The Extremal Function for Three Disjoint Theta Graphs *

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Abstract

A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. We show that every graph of order $n \geq 12$ and size at least $\max\{\lceil \frac{3n+79}{2} \rceil, \lfloor \frac{11n-33}{2} \rfloor\}$ contains three disjoint theta graphs. As a corollary, every graph of order $n \geq 12$ and size at least $\max\{\lceil \frac{3n+79}{2} \rceil, \lfloor \frac{11n-33}{2} \rfloor\}$ contains three disjoint cycles of even length. The lower bound on the size is sharp in general.

Key Words: Disjoint theta graphs; Extremal function; Minimum degree
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1 Terminology and Introduction

In this paper, we only consider finite undirected graphs, without loops or multiple edges. We use [1] for the notation and terminology not defined here. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. Let n be a positive integer, let K_n denote the complete graph of order n and K_4^- be the graph obtained by removing exactly one edge from K_4. For a graph G, we denote its vertex set, edge set, minimum degree by $V(G)$, $E(G)$ and $\delta(G)$, respectively. The order and size of a graph G, is defined by $|V(G)|$ and $|E(G)|$, respectively. A set of subgraphs is said to be vertex-disjoint or independent, if no two of them have any common vertex in G, and we use disjoint to stand for vertex-disjoint throughout this paper. If u is a vertex of G and H is either a subgraph of G or a subset of $V(G)$, we define $N_H(u)$ to be the set of neighbors of u contained in H, and $d_H(u) = |N_H(u)|$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. In particular, we often let $[U]$ stand for $G[U]$, when $U = \{x_1, x_2, \cdots, x_t\}$, we may also use $[x_1, x_2, \cdots, x_t]$ to denote

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$[\{x_1, x_2, \cdots, x_t\}]$. Let S denote a subgraph of G, we write $G \supseteq S$, it means that S is isomorphic to a subgraph of G, in particular, we use mS to represent a set of m vertex-disjoint copies of S. Let V_1, V_2 be two disjoint subsets or subgraphs of G, we use $E(V_1, V_2)$ to denote the set of edges in G with one end-vertex in V_1, while the other in V_2, for simplicity, let $E(x, V_2)$ stand for $E(\{x\}, V_2)$, $E(V_1, x)$ for $E(V_1, \{x\})$,respectively. A path of order n is denoted by P_n.

Corrádi and Hajnal [3] proved the following well-known result on the existence of vertex-disjoint cycles in graphs.

Theorem 1.1 *([3]) Let k be a positive integer and G be a graph with order $n \geq 3k$. If $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Later, Wang [11] and independently Enomoto [5] proved a result stronger than Theorem 1.1 as follows.

Theorem 1.2 *([11]) Let k be a positive integer and G be a graph with order $n \geq 3k$. Suppose for any pair of nonadjacent u and v in G, $d_G(u) + d_G(v) \geq 4k - 1$, then G contains k disjoint cycles.*

Given a cycle C of a graph G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C. A cycle is called a chorded cycle if it has at least one chord. A chorded cycle is a simple example of a theta graph but, in general, a theta graph need not be a chorded cycle. It is obvious that K_4^- is the theta graph with minimum order and every theta graph contains a cycle of even length. Pósa [10] proved that any graph with minimum degree at least three contains a chorded cycle. Motivated by these results, Finkel et al. [6] and Chiba et al. [3] obtained the following results analogous to Theorem 1.2, respectively.

Theorem 1.3 *([6]) If G is a graph of order $n \geq 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.*

Theorem 1.4 *([3]) Let r,s be two nonnegative integers and let G be a graph with order $n \geq 3r + 4s$. Suppose for any pair of nonadjacent u and v in G, $d_G(u) + d_G(v) \geq 4r + 6s - 1$, then G contains $r + s$ disjoint cycles, such that s of them are chorded cycles.*

In particular, Kawarabayashi [9] considered the minimum degree to ensure the existence of disjoint copies of K_4^- in a general graph G, which can be seen the specified version of disjoint theta graphs.

Theorem 1.5 *([9]) Let k be a positive integer and G be a graph with order $n \geq 4k$. If $\delta(G) \geq \lceil \frac{n+k}{2} \rceil$, then G contains k disjoint copies of K_4^-.*

In this paper, we determine the edge number for a graph to contain three disjoint theta graphs. Our research is motivated by the following conjecture which was proposed by Gao and Ji [7]. They verified Conjecture 1.6 for the case $k = 2$.

Conjecture 1.6 *([7]) Let $k \geq 2$ be an integer. Every graph of order n and size at least $f(n,k)+1$ contains k disjoint theta graphs, when*

$$f(n,k) = \max\left\{\binom{4k-1}{2} + \frac{3}{2}(n-4k+1),\right.$$
$$\left.\left\lfloor\frac{2(k-1)(2k-1)+(4k-1)(n-2k+1)}{2}\right\rfloor\right\}$$

Theorem 1.7 *([7]) Every graph of order $n \geq 8$ and size at least $f(n)$ contains two disjoint theta graphs, if*

$$f(n) = \begin{cases} 23 & \text{if } n = 8 \\ \lfloor\frac{7n-13}{2}\rfloor & \text{if } n \geq 9 \end{cases}$$

In current paper, we completely solve Conjecture 1.6 for the case $k = 3$.

Theorem 1.8 *Every graph of order $n \geq 12$ and size at least*

$$\max\left\{\left\lceil\frac{3n+79}{2}\right\rceil, \left\lfloor\frac{11n-33}{2}\right\rfloor\right\}$$

contains three disjoint theta graphs.

Corollary 1.9 *Every graph of order $n \geq 12$ and size at least*

$$\max\left\{\left\lceil\frac{3n+79}{2}\right\rceil, \left\lfloor\frac{11n-33}{2}\right\rfloor\right\}$$

contains three disjoint cycles of even length.

Note that Gao and Ma [8] obtained the following result.

Theorem 1.10 *([8]) Every graph of order $n \geq 12$ and size at least $\lfloor\frac{11n-18}{2}\rfloor$ contains three disjoint theta graphs.*

2 Basic Lemma

Lemma 2.1 *Let G be a graph of order 12 and size at least 57, then G contains three disjoint copies of K_4^-.*

Proof This is obviously true by Theorem 1.10. □

Lemma 2.2 *Let G be a graph of order 13 and size at least 59, then G contains three disjoint theta graphs.*

Proof Suppose that G does not contain three disjoint theta graphs. If $\delta(G) \geq 8$, then by Theorem 1.5, $G \supseteq 3K_4^-$, a contradiction. Hence, we may assume that $\delta(G) \leq 7$. Let $v_0 \in V(G)$ such that $d_G(v_0) = \delta(G) = l$. If $l \leq 2$, then $|E(G - \{v_0\})| \geq 59 - 2 = 57$, it follows from Lemma 2.1 that $[V(G) - \{v_0\}] \supseteq 3K_4^-$, a contradiction. Hence, we may assume that $l \geq 3$ and let $G' = G - \{v_0\}$. It is obvious that G' can be obtained from K_{12} by removing at most

$$66 - (59 - l) = 7 + l \leq 14$$

edges. We divide the proof into two cases.

Case 1 *$G[U \cup \{v_0\}] \supseteq K_4^-$ for some $U \in V(G')$ with $|U| = 3$.*

In this case, for $y_1, y_2, y_3 \in U$ such that $[v_0, y_1, y_2, y_3] \supseteq K_4^-$, we have

$$\begin{aligned} |E(G' - U))| &\geq |E(G)| - l - (\sum_{i=1}^{3} d_{G'}(y_i) - 3) \\ &\geq |E(G)| - l - (11 + 10 + 9) \\ &\geq 29 - l. \end{aligned} \qquad (1)$$

If $l \leq 4$, then by (1), $|E(G' - U)| \geq 25$, applying Theorem 1.7 to $G' - U$, whose order is nine, $G' - U$ contains two disjoint theta graphs, so G contains three disjoint theta graphs, a contradiction. Hence, $5 \leq l \leq 7$. Label $G^* = G' - U$, that is, G^* denotes the graph which obtained from G by removing $U \cup \{v_0\}$. Since G^* does not contain two disjoint theta graph, by Theorem 1.7, we have $|E(G^*)| \leq 24$ (remove at least 12 edges from K_9). If $l = 5$, recall that G' can be obtained from K_{12} by removing at most $66 - (59 - l) = 7 + l \leq 12$ edges, we obtain

- $|E([U])| = 3$;
- for each $x \in U$, $|E(x, V(G^*))| = 9$;
- $|E(G^*)| = 24$;
- for each $U \in V(G')$ with $|U| = 3$ such that $G[U \cup \{v_0\}] \supseteq K_4^-$, we have $\sum_{u \in U} d_{G'}(u) = 33$ by (1).

As $l = 5$, label u_1, u_2 be two neighbors of v_0 in $V(G^*)$ and $U = \{y_1, y_2, y_3\}$. By above, u_i is adjacent to each vertex in $V(G) - u_i$, when $i \in \{1, 2\}$. We prove that $G^* - \{u_1, u_2\}$ contains a path of order three. If not, then $G^* - \{u_1, u_2\}$ has at most three independent edges, so

$$|E(G^*)| \leq 3 + 15 = 18 < 24,$$

a contradiction. Without loss of generality, let P^* be one path of order three in $G^* - \{u_1, u_2\}$ and let $\mathcal{W} = V(G^*) - V(P^*) - \{u_1, u_2\}$. Suppose that there exist $x_1, x_2 \in \mathcal{W}$, such that

$$G^* - (V(P^*) \cup \{u_1, u_2, x_1, x_2\}) \text{ contains a theta graph,}$$

as $[x_1, x_2, y_1, y_2] \supseteq K_4^-$ and $[u_1, u_2, v_0, y_3] \supseteq K_4^-$, then G contains three disjoint theta graphs, a contradiction. This implies that for each $x \in \mathcal{W}$,

$$|E(x, V(P^*))| \leq \begin{cases} 2, & \text{if } G^*[V(P^*)] \cong K_3 \\ 1. & \text{otherwise} \end{cases}$$

Therefore, there are at most 4 edges between $V(P^*)$ and $\mathcal{W}$ if $G^*[V(P^*)] \cong K_3$, otherwise, there are at most 5 edges between $V(P^*)$ and $\mathcal{W}$. Furthermore, as $G \not\supseteq 3K_4^-$, it follows that $G^*[\mathcal{W}]$ contains no path of order three. To summarize, we have

$$|E(G^*)| \leq \begin{cases} 3+4+2+14+1, & \text{if } G^*[V(P^*)] \cong K_3 \\ 2+5+2+14+1. & \text{Otherwise} \end{cases}$$

Thus, $G^*[\mathcal{W}]$ is isomorphic to two independent edges, denoted by e_1 and e_2, then $[V(e_1) \cup \{y_1, u_1\}] \supseteq K_4^-$, $[V(e_2) \cup \{u_2, v_0, y_2\}] \supseteq K_4^-$ and $[V(P^* \cup \{y_3\}] \supseteq K_4^-$, a contradiction.

Suppose $l = 7$. Then recall that G' can be obtained from K_{12} by removing at most 14 edges, it follows that H contains a cycle of length 7. Clearly, $|E(w, H)| \geq 5$, for otherwise, $|E(G)| \leq 7 + 21 + 10 + 20 = 58$, a contradiction. By the pigeonhole principle, we may assume that x_1, x_2, x_3 are three neighbors of w. By (1), if $d_{G'}(x_1) + d_{G'}(x_2) + d_{G'}(w) \leq 30$, then it follows from (1) that $G \supseteq 3K_4^-$, a contradiction. Thus, we may assume that $d_{G'}(x_1) + d_{G'}(x_2) + d_{G'}(w) \geq 31$. Similarly, $\sum_{i=3}^{5} d_{G'}(x_i) \geq 31$ and $\sum_{i=5}^{7} d_{G'}(x_i) \geq 31$. Let $z_1, z_2, z_3, z_4 \in V(G') - V(H) - \{w\}$. Without loss of generality, assume that $z_1, z_2 \in N_H(x_4) \cap N_H(x_5)$. Then for each $i \in \{3, 4\}$, z_i can not be both neighbors of x_6 and x_7, otherwise, $[v_0, x_6, x_7, z_i] \supseteq K_4^-$, $[x_1, x_2, x_3, w] \supseteq K_4^-$ and $[x_4, x_5, z_1, z_2] \supseteq K_4^-$, a contradiction. Thus, $\{z_3, z_4\} \subseteq N_H(x_5) \cap N_H(x_1)$ and $\{z_1, z_2\} \subseteq N_H(x_6) \cap N_H(x_7)$. If $x_2z_4 \in E(G)$, then $[x_1, x_2, z_4, w] \supseteq K_4^-$, $[v_0, x_3, x_4, x_5] \supseteq K_4^-$ and $[z_1, z_2, x_6, x_7] \supseteq K_4^-$, a contradiction. Thus, $x_2z_4 \notin E(G)$ and $x_2z_3 \notin E(G)$. Then $[w, z_3, z_4, x_1] \supseteq K_4^-$ since $d_{G'}(x_1) + d_{G'}(x_2) + d_{G'}(w) \geq 31$. Since $[v_0, x_3, x_4, x_5] \supseteq K_4^-$ and $[z_1, z_2, x_6, x_7] \supseteq K_4^-$, a contradiction.

Suppose $l = 6$. Then for each $U \in V(G')$ with $|U| = 3$ such that $G[U \cup \{v_0\}] \supseteq K_4^-$, we have $\sum_{u \in U} d_{G'}(u) \geq 32$ by (1). Recall that G' can be obtained from K_{12} by removing at most 13 edges, this implies that H contains a cycle of length six. In this case, $|E(w, H)| \geq 2$ and we may assume that $x_1w, x_2w \in E(G)$. Furthermore, it is easy to see that there exist four vertices in $V(G' -$

$V(H)-\{w\})$, say z_1, z_2, z_3, z_4, such that $z_ix_3, z_ix_4, z_ix_5, z_ix_6 \in E(G)$ for each $i \in \{1,2,3,4\}$. Thus, $[V(G') - \{x_1, x_2, w\}] \supseteq 2K_4^-$, since $[v_0, x_1, x_2, w] \supseteq K_4^-$, a contradiction.

Case 2 $G[U \cup \{v_0\}] \not\supseteq K_4^-$ *for each* $U \in V(G')$ *with* $|U| = 3$.

In this case, $[N_G(v_0)] \not\supseteq P_3$, then $[N_G(v_0)]$ contains at most $\lfloor \frac{l}{2} \rfloor$ edges and so at least $\binom{l}{2} - \lfloor \frac{l}{2} \rfloor$ edges does not exist in $[N_G(v_0)]$. Then

$$\binom{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor \leq 14, \tag{2}$$

it follows from (2) that $l \leq 6$.

Furthermore, Let $x_1, x_2 \in N_G(v_0)$ such that $x_1x_2 \in E(G)$, then x_1 and x_2 has no common neighbor in $V(G')$, that is,

$$d_{G'}(x_1) + d_{G'}(x_2) \leq 12.$$

That is, x_1 and x_2 are incident to at most 11 edges and so there are at least 10 edges removed from x_1 and x_2. Therefore, the lost numbers of x_1 and x_2 are at least

$$(10 - l + 2) + \binom{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor. \tag{3}$$

If $l = 6$, then $[N_G(v_0)]$ is empty, for otherwise, let $x_1, x_2 \in N_G(v_0)$ with $x_1x_2 \in E(G)$, then by (3), $(10 - l + 2) + \binom{6}{2} - \lfloor \frac{6}{2} \rfloor \leq 14$, a contradiction. However, then $[N_G(v_0)]$ misses 15 edges, which contradicts the fact $l + 7 \leq 14$. If $l = 5$, it follows from (3) that $[N_G(v_0)]$ is empty, thus, $G' - N_G(v_0)$ has at least 9 edges, thus, $G' - N_G(v_0) \supseteq P_7$, as at most one vertex in $N_G(v_0)$ misses at most four edges, there exist two vertices in $N_G(v_0)$ and two vertices in $G' - N_G(v_0)$, such that these four vertices forms a subgraphs $G^* \supseteq K_4^-$, however, $|E(G - G^*)| \geq 59 - (11 + 10 + 6 + 6) = 26$, by Theorem 1.7, $G - G^*$ contains two disjoint theta graphs, so G contains three disjoint theta graphs, a contradiction. If $l = 4$, it follows from (3) that $[N_G(v_0)]$ contains at most one edge, and so it is easy to find a subgraph G^* of order 4 such that $G^* \supseteq K_4^-$, such that $|E(G - G^*)| \geq 25$, a contradiction. The case $l = 3$ is similar. This proves Lemma 2.2. □

Similarly, by applying Lemma 2.3 and the arguments likewise in proof of Lemma 2.2, we can show the following lemma.

Lemma 2.3 *Let G be a graph of order 14 and size at least 61, then G contains three disjoint theta graphs.*

3 Proof of Theorem 1.8

If $n = 12, 13, 14$, then Lemmas 2.1, 2.2 and 2.3 give us the required conclusion. Hence, $n \geq 15$. Since

$$\begin{aligned} \lfloor \frac{11n-33}{2} \rfloor &\geq \frac{11n-34}{2} \\ &\geq \frac{3n+80}{2} \\ &\geq \lceil \frac{3n+79}{2} \rceil, \end{aligned}$$

it is sufficient to prove that every graph of order $n \geq 15$ and size at least $\lfloor \frac{11n-33}{2} \rfloor$ contains three disjoint theta graph. We employ induction on n.

Assume that for all integers k with $14 \leq k < n$, every graph of order k and size at least $\lfloor \frac{11k-33}{2} \rfloor$ contains three disjoint theta graphs. In the following proof, we always let G be any graph of order n and size at least $\lfloor \frac{11n-33}{2} \rfloor$. By way of contradiction, we suppose that

$$G \text{ does not contain three disjoint theta graphs.} \tag{4}$$

Claim 3.1 $6 \leq \delta(G) \leq 8$.

Proof By Theorem 1.3, we have $\delta(G) \leq 8$. Suppose that $\delta(G) \leq 5$ and let $v_0 \in V(G)$ such that $d_G(v_0) = \delta(G)$. The graph $G - v_0$ is of order $n-1$ and size $\lfloor \frac{11n-33}{2} \rfloor - d_G(v_0) \geq \lfloor \frac{11n-33}{2} \rfloor - 5 \geq \frac{11n-34-10}{2} = \frac{11(n-1)-33}{2} \geq \left\lfloor \frac{11(n-1)-33}{2} \right\rfloor$, by induction hypothesis, $G - v_0$ contains three disjoint theta graphs, and so does G, which contradicts (1). Therefore, $\delta(G) \geq 6$. □

Let v_0 be a vertex in G such that $d_G(v_0) = \delta(G)$. In what follows, we always assume that $N_G(v_0) = \{v_1, ..., v_l\}$ and $H = [v_1, ..., v_l]$, where $l = d_G(v_0)$. By Claim 3.1, $6 \leq l \leq 8$. If $l = 6$, then let $\varepsilon_l = 1$; if $l = 7$, then let $\varepsilon_l = 2$; if $l = 8$, then let $\varepsilon_l = 3$. Note that $l = 5 + \varepsilon_l$.

Claim 3.2 *For each* $1 \leq i \leq l$, $d_H(v_i) \geq l - \varepsilon_l$.

Proof Suppose that there exists $1 \leq i \leq l$ such that $d_H(v_i) \leq l - \varepsilon_l - 1 = (l-1) - \varepsilon_l$. Without loss of generality, we may assume that $i = l$, and we may also assume that $v_j v_l \notin E(G)$ for each $1 \leq j \leq \varepsilon_l$ (otherwise, we can relabel the index of $V(H)$). Define the edge set $X = \{v_j v_l : 1 \leq j \leq \varepsilon_l\}$ and construct the graph $G' = (G - v_0) + X$, which is a graph with order $n-1$ and $|E(G')| = \lfloor \frac{11n-33}{2} \rfloor - l + \varepsilon_l \geq \frac{11n-34}{2} - l + \varepsilon_l = \frac{11(n-1)-33}{2} \geq \lfloor \frac{11(n-1)-33}{2} \rfloor$, because of $l = 5 + \varepsilon_l$. By induction hypothesis, G' contains three disjoint theta graphs, say T_1, T_2 and T_3, respectively. Clearly, at least two of them, say T_1 and T_2, do not contain vertex v_l, since T_1, T_2 and T_3 are disjoint theta graphs, then $E(T_1) \cap X = \emptyset$, $E(T_2) \cap X = \emptyset$ and by (1), $E(T_3) \cap X \neq \emptyset$.

Suppose that $|E(T_3) \cap X| = 1$, we may assume that $E(T_3) \cap X = \{v_l v_1\}$. Then $T_3{}' = (T_3 - \{v_l v_1\}) + \{v_1 v_0, v_l v_0\}$ is a theta graph in G, T_1,T_2 and $T_3{}'$ are disjoint in G, which contradicts (1). Therefore, it remains the case $E(T_3) \cap X = \{v_1 v_l, v_2 v_l\}$ or $E(T_3) \cap X = \{v_1 v_l, v_2 v_l, v_3 v_l\}$, as $\varepsilon_l \leq 3$. Let

$$T_3' = \begin{cases} (T_3 - \{v_1 v_l, v_2 v_l\}) + \{v_0 v_1, v_0 v_2\}, & \text{if } d_{T_3}(v_l) = 2 \\ (T_3 - \{v_1 v_l, v_2 v_l\}) + \{v_0 v_1, v_0 v_2, v_0 v_3\}, & \text{else } E(T_3) \cap X = \{v_1 v_l, v_2 v_l\} \\ (T_3 - \{v_1 v_l, v_2 v_l, v_3 v_l\}) + \{v_0 v_1, v_0 v_2, v_0 v_3\}, & \text{else.} \end{cases}$$

It is obvious that T_1, T_2 and $T_3{}'$ are three disjoint theta graphs in G, which contradicts (1). □

By Claim 3.2, Theorem 1.5 and the definition of ε_l, when $7 \leq l \leq 8$, for each subset S of $V(H)$ with $|S| \geq 7$, we obtain

$$[\{v_0\} \cup S] \supseteq 2K_4^-. \tag{5}$$

In particular, if $l = 6$, then

$$[\{v_0\} \cup V(H)] \cong K_7. \tag{6}$$

We take a vertex $v \in V(G - H - \{v_0\})$ such that $|E(v, V(H))|$ is maximum and fix it. When $l = 6$, by (3) and the definition of v, denote $W = V(H) \cup \{v\}$, we claim that

$$[\{v_0\} \cup W] \supseteq 2K_4^-. \tag{7}$$

Proof By way of contradiction, suppose that $[\{v_0\} \cup W]$ does not contain two disjoint K_4^-. By (6) and the assumption that $[\{v_0\} \cup W] \not\supseteq 2K_4^-$, for each $w \in V(G - \{v_0\} - V(H))$, there is at most one edge between w and $V(H)$. If $n = 13$, then $55 \leq |E(G)| \leq \frac{7\times 6}{2} + 6 + \frac{6\times 5}{2} = 42$, a contradiction. If $n = 15$, then $66 \leq |E(G)| \leq \frac{7\times 6}{2} + 8 + \frac{8\times 7}{2} = 57$, a contradiction. If $n = 16$, then $71 \leq |E(G)| \leq \frac{7\times 6}{2} + 9 + \frac{9\times 8}{2} = 66$, a contradiction. Therefore, we see that $n \geq 17$. By Theorem 1.7, we have

$$\begin{aligned}|E(G)| &= |E(G-\{v_0\}-V(H))|+21+(n-7)\\ &< \lfloor\frac{7(n-7)-13}{2}\rfloor+n+14\\ &\le \frac{9n-34}{2}\\ &< \frac{11n-34}{2}\\ &\le \lfloor\frac{11n-33}{2}\rfloor,\end{aligned}$$

this is an obvious contradiction. □

Let

$$G^* = \begin{cases} G-(\{v_0\}\cup V(H)), & \text{if } 7\le l\le 8\\ G-(\{v_0,v\}\cup V(H)), & \text{if } l=6.\end{cases}$$

Let F^* be the set of components of G^*. By (5) and (7), it follows from (4) that every graph in F^* contains no theta graph. In the following proof, let F denote arbitrary component in F^*, then, each block of F is either a K_2 or a cycle. Likewise in the same proof in [8], the following Claim 3.3 is obvious.

Claim 3.3 $|V(F)|\le 2$ *for each* $F\in F^*$.

Claim 3.4 *For each graph* $F\in\mathcal{F}$ *such that* $|V(F)|=2$, *there exists* $S\subset V(H)$ *with* $|S|=2$ *and* $[V(F)\cup S]\supseteq K_4^-$.

Proof Let $F\in\mathcal{F}$ such that $|V(F)|=2$, label $V(F)=\{u_1,u_2\}$. Since $|E(u_i,V(H))|\ge l-1$ if $7\le l\le 8$ and $|E(u_i,V(H)\cup\{v\})|\ge l-1$ for each i with $1\le i\le 2$ if $l=2$, it follows from the pigeonhole principle that there exists a subset $S\subset V(H)$ with $|S|=2$ and $S\subseteq N_H(u_1)\cap N_H(u_2)$. By (6), we know $[V(F)\cup S]\supseteq K_4^-$. □

Claim 3.5 *For any* $u\in V(G^*)$, $|E(u,\{v_0\}\cup V(H))|=|E(u,V(H))|\le l-1$ *if* $7\le l\le 8$; $|E(u,V(H)\cup\{v\})|\le l$ *if* $l=6$.

Proof Suppose that there exists $u\in V(G^*)$ such that $|E(u,V(H))|\ge l$ if $7\le l\le 8$, and $|E(u,V(H)\cup\{v\})|\ge l+1$ if $l=6$. By Claim 3.3, we may assume that F^* contains two components F_1 and F_2 with $|V(F_i)|\le 2$ for each $1\le i\le 2$, such that $u\in V(F_1)$. Suppose that $|V(F_2)|=2$ and label $F_2=u_2u_3$. Note that $|E(u_i,V(H))|\ge l-1$ for each $i\in\{2,3\}$. By Claim 3.4, there exists $v_i,v_j\in V(H)$ such that $[u_2,u_3,v_i,v_j]\supseteq K_4^-$. If $7\le l\le 8$, combining with (5) and (6), $[V(H-\{v_i,v_j\})\cup\{u,v_0\}]\supseteq 2K_4^-$, which contradicts (4). Therefore, $l=6$. By the choice of v, $|E(v,V(H))|=6$. Since $F^*\setminus(F_1\cup F_2)\neq\emptyset$, choose $u_4\in V(F^*\setminus(F_1\cup F_2))$. By Claim 3.3, $|E(u_4,V(H))|\geq 4$, choose $\{v_p,v_q\}\subseteq N_H(u_4)\cap N_H(v)-\{v_i,v_j\}$ such that $p\neq q$. Now, $[v_p,v_q,u_4,v_0]\supseteq K_4^-$ and $[V(H-\{v_i,v_j,v_p,v_q\})\cup\{u,v\}]\supseteq K_4^-$, which contradicts (4). This shows the order of each component of $F^*\setminus F_1$ is one. Now, note that $|F^*\setminus F_1|\geq 3$, we can choose three different vertices u_1,u_2,u_3, such that $|E(u_i,V(H))|\geq 5$ for each $1\leq i\leq 3$. As above, it is obvious that $[V(H)\cup\{v,u,v_0,u_1,u_2,u_3\}]\supseteq 3K_4^-$, a contradiction. □

Claim 3.6 *For any $u\in V(G^*)$, $|E(u,V(H)\cup\{v\})|\leq l-1$ if $l=6$.*

Proof Suppose that there exists $u\in V(G^*)$ such that $|E(u,V(H)\cup\{v\})|\geq l$ if $l=6$. By Claim 3.3, $|F^*|\geq 4$ and let $F_1,\ldots,F_4$ denote four components in F^*, such that $|V(F_i)|\leq 2$ for each $1\leq i\leq 4$ and $u\in V(F_1)$. By the choice of v, $|E(v,V(H))|\geq 5$.

We show that the order of each component in F^* is two. On the contrary, without loss of generality, we may assume that $F_2=u_1$. By Claim 3.5, $|E(u_1,V(H)\cup\{v\})|=6$. Swap the role of u_1 and v_0, it follows from induction hypothesis and (6) that

$$[\{u_1\}\cup N_G(u_1)]\cong K_7. \tag{8}$$

In this section, we always choose $u_3\in V(F_3)$ and choose u_4 as follows: If $|F_3|=2$, then let $U_4=V(F_3)-\{u_3\}$; otherwise, let $u_4\in V(F_4)$. If $u_1v\notin E(G)$, then $V(H)=N_G(u_1)$ and so $V(H)\subseteq N_G(v)$ by the maximality of $|E(v,V(H))|$. Suppose that $|F_3|=2$, then by Claim 3.4, $[u_3,u_4,v_i,v_j]\supseteq K_4^-$ for some $v_i,v_j\in V(H)$. Since $[V(H)\cup\{v,u,u_1,v_0\}]\supseteq 2K_4^-$, then $G\supseteq 3K_4^-$, which contradicts (4). Thus, $F_3=u_3$ and similarly $F_4=u_4$, that is, $|E(u_3,V(H))|\geq 5$ and $|E(u_4,V(H))|\geq 5$, so $|N_H(u_3)\cap N_H(u_4)|\geq 4$. Thus, there exists $\mathcal{X}\in V(H)$ with $|\mathcal{X}|=4$ such that $[\mathcal{X}\cup\{u,u_1,u_3,u_4\}]\supseteq 2K_4^-$, since $[V(H\setminus\mathcal{X})\cup\{v,v_0\}]\supseteq K_4^-$, a contradiction. This show that $u_1v\in E(G)$ and we may assume that $u_1v_i\in E(G)$ for each $i\in\{1,2,\ldots,5\}$. By (8), $vv_i\in E(G)$ for each $i\in\{1,2,\ldots,5\}$. Swap the role of u_1 and u, by the similar arguments as above, we see that $uv\in E(G)$. Suppose that $|F_3|=2$ and so $F_3=u_3u_4$ by our choice. Then by Claim 3.4, there exist $v_i,v_j\in V(H)$ such that $[v_i,v_j,u_3,u_4]\supseteq K_4^-$. As $|E(u,V(H))|\geq 5$, we choose $v_k\in V(H)-\{v_6,v_i,v_j\}$ such that $uv_k\in E(G)$, then $[v,u,u_1,v_k]\supseteq K_4^-$, furthermore, as $[\{v_0\}\cup V(H)\setminus\{v_i,v_j,v_k\}]\supseteq K_4^-$, which contradicts (4). Thus, $F_3=u_3$ and so $|E(u_3,V(H)\cup\{v\})|=6$ by Claim 3.5. Similarly, we can show that $F_1=u$ and $F_4=u_4$. Swap the role of u_1 and u_3, we have $u_3v\in E(G)$ and similarly $u_4v\in E(G)$. Since $|N_H(u_3)\cap N_H(u_4)|\geq 4$, we choose $v_i\in N_H(u_3)\cap N_H(u_4)$ with $1\leq i\leq 5$, such that $u_3v_i,u_4v_i\in E(G)$, that is, $[u_3,u_4,v,v_i]\supseteq K_4^-$. Since $|N_H(u)\cap N_H(u_1)|\geq 4$, we can choose $v_j,v_k\in N_H(u)\cap N_H(u_1)\setminus\{v_i,v_6\}$, then

$[v_j, v_k, u, u_1] \supseteq K_4^-$, as $[\{v_0\} \cup V(H) \setminus \{v_i, v_j, v_k\}] \supseteq K_4^-$, which contradicts (4). This shows that the order of each component in F^* is two, as required.

Label $F_i = u_i u_i'$ for each $1 \leq i \leq 4$ and $u_1 = u$, then $|E(u_i, V(H))| \geq 4$ and $|E(u_i', V(H))| \geq 4$. Suppose that $u_1 v \notin E(G)$. Then $V(H) \subseteq N_G(u)$ and so $|E(v, V(H))| = 6$. In this situation, by Claim 3.4, it is easy to check that $[V(H) \cup V(F_1) \cup V(F_2) \cup \{v, v_0\}] \supseteq 3K_4^-$, a contradiction. Thus, $u_1 v \in E(G)$ and so $|E(u_1, V(H))| \geq 5$, $|E(v, V(H))| \geq 5$. By symmetry, say $\{v_2, v_3, v_4, v_5\} \subseteq N_H(v) \cap N_H(u_1)$ and $vv_1 \in E(G)$. By Claim 3.4, there exist $v_i, v_j \in V(H)$, such that $[v_i, v_j, u_2, u_2'] \supseteq K_4^-$, furthermore, since $|N_H(v) \cap N_H(u_1')| \geq 3$, we can choose $v_k \in \{v_2, v_3, v_4, v_5\} \setminus \{v_i, v_j\}$, such that $u_1' v_k \in E(G)$, thus, $[v, u_1, u_1', v_k] \supseteq K_4^-$, as $[\{v_0\} \cup V(H) \setminus \{v_i, v_j, v_k\}] \supseteq K_4^-$, which contradicts (4). This proves Claim 3.6 □

Now we are in the position to complete the proof of Theorem 1.8. By Claim 3.3 and Claim 3.6, $|V(F)| = 2$ for all $F \in F^*$, we have

$$\sum_{F \in F^*} |E(F)| = \begin{cases} \frac{n-1-l}{2}, & \text{if } 7 \leq l \leq 8 \\ \frac{n-8}{2}, & \text{if } l = 6. \end{cases}$$

Suppose that $7 \leq l \leq 8$. We may assume that $u_1 u_2$ and $u_3 u_4$ are two component of G^*, since $|E(u_i, V(H))| \geq l - 1$, by Claim 3.2, it is obvious that $[V(H) \cup \{v_0, u_1, u_2, u_3, u_4\}] \supseteq 3K_4^-$, a contradiction. Thus, $l = 6$, and according to Claim 3.6, we obtain

$$\begin{aligned} |E(G)| &= |E([\{v_0, v\} \cup V(H)])| + |E(V(G^*), \{v_0, v\} \cup V(H))| + \sum_{F \in F^*} |E(F)| \\ &\leq 27 + 5|V(G^*)| + \sum_{F \in F^*} |E(F)| \\ &= 27 + 5(n-8) + \frac{n-8}{2} \\ &= \frac{11n - 34}{2}, \end{aligned} \tag{9}$$

since $|E(G)| \geq \lfloor \frac{11n-33}{2} \rfloor \geq \frac{11n-34}{2}$, thus, the equality in (9) holds each place. That is, for any $u \in V(G^*)$, $|E(u, V(H) \cup \{v\})| = 5$. Let $F_i = u_i u_i'$ denote the component of F^*, where $i \geq 1$.

Suppose that $u_1 v \notin E(G)$. Then $|E(u_1, V(H))| = 5$ and we may assume that $v_1, v_2, \ldots, v_5$ are the neighbors of u_1. By the induction hypothesis and (6), $N(u_1', V(H) \cup \{v\}) = V(H) - \{v_6\}$. If $u_2 v \in E(G)$, then by the induction hypothesis and (6) again, $u_2' v \in E(G)$. Then we can find a common neighbor of v, u_2 and u_2' in $V(H)$, denoted by v_k. Since $[v, u_2, u_2', v_k] \supseteq K_4^-$ and there exist $v_i, v_j \in N_H(u_1) \cap N_H(u_1')$ such that $i \neq k$ and $j \neq k$, then $[v_i, v_j, u_1, u_1'] \supseteq K_4^-$ and $[\{v_0\} \cup V(H) \setminus \{v_i, v_j, v_k\}] \supseteq K_4^-$, which contradicts (4). Thus, $u_2 v \notin E(G)$

and $u_2'v \notin E(G)$ by symmetry. Similarly, for each $F_i \in F^*$, $u_iv \notin E(G)$ and $u_i'v \notin E(G)$. Thus, $N_G(v) = V(H)$ and by Claim 3.4, there exist $v_i, v_j \in N_H(u_2) \cap N_H(u_2')$ such that $[v_i, v_j, u_2, u_2'] \supseteq K_4^-$. Since we can choose two distinct vertex in $\{v_1, v_2, v_3, v_4, v_5\} \setminus \{v_i, v_j\}$, say v_k, v_f, such that $[v_k, v_f, u_1, u_1'] \supseteq K_4^-$, thus, $G \supseteq 3K_4^-$, because of $[\{v_0, v\} \cup V(H) \setminus \{v_i, v_j, v_k, v_f\}] \supseteq K_4^-$, this is a contradiction. This proves that $u_1v \in E(G)$ and by symmetry, $u_iv, u_i'v \in E(G)$ for each $i \geq 1$.

By Claim 3.4, we may assume that $[v_2, v_3, u_1, u_1'] \supseteq K_4^-$. If v and u_2' has one common neighbor in $V(H) \setminus \{v_2, v_3\}$, without loss of generality, say v_1, then $[v, v_1, u_2, u_2'] \supseteq K_4^-$, since $[v_0, v_4, v_5, v_6] \supseteq K_4^-$, a contradiction, thus, v and u_2' has no common neighbor in $V(H) \setminus \{v_2, v_3\}$. By symmetry, v and u_i' has no common neighbor in $V(H) \setminus \{v_2, v_3\}$ for each $2 \leq i \leq 4$, and so does v and u_i for $2 \leq i \leq 4$. This implies that $v_2v, v_3v \in E(G)$ and also

$$|E(v, V(H))| = 4, |E(u_2, V(H))| = 4 \text{ and } |E(u_2', V(H))| = 4.$$

Without loss of generality, say $vv_1, vv_4 \in E(G)$. Then, $\{v_5, v_6\} \subseteq N_H(u_2) \cap N_H(u_2')$ and so $[v_5, v_6, u_2, u_2'] \supseteq K_4^-$. Note that $[v_0, v, v_1, v_4] \supseteq K_4^-$, then $G \supseteq 3K_4^-$, which contradicts (4). This proves Theorem 1.8. □

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