,s_{i+1}\}\}$ of directed $(2i)$-cycles,
- the set $\{(1,1) \to (1,n) \to (i,n-1) \to (j,n) \to (1,1) : i \in \{2,3,\ldots,s_{n-1}\}$ and $j \in \{2,3,\ldots,s_n\}\}$ of directed 4-cycles, and
- for $i \in \{\lfloor \frac{n}{2} \rfloor + 3, \lfloor \frac{n}{2} \rfloor + 4, \ldots, n-1\}$, the set $\{(1,1) \to (1,n) \to (1,n-1) \to \cdots \to (1,i) \to (j,i-1) \to (2,i) \to (2,i+1) \to \cdots \to (2,n) \to (1,1) : j \in \{1,2,\ldots,s_{i-1}\}\}$ of directed $(2n-2i+4)$-cycles in D proves the claim.

This completes the proof. ∎

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