# On the half of a Riordan array

Sheng-Liang Yang, Yan-Xue Xu, Xiao Gao

Department of Applied Mathematics

Lanzhou University of Technology

Lanzhou, 730050, Gansu, PR China

## Abstract

The half of an infinite lower triangular matrix $G = (g_{n,k})_{n,k\geq 0}$ is defined to be the infinite lower triangular matrix $G^{(1)} = \left(g_{n,k}^{(1)}\right)_{n,k\geq 0}$ such that $g_{n,k}^{(1)} = g_{2n-k,n}$ for all $n \geq k \geq 0$. In this paper, we will show that if $G$ is a Riordan array then its half $G^{(1)}$ is also a Riordan array. We use Lagrange inversion theorem to characterize the generating functions of $G^{(1)}$ in terms of the generating functions of $G$. Consequently, a tight relation between $G^{(1)}$ and the initial array $G$ is given, hence it is possible to invert the process and rebuild the original Riordan array $G$ from the array $G^{(1)}$. If the process of taking half of a Riordan array $G$ is iterated $r$ times, then we obtain a Riordan array $G^{(r)}$. The further relation between the result array $G^{(r)}$ and the initial array $G$ is also considered. Some examples and applications are presented.

**Keywords:** Riordan array; central coefficient; Catalan number; Generating function; generalized binomial series

**AMS Classification:** 05A05, 05A15, 05A10, 15A09

# 1 Introduction

The concept of Riordan array was first introduced in [18, 20] as a generalization of the Pascal triangle and Catalan triangle. It has been proved that Riordan arrays constitute a natural way of describing several combinatorial situations. Some of the main results on the Riordan arrays and their applications to combinatorial sums and identities can be found in [8, 9, 13, 21]. An infinite lower triangular array $D = (d_{n,k})_{n,k\geq 0}$ is called a Riordan array if its column $k$ has generating function $g(t)f(t)^k$, $k = 0, 1, 2, \cdots$, where $g(t)$ and $f(t)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. That is, the general term of array $D$ is given by

$$d_{n,k} = [t^n]g(t)f(t)^k, \tag{1}$$

where $[t^n]$ denotes the coefficient operator. The array corresponding to the pair $g(t)$, $f(t)$ is denoted by $(g(t), f(t))$. The set $\mathcal{R}$ of all Riordan arrays forms a group under ordinary matrix multiplication. The group law is then given by

$$(g(t), f(t))(d(t), h(t)) = (g(t)d(f(t)), h(f(t))). \tag{2}$$

The identity is $(1, t)$ and the inverse of $(g(t), f(t))$ is

$$(g(t), f(t))^{-1} = (1/g(\bar{f}(t)), \bar{f}(t)), \tag{3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, i.e., $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Many properties of Riordan arrays have been studied in the literature in recent years, in particular their connection with combinatorial sums. Actually if we multiply the array $D = (g(t), f(t))$ by a column vector $(b_0, b_1, b_2, \ldots)^T$ with generating function $b(t)$, then we get a column vector whose generating function is given by $g(t)b(f(t))$. If we identify a sequence with its generating function, the composition rule can be rewritten as

$$(g(t), f(t))b(t) = g(t)b(f(t)). \tag{4}$$

Important subgroups of Riordan group $\mathcal{R}$ are [12, 19]:

- Appell subgroup $\mathcal{R}_A$, the Riordan arrays are of form $(g(t), t)$;
- Lagrange subgroup $\mathcal{R}_L$, the Riordan arrays are of form $(1, f(t))$;
- Renewal or Bell subgroup $\mathcal{R}_N$, the Riordan arrays are of form $(g(t), tg(t))$;
- Derivative subgroup $\mathcal{R}_D$, the Riordan arrays are of form $(f'(t), f(t))$;
- Hitting-time subgroup $\mathcal{R}_H$, the Riordan arrays are of form $\left(\frac{tf'(t)}{f(t)}, f(t)\right)$;
- Generalized hitting-time subgroup $\mathcal{H}[r, s]$, the Riordan arrays are of form $\left(\left(\frac{f(t)}{t}\right)^r f'(t)^s, f(t)\right)$.

An important feature of Riordan arrays is that one can extracts new Riordan arrays from a given Riordan array. There are many related work can be found from the references [1–4, 24]. Particularly, given a Riordan array $D = (d_{n,k})_{n,k\geq 0}$, for any integers $p \geq 2$, $r \geq 0$, $\tilde{d}_{pn+r,(p-1)n+r+k}$, for $n, k \geq 0$, defines a new Riordan array [3]. In this paper, we first introduce the notion of half array of a Riordan array. Then, by iterating $r$ times, we obtain the $r$-half array of a Riordan array, for any positive integer $r$. In next section, for a Riordan array $G = (p(t), tq(t))$, we obtain an explicit representation for the $r$-half Riordan array $G^{(r)}$ in terms of a Riordan array $G$. Meanwhile, the $r$-half array of some classical examples, such as Pascal matrix, Catalan matrices, are investigated. In section 3, as applications, we consider enumeration of a kind of generalized Dyck paths by using Riordan arrays.

We begin by recalling the Lagrange Inversion Formula, which is an important element needed in our study. Several forms of the Lagrange Inversion Formula exist in literatures (see [15,22]). We summarize some of them below.

**Lemma 1.1**( LIF [6, 11, 22]). Let $w = w(t)$ be a formal power series which is implicitly defined by a functional equation $w = t\phi(w)$, where $\phi(t)$ is a formal power series such that $\phi(0) \neq 0$, and let $F(t)$ be any formal power series. Then we have
$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n = [t^n]F(t)\phi(t)^{n-1}(\phi(t) - t\phi'(t))$.

# 2 Half of Riordan array

Let $G = (g_{n,k})_{n,k\geq 0}$ be an infinite lower triangular matrix. By the central coefficients of this matrix we understand the terms $g_{2n,n}$. The method of obtaining a generating function for central coefficients of a given matrix is studied by several authors recently [1, 10, 23]. We define the half array of $G = (g_{n,k})_{n,k\geq 0}$ to be the lower triangular matrix $G^{(1)} = (g^{(1)}_{n,k})_{n,k\geq 0}$ such that $g^{(1)}_{n,k} = g_{2n-k,n}$ for all $n \geq k \geq 0$.

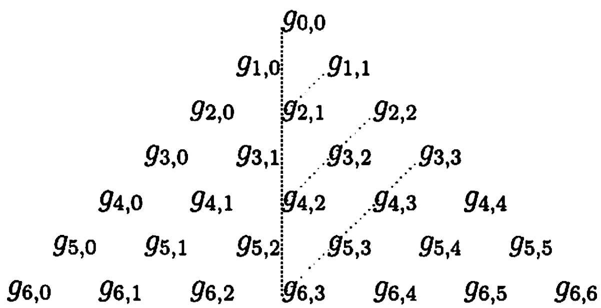


Figure 1: The isosceles triangle of matrix $G$

If the matrix $G$ is represented as in Figure 1, then the central coefficients $(g_{2n,n})_{n\geq 0}$ are located in the central column of the isosceles triangle, and $G^{(1)}$, the half array of $G$, is the right part of $G$, i.e.,

$$G^{(1)} = (g_{2n-k,n}) = \begin{pmatrix} g_{0,0} & 0 & 0 & 0 & 0 & \cdots \\ g_{2,1} & g_{1,1} & 0 & 0 & 0 & \cdots \\ g_{4,2} & g_{3,2} & g_{2,2} & 0 & 0 & \cdots \\ g_{6,3} & g_{5,3} & g_{4,3} & g_{3,3} & 0 & \cdots \\ g_{8,4} & g_{7,4} & g_{6,4} & g_{5,4} & g_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For a positive integer $r$, the $r$-half array of $G = (g_{n,k})_{n,k\geq 0}$ is defined as $G^{(r)} = (G^{(r-1)})^{(1)}$, here $G^{(0)} = G$, that is, $G^{(r)}$ is obtained by iterating the half process $r$ times. The generic element of $G^{(r)}$ is given by

$$g_{n,k}^{(r)} = g_{(r+1)n-rk,rn-(r-1)k}. \tag{5}$$

For example, the twice half array of $G = (g_{n,k})_{n,k\geq 0}$ is

$$G^{(2)} = (g_{3n-2k,2n-k}) = \begin{pmatrix} g_{0,0} & 0 & 0 & 0 & 0 & \cdots \\ g_{3,2} & g_{1,1} & 0 & 0 & 0 & \cdots \\ g_{6,4} & g_{4,3} & g_{2,2} & 0 & 0 & \cdots \\ g_{9,6} & g_{7,5} & g_{5,4} & g_{3,3} & 0 & \cdots \\ g_{12,8} & g_{10,7} & g_{8,6} & g_{6,5} & g_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

**Theorem 2.1.** Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $G^{(1)}$ denote the half array of $G$. Then $G^{(1)} = \left(\frac{tf'(t)p(f(t))}{f(t)}, f(t)\right)$, where $f(t)$ is the compositional inverse of $\frac{t}{q(t)}$. Consequently, $(G^{(1)})^{-1} = \left(\frac{th'(t)}{h(t)p(t)}, h(t)\right)$ with $h(t) = \frac{t}{q(t)}$.

**Proof.** Differentiating the both sides of the equation $f(t) = tq(f(t))$ we get $f'(t) = tq'(f(t))f'(t) + q(f(t))$, from which we have $f'(t) = \frac{q(f(t))}{1-tq'(f(t))} = \frac{f(t)}{t(1-tq'(f(t)))}$.

Considering the relation $f(t) = tq(f(t))$ and using Lemma 1.1, we have $[t^n]\frac{tf'(t)p(f(t))}{f(t)}f(t)^k = [t^n]\frac{p(f(t))}{1-tq'(f(t))}f(t)^k$

$= [t^n]\frac{p(f(t))q(f(t))f(t)^k}{q(f(t))-f(t)q'(f(t))} = [t^n]\frac{p(t)q(t)t^k}{q(t)-tq'(t)}q(t)^{n-1}(q(t)-tq'(t))$
$= [t^{n-k}]p(t)q(t)^n = [t^{2n-k}]p(t)(tq(t))^n = g_{2n-k,n}$. Since $\bar{f}(t) = \frac{t}{q(t)}$, we have $(G^{(1)})^{-1} = \left(\frac{q(t)-tq'(t)}{p(t)q(t)}, \frac{t}{q(t)}\right) = \left(\frac{th'(t)}{h(t)p(t)}, h(t)\right)$ with $h(t) = \frac{t}{q(t)}$. □

From this theorem, we can obtain the following results immediately.

**Corollary 2.2.** Let $G = (p(t), tq(t))$ and $G^{(1)}$ be the half array of $G$, and let $f(t)$ be the compositional inverse of $\frac{t}{q(t)}$. Then

(1) If $G \in \mathcal{R}_A$, i.e., $q(t) = 1$, then $G^{(1)} = G$.

(2) If $G \in \mathcal{R}_L$, i.e., $p(t) = 1$, then $G^{(1)} = \left(\frac{tf'(t)}{f(t)}, f(t)\right)$, hence $G^{(1)} \in \mathcal{R}_H$.

(3) If $G \in \mathcal{R}_N$, $p(t) = q(t)$, then $G^{(1)} = (f'(t), f(t))$, hence $G^{(1)} \in \mathcal{R}_D$.

(4) If $p(t) = \frac{q(t)-tq(t)'}{q(t)}$, then $G^{(1)} = (1, f(t))$, hence $G^{(1)} \in \mathcal{R}_L$.

(5) If $p(t) = q(t) - tq(t)'$, then $G^{(1)} = \left(\frac{f(t)}{t}, f(t)\right)$, hence $G^{(1)} \in \mathcal{R}_N$.

(6) If $p(t) = q(t)^{r+1}$, then $G^{(1)} = \left((\frac{f(t)}{t})^r f'(t), f(t)\right)$, hence $G^{(1)} \in \mathcal{H}[r, 1]$.

**Example 2.1.** Let $P = (p(t), tq(t)) = (\frac{1}{1-t}, \frac{t}{1-t})$ be the well-known Pascal triangle. From $f(t) = tq(f(t)) = \frac{t}{1-f(t)}$, we get $f(t) = \frac{1-\sqrt{1-4t}}{2} = tC(t)$, and $g(t) = \frac{tf'(t)p(f(t))}{f(t)} = \frac{1}{\sqrt{1-4t}} = \frac{C(t)}{1-tC(t)^2}$, where $C(t)$ denote the generating function for the Catalan numbers (see [22]). Therefore, $P^{(1)} = \left(\frac{1}{\sqrt{1-4t}}, tC(t)\right)$, and $(P^{(1)})^{-1} = (1-2t, t(1-t))$. Since $P^{-1} = (\frac{1}{1+t}, \frac{t}{1+t})$ and $P^2 = (\frac{1}{1-2t}, \frac{t}{1-2t})$ and by routine computation, we get $(P^{-1})^{(1)} = \left(\frac{1}{\sqrt{1+4t}}, \frac{\sqrt{1+4t}-1}{2}\right)$, and $(P^2)^{(1)} = \left(\frac{1}{\sqrt{1-8t}}, \frac{1-\sqrt{1-8t}}{4}\right)$.

In a sense, Theorem 2.1 is a characterization of half array of a Riordan array and we can also prove a sort of inverse property.

**Theorem 2.3.** Let $H = (g(t), f(t))$ be a Riordan array with inverse $H^{-1} = (d(t), h(t))$. If $q(t) = \frac{t}{h(t)}$, $p(t) = \frac{th'(t)}{h(t)d(t)}$, then $H$ is the half array of $G = (p(t), tq(t))$.
**Proof.** Let $G^{(1)}$ be half of $G = (p(t), tq(t))$ . Then, by Theorem 2.1, $(G^{(1)})^{-1} = \left(\frac{th'(t)}{h(t)p(t)}, h(t)\right) = (d(t), h(t))$. Therefore, $G^{(1)} = H$. □

**Remark 1.** From Theorem 2.1 and Theorem 2.3, the assignment to every Riordan array its corresponding half is a bijection. One referee remain us to check if this map is a group isomorphism. From Example 2.2, we conclude that this map is not group homomorphism.

**Example 2.2.** Let $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$, then $P^{-1} = \left(\frac{1}{1+t}, \frac{t}{1+t}\right)$. From Theorem 2.3, $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is the half of $H = (1, t + t^2)$.

**Example 2.3.** Let $H = (1, t + t^2)$, then $H^{-1} = \left(1, \frac{\sqrt{1+4t}-1}{2}\right)$. Hence, $H = (1, t + t^2)$ is the half of $G = \left(\frac{\sqrt{1+4t}+1}{2\sqrt{1+4t}}, \frac{t(\sqrt{1+4t}+1)}{2}\right)$.

**Theorem 2.4.** Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the compositional inverse of $\frac{t}{q(t)^r}$. Then the $r$-half of Riordan array $G$ is given by

$$G^{(r)} = \left(\frac{tf'(t)p(f(t))}{f(t)}, tq(f(t))\right).$$

**Proof.** Since $f(t)$ is the compositional inverse of $\frac{t}{q(t)^r}$, it follows that $f(t) = tq(f(t))^r$. From this relation we get $f'(t) = \frac{q(f(t))^r}{1-rtq(f(t))^{r-1}q'(f(t))}$. Applying Lemma 1.1, we have
$[t^n]f'(t)p(f(t))\frac{t}{f(t)}\,(tq(f(t)))^k = [t^{n-k}]\frac{q(f(t))^{k+1}p(f(t))}{q(f(t))-rf(t)q'(f(t))}$
$= [t^{n-k}]\frac{p(t)q(t)^{k+1}}{q(t)-rtq'(t)}q(t)^{r(n-k-1)}(q(t)^r - rtq(t)^{r-1}q'(t))$
$= [t^{n-k}]p(t)q(t)^{rn-(r-1)k} = g_{(r+1)n-rk,rn-(r-1)k}.$

Theorem now directly follows form (1) and (5) □

Let $\mathcal{B}_m(t)$ be the generalized binomial series given by [7]

$$\mathcal{B}_m(t) = \sum_{n=0}^{\infty} \frac{1}{mn+1}\binom{mn+1}{n} t^n,$$

for which we have

$$\mathcal{B}_m(t)^r = \sum_{n=0}^{\infty} \frac{r}{mn+r}\binom{mn+r}{n} t^n,$$

for any real number $r$. In the special cases $m = 0, 1$, and 2, we have $\mathcal{B}_0(t) = 1+t$, $\mathcal{B}_1(t) = \frac{1}{1-t}$, and $\mathcal{B}_2(t) = C(t)$ is the generating function for Catalan numbers.

**Example 2.4.** Let $G = \left(\frac{\sqrt{1+4t}+1}{2\sqrt{1+4t}}, \frac{t(\sqrt{1+4t}+1)}{2}\right)$. Then it can be represented as $G = \left(\frac{1}{2-\mathcal{B}_{-1}(t)^{-1}}, t\mathcal{B}_{-1}(t)\right)$, and its generic element is $g_{n,k} = \binom{2k-n}{n-k}$. By Example 2.1-2.3, we have $G^{(1)} = (1, t(1+t))$, $G^{(2)} = P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is the Pascal matrix. Thus $G^{(3)} = P^{(1)} = \left(\frac{C(t)}{1-tC(t)^2}, tC(t)\right)$. By applying Theorem 2.4, we obtain $G^{(4)} = P^{(2)} = \left(\frac{\mathcal{B}_3(t)}{1-2t\mathcal{B}_3(t)^3}, t\mathcal{B}_3(t)\right)$, where $\mathcal{B}_3(t)$ is the generalized binomial series defined by $\mathcal{B}_3(t) = \sum_{n=0}^{\infty} \frac{1}{3n+1}\binom{3n+1}{n} t^n$. The first few rows of $P^{(1)}$ and $P^{(2)}$ are

$$P^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 20 & 10 & 4 & 1 & 0 \\ 70 & 35 & 15 & 5 & 1 \end{pmatrix}, P^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 15 & 4 & 1 & 0 & 0 \\ 84 & 21 & 5 & 1 & 0 \\ 495 & 120 & 28 & 6 & 1 \end{pmatrix}.$$

Using the generalized binomial series, we can give following characterization for the $r$-half array of Pascal matrix.

**Theorem 2.5.** Let $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$, then

$$P^{(r)} = \left(\frac{\mathcal{B}_{r+1}(t)}{1 - rt\mathcal{B}_{r+1}(t)^{r+1}}, t\mathcal{B}_{r+1}(t)\right)$$

and its inverse is

$$(P^{(r)})^{-1} = \left( \frac{1 - rt\mathcal{B}_r(t)^r}{\mathcal{B}_r(t)}, \frac{t}{\mathcal{B}_r(t)} \right).$$

**Proof.** We apply Theorem 2.4 for $p(t) = q(t) = \frac{1}{1-t}$. Let $f(t) = tq(f(t))^r = \frac{t}{(1-f(t))^r}$. Then $1 - (1 - f(t)) = \frac{t}{(1-f(t))^r}$, and hence $\frac{1}{1-f(t)} = 1 + \frac{t}{(1-f(t))^{r+1}}$. Consequently, $\frac{1}{1-f(t)} = \mathcal{B}_{r+1}(t)$, and $f(t) = t\mathcal{B}_{r+1}(t)^r$. It follows from Theorem 2.4 that $P^{(r)} = \left(\frac{tf'(t)p(f(t))}{f(t)}, t(\frac{f(t)}{t})^{\frac{1}{r}}\right) = \left(\frac{\mathcal{B}_{r+1}(t)}{1-rt\mathcal{B}_{r+1}(t)^{r+1}}, t\mathcal{B}_{r+1}(t)\right)$.

Since $P^{(r)}$ is the half of $P^{(r-1)} = \left(\frac{\mathcal{B}_r(t)}{1-(r-1)t\mathcal{B}_r(t)^r}, t\mathcal{B}_r(t)\right)$, it follows from Theorem 2.1 that $(P^{(r)})^{-1} = \left(\frac{1-(r-1)t\mathcal{B}_r(t)^r}{\mathcal{B}_r(t)}\frac{th'(t)}{h(t)}, h(t)\right)$ with $h(t) = \frac{t}{\mathcal{B}_r(t)}$. By a straightforward computation we obtain $(P^{(r)})^{-1} = \left(\frac{1-rt\mathcal{B}_r(t)^r}{\mathcal{B}_r(t)}, \frac{t}{\mathcal{B}_r(t)}\right)$. □

The generic term of $(P^{(r)})^{-1}$ is $\frac{n(k+1)-r^2(n-k)}{(rn-rk-k)(rn-rk-k-1)}\binom{rn-rk-k}{n-k}$, while generic term of $P^{(r)}$ is $\binom{n(r+1)-rk}{n-k}$. Hence, we get the following inverse relation

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \binom{n(r+1)-rk}{n-k} b_k, \\ b_n &= \sum_{k=0}^{n} \frac{n(k+1)-r^2(n-k)}{(rn-rk-k)(rn-rk-k-1)} \binom{rn-rk-k}{n-k} a_k. \end{aligned}$$

Using Lemma 1.1 and Theorem 2.4, we can obtain the following results:

**Corollary 2.6.** Let $A = \left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right) = (C(t), tC(t))$, then

$$A^{(r)} = \left( \frac{1 - t^2\mathcal{B}_{r+2}(t)^{2r+4}}{1-(r+1)t\mathcal{B}_{r+2}(t)^{r+2}}, t\mathcal{B}_{r+2}(t) \right),$$

$$(A^{(r)})^{-1} = \left( \frac{1-(r+1)t\mathcal{B}_{r+1}(t)^{r+1}}{1 - t^2\mathcal{B}_{r+1}(t)^{2r+2}}, \frac{t}{\mathcal{B}_{r+1}(t)} \right).$$

**Corollary 2.7.** Let $B = (C(t)^2, tC(t)^2)$, then

$$B^{(r)} = \left(\frac{\mathcal{B}_{2r+2}(t)^2(1 - t\mathcal{B}_{2r+2}(t)^{2r+2})}{1 - (2r+1)t\mathcal{B}_{2r+2}(t)^{2r+2}}, t\mathcal{B}_{2r+2}(t)^2\right),$$

$$(B^{(r)})^{-1} = \left(\frac{(2r+2) - (2r+1)\mathcal{B}_{2r+1}(t)}{\mathcal{B}_{2r}(t)^2(2 - \mathcal{B}_{2r}(t))}, \frac{t}{\mathcal{B}_{2r}(t)^2}\right).$$

Particularly, the half array of $(C(t), tC(t))$ is $((t\mathcal{B}_3(t))', t\mathcal{B}_3(t))$, while the half array of $(C(t)^2, tC(t)^2)$ is $((t\mathcal{B}_4(t)^2)', t\mathcal{B}_4(t)^2)$.

# 3 Applications

In this section, we present an application of our construction to count those lattice paths starting from $(0, 0)$ that use the step set $S_4 = \{E = (1, 0), N = (0, 1), U = (1, 1), D = (1, -1)\}$, where each step is labeled with weights $a$, $b$, $u$ and $v$, respectively. These paths generalize both Dyck paths which consist of steps $U$ and $D$ and Delannoy paths which consist of steps $E$, $N$ and $U$. Let $G(n, k)$ be the set of all weighted lattice paths ending at the point $(k, n - 2k)$. Let $g_{n,k}$ be the sum of all weights $w(P)$ with $P$ in $G(n, k)$. The first few rows of matrix $(g_{n,k})_{n,k\in\mathbb{N}}$ are illustrated in Figure 2.

The last step of any path from $G(n, k)$ is one of $S_4 = \{E = (1, 0), N = (0, 1), U = (1, 1), D = (1, -1)\}$, as shown in Figure 3. Therefore, the number $g_{n,k} = |G(n, k)|$ satisfies the following recurrence relation

$$g_{n+1,k+1} = vg_{n,k} + ag_{n-1,k} + ug_{n-2,k} + bg_{n,k+1}, \tag{6}$$

with $n, k \geq 0$ and boundary conditions $g_{n,0} = b^n$ and $g_{n,n} = v^n$.

For $k \geq 0$, let $g_k(t) = \sum_{n=k}^{\infty} g_{n,k}t^n$. Then $g_0(t) = \frac{1}{1-bt}$ and by (6), we have

$$g_{k+1}(t) = vtg_k(t) + at^2g_k(t) + ut^3g_k(t) + btg_{k+1}(t).$$

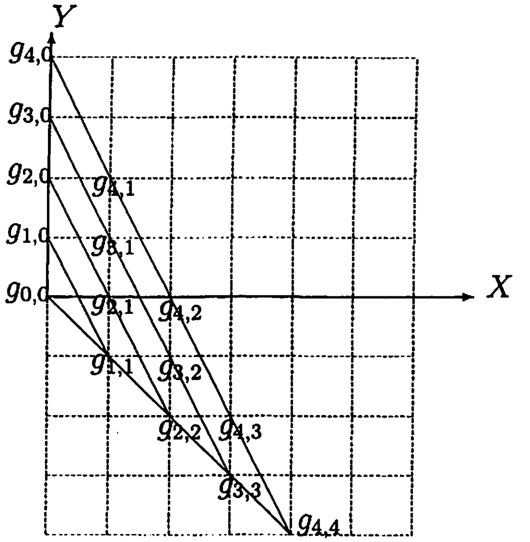


Figure 2: Combinatorial interpretation of the matrix $(g_{n,k})_{n,k\in\mathbb{N}}$

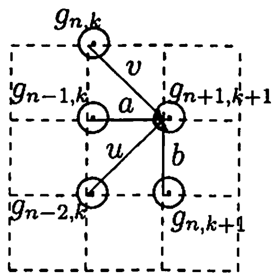


Figure 3: Recurrence relation of $(g_{n,k})$

By iterating, we obtain

$$g_{k+1}(t) = \frac{vt + at^2 + ut^3}{1 - bt} g_k(t) = \left(\frac{vt + at^2 + ut^3}{1 - bt}\right)^{k+1} \frac{1}{1 - bt}.$$

In view of definition (1), we have the following theorem.

**Theorem 3.1.** The infinite lower triangular array $(g_{i,j})_{i,j\in\mathbb{N}}$

has a Riordan array expression given by

$$G = \left(\frac{1}{1-bt}, \frac{vt+at^2+ut^3}{1-bt}\right).$$

**Theorem 3.2.** The general terms of the arrays $G$ is given by

$$g_{n,k} = \sum_{m=0}^{2k}\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\binom{n-m}{k}\binom{k}{m-i}\binom{m-i}{i}v^{k+i-m}a^{m-2i}u^{i}b^{n-k-m}.$$

**Proof.** By the definition and the binomial theorem, we have

$$\begin{aligned}
g_{n,k} &= [t^n]\frac{1}{1-bt}\left(\frac{vt+at^2+ut^3}{1-bt}\right)^k \\
&= [t^{n-k}]\left(\frac{1}{1-bt}\right)^{k+1}(v+at+ut^2)^k \\
&= [t^{n-k}](v+at+ut^2)^k(1-bt)^{-(k+1)} \\
&= [t^{n-k}]\sum_{m=0}^{2k}\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\binom{k}{m-i}\binom{m-i}{i}v^{k+i-m}a^{m-2i}u^{i}\sum_{j=0}^{\infty}\binom{k+j}{j}b^j t^{m+j} \\
&= \sum_{m=0}^{2k}\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\binom{n-m}{n-k-m}\binom{k}{m-i}\binom{m-i}{i}v^{k+i-m}a^{m-2i}u^{i}b^{n-k-m} \\
&= \sum_{m=0}^{2k}\sum_{i=0}^{\lfloor\frac{m}{2}\rfloor}\binom{n-m}{k}\binom{k}{m-i}\binom{m-i}{i}v^{k+i-m}a^{m-2i}u^{i}b^{n-k-m}.
\end{aligned}$$

This complete the proof. □

Obviously, when $a = u = 0$, we obtain the generalized Pascal matrix $\left(\frac{1}{1-bt}, \frac{vt}{1-bt}\right)$; when $u = 0$, we obtain the Riordan array with weighted Delannoy numbers [5, 16, 23] $\left(\frac{1}{1-bt}, \frac{vt+at^2}{1-bt}\right)$. If

$a = b = v = u = 1$, then the first few terms of this array are

$$G = \left(\frac{1}{1-t}, \frac{t+t^2+t^3}{1-t}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 6 & 5 & 1 & 0 & 0 & \cdots \\ 1 & 9 & 15 & 7 & 1 & 0 & \cdots \\ 1 & 12 & 33 & 28 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $D(n,k)$ be the set of all weighted lattice paths starting from $(0,0)$ and ending at the point $(n,-k)$, see Figure 4 below. Let $d_{n,k}$ be the sum of all $w(P)$ with $P$ in $D(n,k)$. Then $d_{n,k} = g_{2n-k,n}$, hence the matrix $(d_{n,k})_{n,k\geq 0}$ is the half array of $G$. By Theorem 3.2,
$d_{n,k} = \sum_{m=0}^{2n} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{2n-k-m}{n} \binom{n}{m-i} \binom{m-i}{i} v^{n+i-m} a^{m-2i} u^i b^{n-k-m}.$

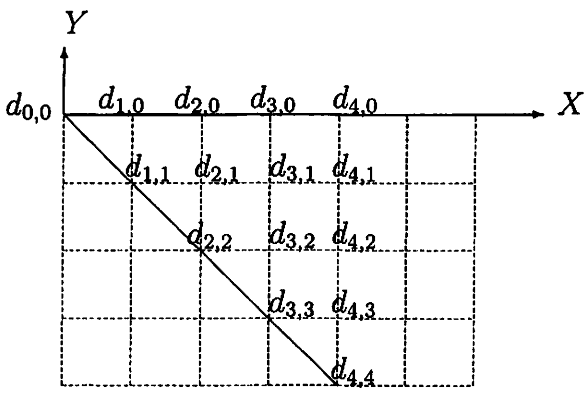


Figure 4: Combinatorial interpretation of the matrix $(d_{n,k})_{n,k\in\mathbf{N}}$

**Theorem 3.4.** The array $D = (d_{n,k})_{n,k\geq 0}$ is given by the following Riordan array
$D = \left(\frac{1}{\sqrt{(1-at)^2-4vt(b+ut)}}, \frac{1-at-\sqrt{(1-at)^2-4vt(b+ut)}}{2(b+ut)}\right).$

**Proof.** Since $(d_{n,k})_{n,k\geq 0}$ is the half array of $G = \left(\frac{1}{1-bt}, \frac{vt+at^2+ut^3}{1-bt}\right)$, it follows from Theorem 2.1 that $D = \left(\frac{tf'(t)}{f(t)(1-bf(t))}, f(t)\right)$, where $f(t)$ is the compositional inverse of $\frac{(1-bt)t}{v+at+ut^2}$. Then we get $f(t) = \frac{1-at-\sqrt{(1-at)^2-4vt(ut+b)}}{2(ut+b)}$, and $\frac{tf'(t)}{f(t)(1-bf(t))} = \frac{1}{\sqrt{(1-at)^2-4vt(ut+b)}}$. □

If $u = 0$, then $D = \left(\frac{1}{\sqrt{1-(2a+4bv)t+a^2t^2}}, \frac{1-at-\sqrt{1-(2a+4bv)t+a^2t^2}}{2b}\right)$, the first column is consist of generalized central Delannoy numbers [5, 23].

If $b = 0$, then $D = \left(\frac{1}{\sqrt{1-2at+(a^2-4uv)t^2}}, \frac{1-at-\sqrt{1-2at+(a^2-4uv)t^2}}{2ut}\right)$, the first column is consist of generalized central trinomial coefficients [8, 14, 17].

If $a = b = v = u = 1$, then $D = \left(\frac{1}{\sqrt{(1-6t-3t^2}}, \frac{1-t-\sqrt{1-6t-3t^2}}{2(1+t)}\right)$ and the first few terms of this array are

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 15 & 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 81 & 28 & 7 & 1 & 0 & 0 & 0 & \cdots \\ 459 & 161 & 45 & 9 & 1 & 0 & 0 & \cdots \\ 2673 & 946 & 281 & 66 & 11 & 1 & 0 & \cdots \\ 15894 & 5642 & 1742 & 449 & 91 & 13 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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