List vertex arboricity of planar graphs with 5-cycles not adjacent to 3-cycles and 4-cycles *

Ling Xue

Department of Information Engineering, Taishan Polytechnic,

Tai'an, 271000, China

Abstract

A graph G is list k-arborable if for any sets $L(v)$ of cardinality at least k at its vertices, one can choose an element (color) for each vertex v from its list $L(v)$ so that the subgraph induced by every color class is an acyclic graph (a forest). In the paper, it is proved that every planar graph with 5-cycles not adjacent to 3-cycles and 4-cycles is list 2-arborable.

Key words: planar graph; cycle; vertex arboricity; arborable; list coloring

1 Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [1] for terminologies and notations not defined here. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v and let $d(v) = |N(v)|$ denote the degree of v. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. A *k-vertex*, *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. If a vertex v is adjacent to a d-vertex u, we say that u is a *d-neighbor* of v. We denote by $n_d(v)$ the number of d-neighbors of v. A *k-cycle* is a cycle of length k.

Let G be a plane graph. Denote by F or $F(G)$ the face set of G. For a face $f \in F$, the *degree* $d(f)$ of f is the length of the boundary walk of f. A *k-face*, *k^--face* or a *k^+-face* is a face of degree k, at most k or at least k, respectively. For convenience, a k-face $f = (v_1, v_2, \cdots, v_k)$ with

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consecutive vertices $v_1, v_2, \cdots, v_k$ along its boundary in the clockwise order is often said to be a $(d(v_1), d(v_2), \cdots, d(v_k))$-face. For a face f, let $n_i(f)$ and $n_{i^+}(f)$ denote the number of i-vertices and i^+-vertices incident with f, respectively. Denote by $f_d(v)$ and $f_{d^+}(v)$ the number of d-faces and d^+-faces incident with v, respectively. We say that two cycles (or faces) are *intersecting* if they share at least one common vertex or *adjacent* if they share at least one common edge.

A *forest k-coloring* of a graph G is a mapping ϕ from the vertex set $V(G)$ to the set $\{1, 2, \cdots, k\}$ such that each color class induces an acyclic subgraph, i.e., a forest. The *vertex arboricity* $va(G)$ of G is the smallest integer k such that G has a forest k-coloring. This version of vertex arboricity was first introduced by Chartrand et al. [5] in 1968, who named it point-arboricity. They proved that $va(G) \leq \lceil \frac{1+\Delta(G)}{2} \rceil$ for any graph G in [5] and $va(G) \leq 3$ for any planar graph in [6] . A graph G is called *d-degenerate* if every subgraph H of G contains a vertex of degree at most d. It is easy to see that $va(G) \leq \lfloor (d+1)/2 \rfloor$ for any d-degenerate graph G. So $va(G) \leq \lceil \frac{1+\Delta(G)}{2} \rceil$ for any graph G. Since every planar graph has a vertex of degree at most 5, $va(G) \leq 3$ for any planar graph G. It is well known that every planar graph without 3-cycles is 3-degenerate. It was shown in [13] that every planar graph without 5-cycles is 3-degenerate and in [8] that every planar graph without 6-cycles is 3-degenerate. These facts imply that $va(G) \leq 2$ if G is a planar graph without 3-,5- or 6-cycles. Raspaud et al. [12] proved that every planar graph G without 4-cycles has $va(G) \leq 2$ and Huang et al. [9] further proved that every planar graph G without 7-cycles has $va(G) \leq 2$. Raspaud et al. [12] also proved that $va(G) \leq 2$ if G is a planar graph such that any two triangles of G are at distance at least 3. It was shown in [10] that every planar graph G without chordal 6-cycles has $va(G) \leq 2$. Chen et al. [7] proved that $va(G) \leq 2$ if G is a planar graph without intersecting triangles. Cai and Wu [4] proved that $va(G) \leq 2$ if G is a planar graph without intersecting 5-cycles.

We say that L is an assignment for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has a forest k-coloring ϕ such that $|L(v)| \geq k$ and $\phi(v) \in L(v)$ for any vertex v, then we say that G is *forest L-colorable* or ϕ is a *forest L-coloring* of G. The graph G is *list k-arborable* if it is forest L-colorable for every assignment L satisfying $|L(v)| \geq k$ for any vertex v. The *list vertex arboricity* $va_{list}(G)$ of G is the smallest k such that G is list k-arborable. We also have that $va_{list}(G) \leq \lfloor (d+1)/2 \rfloor$ for any d-degenerate graph G, $va_{list}(G) \leq 3$ for any planar graph G and $va_{list}(G) \leq 2$ for any planar graph G without 3-,5- or 6-cycles. Borodin and Ivanova [2] proved that every planar graph with no triangles at distance less than two is list 2-arborable, and later they [3] proved that every planar graph without 4-cycles adjacent to 3-cycles is list 2-arborable. This paper prove that planar graphs without 5-cycles adjacent to 3-cycles and 4-cycles are list 2-arborable.

2 Main result and its proof

Theorem 1. *If G is a planar graph with 5-cycles not adjacent to 3-cycles and 4-cycles, then $va_{list}(G) \leq 2$.*

Proof. Suppose, to the contrary, that Theorem 1 is false. Let G be a counterexample to Theorem 1 with the fewest vertices. Then

(1) $\delta(G) \geq 4$ (see [2]).

(2) G does not contain a 6-cycle $(v_1, v_2, \cdots, v_6)$ such that $v_2v_6 \in E(G)$ and $d(v_i) = 4$ for every $i \in \{1, 2, \cdots, 6\}$. (see [3]).

By the Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0$$

We define ch to be the initial charge by letting $ch(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} ch(x) < 0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum. If we can show that $ch'(x) \geq 0$ for each $x \in V \cup F$, then we get an obvious contradiction $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) < 0$, which completes our proof.

Let $w(x \rightarrow y)$ be the charge transferred from x to y for all $x, y \in V \cup F$. We define the discharging rules as follows.

R1. *Let f be a 3-face (u, v, w) of G. If f is not adjacent to a 3-face, then f receives $\frac{1}{3}$ from each of its adjacent 5^+-faces; Otherwise, without loss of generality, assume that uv is incident with two 3-faces and $d(u) \leq d(v)$. If $d(u) = d(v) = 4$, then f receives $\frac{1}{2}$ from each of its adjacent 6^+-faces; Otherwise, f receives $\frac{1}{3}$ from each of its adjacent 6^+-faces and $\frac{1}{3}$ from v.*

R2. *Let f be a 5-face $(v_1, v_2, \cdots, v_5)$ of G and f_i be the another face incident with v_iv_{i+1} for $i \in \{1, 2, \cdots, 5\}$, where all the subscripts here are taken modulo 5.*

R2.1. *Suppose that for any $i (1 \leq i \leq 5)$, f_i is a 3-face (v_i, v_{i+1}, u_i). If $n_4(f) = 5$, that is, f is a $(4, 4, 4, 4, 4)$-face, then f receives $\frac{1}{6}$ from u_i for any $i (1 \leq i \leq 5)$; Otherwise, f receives $2/(3n_{5^+}(f))$ from each of 5^+-vertices incident with f.*

R2.2. *Suppose that f is adjacent to four 3-faces, without loss of generality, f_i is a 3-face (v_i, v_{i+1}, u_i) of G, where $i = 1, 2, 3, 4$. If $n_4(f) = 5$, then f receives $\frac{1}{6}$ from u_i for any $i(2 \le i \le 4)$; Otherwise, f receives $1/(3n_{5^+}(f))$ from each of 5^+-vertices incident with f.*

In the following, we will check that $ch'(x) \ge 0$ for each $x \in V \cup F$.

Claim 1. *Let $f \in F(G)$. Then $ch'(f) \ge 0$.*

Suppose that $d(f) = 3$. Note that if f is adjacent to another 3-face, then f must be adjacent to two 6^+-faces since every 5-cycle of G is not adjacent to 3-cycles and 4-cycles at the same time. So $ch'(f) \ge ch(f) + \max\{\frac{1}{2}\times 2, \frac{1}{3}\times 3\} = 0$ by R1. If $d(f) = 4$, then $ch'(f) = ch(f) = 0$. Suppose $d(f) = 5$. Note that if f is adjacent to a 3-face f', then f is not adjacent to a 4-cycle and it follows that all faces incident with f' must be 5^+-faces. If f is adjacent to at most three 3-cycles, then $ch'(f) \ge ch(f) - \frac{1}{3} \times 3 = 0$ by R1; Otherwise, $ch'(f) \ge ch(f) + \min\{\frac{1}{6} \times 5 - \frac{1}{3} \times 5, \frac{2}{3n_{5^+}(f)} \times n_{5^+}(f) - \frac{1}{3} \times 5, \frac{1}{6} \times 3 - \frac{1}{3} \times 4, \frac{1}{3n_{5^+}(f)} \times n_{5^+}(f) - \frac{1}{3} \times 4\} = 0$ by R2. Suppose that f is a k-face $(v_1, v_2, \cdots, v_k)$, where $k \ge 6$. We denote by f_i the face adjacent to f and incident with v_iv_{i+1} where all the subscripts are taken modulo k. If $w(f \to f_i) = \frac{1}{2}$, then $d(v_i) = d(v_{i+1}) = 4$ and f_{i-1}(or f_{i+1}) must be a 6^+-face since every 5-cycle of G is not adjacent to 3-cycles or 4-cycles, and this can be equivalent to say that f sends $\frac{1}{3}$ to f_i and $\frac{1}{6}$ to f_{i-1}(or f_{i+1}, respectively). According to this averaging, every f_i receive at most $\frac{1}{3}$ from f. So $ch'(f) \ge ch(f) - \frac{1}{3} \times d(f) \ge 0$.

Claim 2. *Let $v \in V(G)$. Then $ch'(v) \ge 0$.*

If $d(v) = 4$, then $ch'(v) = ch(v) = 0$ by R1 and R2. Suppose $d(v) = k \ge 5$. Let $N(v) = \{v_1, \cdots, v_k\}$ and $f_1, f_2, \cdots, f_k$ be faces incident with v such that f_i is incident with v_i and v_{i+1}, for $i \in \{1, 2, \cdots, k\}$, where all the subscripts here are taken modulo k.

Suppose that $k = 5$. Then $f_3(v) \le 3$, that is, v is incident with at most three 3-faces. If $f_3(v) = 3$, then v is incident with two 6^+-faces, and it follows from R1 and R2 that $ch'(v) \ge ch(v) - \frac{1}{3} \times 2 - \frac{1}{6} > 0$. If $f_3(v) \le 1$, then we also have $ch'(v) \ge ch(v) - \frac{1}{3} \times 2 - \frac{1}{6} > 0$ by R1 and R2.2. So we assume that $f_3(v) = 2$. If f_i and f_{i+1} are two 3-faces for some $i \in \{1, 2, \cdots, 5\}$, then $ch'(v) \ge ch(v) - \frac{1}{3} \times 2 > 0$ by R1; Otherwise, without loss of generality, assume that f_1 and f_3 are the two 3-faces. We denote a 5-face f by 5^k-*face* if f is a $(5, 4, 4, 4, 4)$-face and adjacent to k 3-faces, where $k \ge 4$. If f_2 is a 5^5-face or f_5 is 5^4-face, then the 5^+-face f_{12} incident with v_1v_2 must not be a $(4, 4, 4, 4, 4)$-face for $k \in \{4, 5\}$. This means that if $w(v \to f_2) = \frac{1}{2}$ or $w(v \to f_5) = \frac{1}{3}$, then $w(v \to f_{12}) = 0$. Similarly, if f_2 is a 5^5-face or f_4 is 5^4-face, then the 5^+-face incident with v_3v_4 must not be a $(4,4,4,4,4)$-face. At the same time, at most one in $\{f_4, f_5\}$ is a 5^4-face. So $ch'(v) \geq ch(v) - \max\{\frac{2}{3} + \frac{1}{3}, \frac{1}{3} \times 2 + \frac{1}{6} \times 2\} = 0$.

Suppose $k \geq 6$. By R2.1, if $w(v \to f_i) = \frac{2}{3}$ for some $i(1 \leq i \leq k)$, then f_{i-1}, f_{i+1} are 3-faces and $w(v \to f_{i-1}) = w(v \to f_{i+1}) = 0$, that can be equivalent to say that v sends $\frac{1}{3}$ to f_i, $\frac{1}{6}$ to f_{i-1} and $\frac{1}{6}$ to f_{i+1}. Every charge of $\frac{1}{6}$ by v to a 5^+-face incident with $v_i v_{i+1}$ can be looked at as giving $\frac{1}{6}$ to f_i. According to this averaging, every face receive at most $\frac{1}{3}$ from v. So $ch'(v) \geq ch(v) - d(v) \times \frac{1}{3} \geq 0$.

Hence we complete the proof of the theorem. □

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