On the rainbow connection number of graphs

G. Dror, A. Lev, Y. Roditty and R. Zigdon

School of Computer Sciences
The Academic College of Tel-Aviv-Yaffo
2 Rabeinu Yeruham st., Tel-Aviv
Israel 61083

May 4, 2015

Abstract

Let $G = (V, E)$, $|V| = n$, be a simple connected graph. An edge-colored graph G is *rainbow edge-connected* if any two vertices are connected by a path whose edges are colored by distinct colors. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow edge connected. In this paper we obtain tight bounds for $rc(G)$. We use our results to generalize previous results for graphs with $\delta(G) \geq 3$.
AMS classification code: 05C78
Keywords: rainbow connected coloring

1 Introduction

Graphs in this paper are considered as finite, simple, undirected and connected. Let G be a graph $G = (V, E)$, $|V| = n(G) = n$.

A *colored* graph $G = (V, E)$ is a graph for which there is a function $f : E \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$. A path P between any two vertices u and v, is called a *rainbow path* if no two edges of P have the same color. We denote by $l(P)$ the number of edges on the path. A graph G is *rainbow-connected* (with respect to f) if any two vertices $u, v \in V$ are connected by a rainbow path. In that case f is called a *rainbow coloring* of G. In case k colors are used it is called a *rainbow k-coloring.* The *rainbow connection number* of G, $rc(G)$ (defined in [3]), is the minimum k for which there exists a rainbow $k-$coloring of the edges of G. A rainbow coloring of G using $rc(G)$ colors is called a *minimum rainbow coloring of G.* Clearly, if a graph is rainbow connected, then it is also connected. Any connected graph has a trivial edge coloring that makes it rainbow connected, where each edge is colored with a distinct color. Furthermore an easy observation is that $rc(G) \leq n-1$, since one may color the edges of a given spanning tree with distinct colors and color the remaining edges with one of the already used colors.

The concept of rainbow coloring has received much attention during the years (one may look at the recent survey of Li and Sun [5]). However, since in this paper we deal with upper bounds for the rainbow connection number, we shall concentrate here only on the relevant papers to our results.

Caro, Lev, Roditty, Tuza and Yuster [2] proved:

Theorem 1.1 *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) \leq \frac{5n}{6}$.*

They conjectured also,

Conjecture 1.2 *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) \leq \frac{3n}{4}$.*

Schiermeyer proved in [7] the conjecture by showing:

Theorem 1.3 *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) \leq \frac{3n-1}{4}$.*

Krivelevich and Yuster in [4] determined the behavior of $rc(G)$ as a function of $\delta(G)$ by proving:

Theorem 1.4 *If G is a connected graph with n vertices, then $rc(G) \leq \frac{20n}{\delta(G)}$.*

In this paper we find bounds on $rc(G)$ for general graphs. We start with the following result for 2-edge-connected graphs:

Theorem 1.5 *Let G be a 2-edge connected graph different from C_5. Then $rc(G) \leq \frac{2}{3}(n-1)$.*

We then turn to any connected graph and prove the following bound for $rc(G)$:

Theorem 1.6 *Let G be a connected graph with $n(G) = n \geq 3$, $G \neq C_5$. Let s be the number of vertices that do not lie on a cycle of G. Then:*

$$rc(G) \leq \frac{3}{4}n + \frac{1}{4}s - 1. \tag{1}$$

In particular if $G \neq C_5$ and every vertex of G lies on a cycle then

$$rc(G) \leq \frac{3}{4}n - 1. \tag{2}$$

The above bounds are tight, in the sense that there is an infinite family $\{G_i\}_{i=1}^{\infty}$ of graphs, for which $rc(G_i)$ meets the bound for every $i \in \mathbb{N}$.

Definition 1.7 *We call a maximal 2-edge connected subgraph of G an e-block of G.*

Then we prove the following:

Theorem 1.8 *Let G be a graph different from C_5. Assume G has m e-blocks and s vertices which do not lie on a cycle. Then*

$$rc(G) \leq \frac{2}{3}n + \frac{1}{3}(s+m) - 1. \tag{3}$$

In section 4 we shall apply our results to obtain an extension of Theorem 1.3.

For more basic definitions we follow [8].

2 2-Edge Connected Graphs

The following bound for the rainbow connection number of 2-connected graphs was obtained in [6]:

Theorem 2.1 *Let G be a 2-connected graph of order n ($n \geq 3$). Then $rc(G) \leq \lceil \frac{n}{2} \rceil$, and the upper bound is tight for $n \geq 4$.*

The following simple corollary of this theorem, shall be very useful to the sequel.

Corollary 2.2 *Let G be a 2-connected graph, such that $G \neq C_2, C_5$. Then $rc(G) \leq \frac{2}{3}(n-1)$.*

Proof For $n \geq 4$ and $n \neq 5$ we have $\lceil \frac{n}{2} \rceil \leq \frac{2}{3}(n-1)$.

For $n = 5$, since $G \neq C_5$, it is easy to check that $rc(G) \leq 2 \leq \frac{2}{3}(5-1)$. The result is obvious for $n = 3$. ■

Note that the bound in the corollary is the best possible since,

$$rc(C_7) = 4 = \frac{2}{3}(7-1).$$

Let G be a 2-edge connected graphs. A *block* of G is a maximal 2-connected subgraph of G. We start with the following useful proposition.

Proposition 2.3 *Every 2-edge connected graph G may be presented as a union $G = \bigcup_i^k H_i$, where for each $1 \leq i \leq k$, H_i is a block of G, such that for each $i \neq j$, $|V(H_i) \cap V(H_j)| \leq 1$ and $E(G) = \bigcup_{i=1}^k E(H_i)$ is a disjoint union. This presentation of G as a union $G = \bigcup_i^k H_i$ is unique.*

Proof Let H_1 be a block of G, (namely any vertex added to H_1 would render it a subgraph which is not 2-connected). If $G = H_1$ we are done. Otherwise, let H_2 be another block of G, different from H_1. If H_1 and H_2 share two vertices, then $H_1 \cup H_2$ is 2-connected, contradicting the maximality of H_1. Therefore, $|V(H_1) \cap V(H_2)| \leq 1$. Now, since G is 2-edge connected, every edge lies on some cycle, and therefore belongs to a block. Since we have shown that no two blocks of G have a common edge we obtain $E(G) = \bigcup_{i=1}^k E(H_i)$ is a disjoint union. The uniqueness of the presentation is easy to prove. ∎

Notice that since G is 2-edge-connected a block of G can not be a single edge.

We construct a graph T whose vertices represent the blocks of G, such that two vertices of T are connected by an edge if and only if the corresponding blocks of G have a common vertex. Evidently, T cannot contain a cycle. Indeed, if T contains a cycle, then there are vertices $v_1, v_2, \ldots, v_t \in V(T)$ whose induced subgraph in T is a cycle, and then the union of the corresponding components, $\bigcup_{i=1}^t H_i$ is a 2-connected subgraph of G, contradicting the maximality of the blocks $H_1, H_2, \ldots, H_t$. The tree T is called the *skeleton* of the decomposition $G = \bigcup H_i$. (See Figure 1 for an example).

We are ready now to prove Theorem 1.5:

Proof By proposition 2.3, it is possible to decompose G into maximal 2-connected subgraphs, $G = \bigcup_{i=1}^k H_i$, such that for each $i \neq j$ H_i, H_j have at most one common vertex, and such that the skeleton of the decomposition is a tree T.

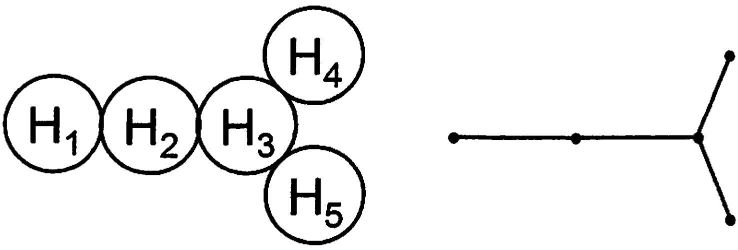
We now concatenate two components H_1 and H_2 of the decomposition, that share a vertex, namely such that $V(H_1) \cap V(H_1) = \{v\} \subset V(G)$. Denote $n_i = |V(H_i)|, i = 1, 2$. If H_1 and H_2 are both different from C_5 we have by corollary 2.2:

$$rc(H_1 \cup H_2) \leq rc(H_1) + rc(H_2) \leq \frac{2}{3}(n_1 - 1) + \frac{2}{3}(n_2 - 1) = \frac{2}{3}((n_1 + n_2 - 1) - 1),$$

yielding,

$$rc(H_1 \cup H_2) \leq \frac{2}{3}(n(H_1 \cup H_2) - 1).$$

Thus, if no maximal 2-connected subgraph H_i is C_5, then using induction we obtain $rc(G) \leq \frac{2}{3}(n - 1)$.



(a) (b)

Figure 1: Decomposition of 2-edge-connected graphs. Each subgraph H_i in (a) represents a block of G. (b) the corresponding graph T.

Now consider the case when $H_1 \neq C_5$ and $H_2 = C_5$. In this case only two new colors are needed to color H_2 to obtain a rainbow coloring (see Figure 2(a)). Then

$$\begin{aligned} rc(H_1 \cup H_2) &\leq rc(H_1) + 2 \leq \frac{2}{3}(n_1 - 1) + \frac{2}{3}(n_2 - 1) \\ &= \frac{2}{3}((n_1 + n_2 - 1) - 1) = \frac{2}{3}(n(H_1 \cup H_2) - 1). \end{aligned}$$

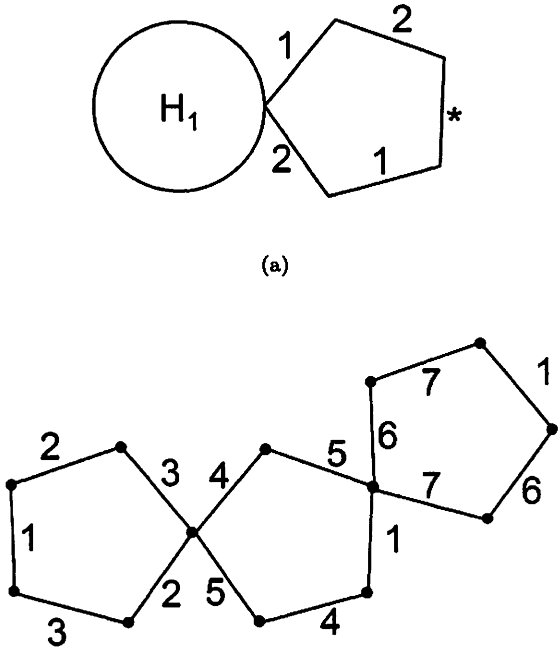
Thus, if $H_1 \neq C_5$, the result $rc(G) \leq \frac{2}{3}(n-1)$ follows similarly by induction.

Assume now that all components in the decomposition of G are C_5. Then H_1 has 5 vertices and 3 colors, and each subsequent component $H_i, 2 \leq i \leq k$ adds 4 new vertices and 2 new colors. Thus we obtain $rc(G) = \frac{1}{2}(n+1)$, which yields $rc(G) \leq \frac{2}{3}(n-1)$, since $n \geq 9$ in this case. (See Figure 2(b)).

Following the above considerations, the proof of the Theorem is easily obtained by induction on the number of blocks. ■

Remark 2.4 *The bound in this theorem is best possible since if we consider a 2-edge-connected graph all whose blocks are C_4 we get*

$$rc(G) = \frac{2}{3}(n-1),$$



(a)

(b)

Figure 2: (a) Concatenation in the case $H_1 \neq C_5$, $H_2 = C_5$. The edge marked with '*' is colored by any of the colors in H_1. (b) An example of coloring a 2-edge connected graph G where all maximal 2-connected subgraphs are C_5.

namely there is an infinite family $\{G_i\}_{i=1}^{\infty}$ of graphs such that $rc(G_i)$ meets the bound of the theorem for all i.

3 General Graphs

In this section we present bounds on the rainbow connection number of any connected graph. We rely on the fact that if a vertex lies on a cycle then it belongs to a maximal 2-edge connected subgraph of G.

The next proposition gives us a useful decomposition:

Proposition 3.1 *If H_1, H_2 are two different e-blocks of G then,* $|V(H_1) \cap V(H_2)| = 0$.

Remark 3.2 *Assume that H is an e-block of G and $H \neq G$. Then we can extend H by attaching to it an edge xy that does not belong to a cycle. Such that $x \in V(H)$ and $y \notin V(H)$.*

Let G be a graph with m e-blocks and s vertices which do not lie on a cycle. Then we can represent G by a tree T with $s+m$ vertices, where m vertices of T represent the e-blocks H_i of G and the remaining s vertices of T represent vertices of G which do not lie on a cycle. The tree T is called the *2-edge-skeleton* of G. The union $\bigcup_{i=1}^{m} H_i$ has a total of $n-s$ vertices, and the 2-edge-skeleton T has $s+m-1$ edges. It is easy to see that for any connected graph there corresponds a unique 2-edge-skeleton.

Proof of Theorem 1.8: Let M be the set of e-blocks of G. Then M can be partitioned as $M = M' \cup M''$, where M' contains all e-blocks different from C_5 and M'' contains all e-blocks that are C_5. We have (by theorem 1.5):

$$rc(H_i) = \frac{2}{3}(n_i - 1) = \frac{2}{3}n_i - \frac{2}{3} \tag{4}$$

for each $H_i \in M'$, and $n_i = |V(H_i)|$.

Assume now that G has at least one e-block different from C_5. Since the 2-edge-skeleton of G, T, is a tree, and since there is at least one e-block which is not C_5, we may attach to each $H_i \in M''$ an adjacent edge e_i as describe in remark 3.2, such that if $H_i \neq H_j$ are in M'' than H_i and H_j are attached distinct edges.

We now apply the following coloring to G: Each e-block is rainbow colored, such that whenever $H_i \neq H_j$ $(H_i, H_j \in M)$ different colors are used. Each e-block $H_i \in M''$ is colored using 3 colors, where the edge we attached to it is colored with one of the three colors, (see Figure 3).

Thus, for each $H_i \in M''$ we obtain

$$rc(H_i) = 3 \leq \frac{2}{3}(5-1) + 1 \leq \frac{2}{3}(n_i - 1) + 1. \tag{5}$$

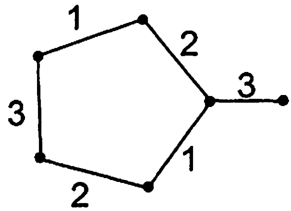


Figure 3: Colorings for the case where the e-block is C_5 and we attache to it an edge.

Now, if we color all subgraphs $H_i \in M$ according to the coloring described above, then for every $H_i \in M''$ the adjacent edge e_i is colored with one of the colors of H_i. The remaining edges that have not yet been colored are colored by additional new colors, so that we obtain a rainbow coloring of G. The total number of colors used follows from (4) and (5) so that,

$$\begin{aligned} rc(G) &\leq \sum_{H_i \in M'} \frac{2}{3}(n_i - 1) + \sum_{H_i \in M''} \left(\frac{2}{3}(n_i - 1) + 1 \right) \\ &\quad (\# \text{ edges not on a cycle, not attached to } M'' \text{ subgraphs}) \\ &\leq \sum_{H_i \in M} \frac{2}{3}(n_i - 1) + s + m - 1 \\ &= \frac{2}{3}(n - s) - \frac{2}{3}m + s + m - 1 \\ &= \frac{2}{3}n + \frac{1}{3}(s + m) - 1 \end{aligned}$$

As required.

We still need to prove the result in the case where all the e-blocks H_i of G are C_5. As in the previous case we attach each e-block a neighboring edge. If each e-block can be attached a different edge, then the previous analysis holds, and we obtain the same result. If there is an e-block that cannot be augmented with a unique edge (there is at most one such e-block) then it is easy to see that we must have $s = 0$ (there are no vertices that do not lie on a cycle) and $m \geq 2$ (we assumed $G \neq C_5$). Therefore, we have

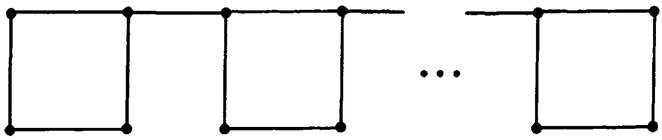


Figure 4: A chain of C_4 subgraphs.

in this case $n = 5m$ and $rc(G) = 3m$, hence:

$$\begin{aligned} \frac{2}{3}n + \frac{1}{3}(s+m) - 1 &= \frac{2}{3}5m + \frac{1}{3}m - 1 \\ &= 3m + \frac{2}{3}m - 1 \\ &> 3m = rc(G) \end{aligned}$$

Remark 3.3 *Notice that if G is 2-edge connected ($m = 1$, $s = 0$) we obtain $rc(G) \leq \frac{2}{3}n + \frac{1}{3} - 1 = \frac{2}{3}(n-1)$, which is the bound in Theorem 1.5*

Remark 3.4 *Let G be a graph composed of a linear chain of C_4 where each C_4 is connected to each neighbor by a single edge, as shown in Figure 4. Then we have $m = \frac{n}{4}$, $s = 0$ and $rc(G) = 2m + m - 1 = \frac{2}{3}n + \frac{1}{3}(s+m) - 1$. That is the bound of theorem 1.8 is tight.*

The following lemma is easy to prove:

Lemma 3.5 *Let G be a graph in which every vertex lies on a cycle then there is a decomposition $V(G) = \bigcup_{i=1}^{m} V_i$ where the subgraph H_i spanned by V_i ($1 \leq i \leq k$) is an e-block of G and all the e-blocks are disjoint.*

For a graph G in which every vertex lies on a cycle the vertices of the 2-edge-skeleton T of G are the e-blocks of G.

In the next theorem we shall focus our attention to connected graphs in which every vertex lies on a cycle.

Theorem 3.6 *If G is a graph where every vertex lies on a cycle, $G \neq C_5$, then $rc(G) \leq \frac{3}{4}n - 1$*

Proof Since every vertex belongs to a cycle, we have by theorem 1.8:

$$rc(G) \leq \frac{2}{3}n + \frac{1}{3}m - 1$$

Recall that we denote $n = n(G)$, m is the number of e-blocks of G, and $V(G) = \bigcup_{i=1}^{m} V_i$ is a disjoint union where, V_i are the vertices of the e-block H_i. Let $m = m_1 + m_2 + m_3$, where m_1 is the number of e-blocks which are C_3, m_2 is the number of e-blocks which are C_5 and m_3 is the number of the remaining e-blocks. Similarly let $n = n_1 + n_2 + n_3$, where $n_1 = 3m_1$ is the number of all vertices in the m_1 e-blocks, $n_2 = 5m_2$ is the number of all vertices in the m_2 e-blocks and n_3 is the number of all vertices in the remaining e-blocks.

If $m_1 = 0$ then $4m \leq n$ and $rc(G) \leq \frac{2}{3}n + \frac{1}{3}m - 1 \leq \frac{3}{4}n - 1$. Thus, we may assume that $m_1 \neq 0$.

Let M be the set of e-blocks which are not C_3 or C_5. Then $m_3 = |M|$. Assume first $m_3 \neq 0$. Since there are $m_1+m_2+m_3-1$ edges in the 2-edge-skeleton of G, we may attach to each component H_i ,which is either C_3 or C_5 an adjacent edge, such that no two components are attached with the same edge. We color each C_3 with one color and the one edge attached to it with another color and we color each C_5 with three colors and the edge attached to it with one of these colors (see Figure 3 for an example). All other components are 2-edge connected different from C_5 and according to theorem 1.5 we use at most additional $\frac{2}{3}(t_i - 1)$ colors to color each 2-edge connected block H_i in M where $t_i = |V(H_i)|$. The remaining $m_3 - 1$ edges are colored with new colors. Thus we get:

$$\begin{aligned} rc(G) &\leq 2m_1 + 3m_2 + \sum_{H_i \in M} \frac{2}{3}(t_i - 1) + m_3 - 1 \\ &= \frac{2}{3}n_1 + \frac{3}{5}n_2 + \frac{2}{3}n_3 - \frac{2}{3}m_3 + m_3 - 1 \\ &\leq \frac{2}{3}n + \frac{1}{3}m_3 - 1 \leq \frac{3}{4}n - 1 \end{aligned}$$

since $4m_3 \leq n$.

The only case left is the one where $m_1 \neq 0$ and $m_3 = 0$.

In this case we attach to each e-component an edge except one of the C_3 components. We color the graph as in the previous case yielding:
$rc(G) \leq 3m_2 + 2(m_1 - 1) + 1 = \frac{3}{5}n_2 + \frac{2}{3}n_1 - 1 \leq \frac{2}{3}n - 1 \leq \frac{3}{4}n - 1$.

■

Remark 3.7 *The bound obtained in Theorem 3.6 is tight in the sense that there is an infinite family of graphs $\{G_i\}_{i=1}^{\infty}$ such that $rc(G_i) = \frac{3}{4}n(G_i) - 1$ for each $i \in N$. A graph G_i in the family is defined by taking a chain of C_4 , connected to each other by a single edge. (See figure 4).*

We are ready now for the proof of theorem 1.6

Proof Define the following contraction on G: In each step of the contraction, one of the s vertices that do not lie on a cycle is replaced by an adjacent vertex of the 2-edge-skeleton T, while the edge connecting the two is deleted.

At the end of this process a graph H with $n - s$ vertices and $|E(G)| - s$ edges is obtained such that each vertex of H lies on a cycle. Then by Theorem 3.6 it follows that $rc(H) \leq \frac{3}{4}(n - s) - 1$.

Since G is obtained from H by adding s edges and s vertices we get:
$rc(G) \leq \frac{3}{4}(n - s) + s - 1 = \frac{3}{4}n + \frac{1}{4}s - 1$
as required ■

4 Extension of the results for $\delta(G) \geq 3$

In this section we extend the results of theorem 1.3.

We start with the following proposition:

Proposition 4.1 *Let G be a 2-edge connected graph. Assume that there is at most one vertex $v \in V(G)$ for which $d(v) = 2$ and $d(u) \geq 3$ for each $u \in V(G) \backslash \{v\}$. Then $rc(G) \leq \frac{3}{4}n - \frac{7}{4}$*

Proof From the assumptions it follows that $G \neq C_5$ and that it is 2-edge connected. Therefore, by Theorem 1.5, $rc(G) \leq \frac{2}{3}(n-1)$. If $n = n(G) \geq 13$, then $\frac{2}{3}(n - 1) \leq \frac{3}{4}n - \frac{7}{4}$ as required.

In table 1 we investigate the cases where $n \leq 12$.

Since $rc(G)$ is always a natural number, we see from the table that the claim is satisfied for $n \leq 12$ except the cases $n = 4, 6, 7, 10$ which should be examined separately. The cases $n = 4$, $n = 6$ and $n = 7$ are simple. The case $n = 10$ is treated in the appendix ■

Let now G be a simple graph satisfying $\delta(G) \geq 3$. Let T be the 2-edge-skeleton of G. In particular, since $\delta(G) \geq 3$ we get that each leaf of T represents an e-block of G. We denote by l the number of leaves of T. Then we have the following:

Theorem 4.2 *Let G be a graph with $\delta(G) \geq 3$. Then*

1. *If G is 2-edge connected then $rc(G) \leq \frac{3}{4}n - \frac{7}{4}$*
2. *If G is not 2-edge connected then (using the notation above):*

$$rc(G) \leq \frac{3}{4}n - \frac{1}{2}l - \frac{3}{2}$$

Table 1:

n	$\frac{2}{3}(n-1)$	$\frac{3}{4}n-\frac{7}{4}$
4	2	$1\frac{1}{4}$
5	$2\frac{2}{3}$	2
6	$3\frac{2}{3}$	$2\frac{3}{4}$
7	4	$3\frac{1}{2}$
8	$4\frac{2}{3}$	$4\frac{1}{4}$
9	$5\frac{1}{3}$	5
10	6	$5\frac{3}{4}$
11	$6\frac{2}{3}$	$6\frac{1}{5}$
12	$7\frac{1}{3}$	$7\frac{1}{4}$

In particular we get the following corollary:

Corollary 4.3 *Let G be a graph with $\delta(G) \geq 3$. Then*

1. *If G is 2-edge connected then $rc(G) \leq \frac{3}{4}n - \frac{7}{4}$*
2. *If G is not 2-edge connected then $rc(G) \leq \frac{3}{4}n - \frac{5}{2}$.*

We start by proving the corollary which follows directly from the theorem:

Proof Case (1) is case (1) of the theorem. In case (2) we observe that the 2-edge-skeleton of G has at least two leaves, hence $rc(G) \leq \frac{3}{4}n - \frac{1}{2} \cdot 2 - \frac{3}{2} = \frac{3}{4}n - \frac{5}{2}$. ■

Case (2) of Corollary 4.3 was also proved in [7].

Remark 4.4 *The bound of case (1) of the theorem is tight, since for $n = 5$, $rc(G) = 2 = \frac{3}{4}5 - \frac{7}{4}$, as demonstrated in figure 5.*

Remark 4.5 *The bound of case (2) of the corollary is tight. Indeed, let $G = (V, E)$ be the graph as in Figure 6, where $|V| = 4k + 10$. One can easily observe that $rc(G) = 2 + 1 + 3k + 2 = 3k + 5 = \frac{3}{4}n - \frac{5}{2}$.*

We are ready now for the proof of theorem 4.2:

Proof Part (1) of the theorem (when G is 2-edge connected) follows from proposition 4.1.

We now prove part (2). Let T be the 2-edge-skeleton of G. Denote by k the number of vertices of T ($k \geq 2$). Each vertex of T is either a single vertex of G or an e-block of G. Let l be the number of leaves of T and m the number of vertices in T which are not leaves (namely $k = l + m$). We

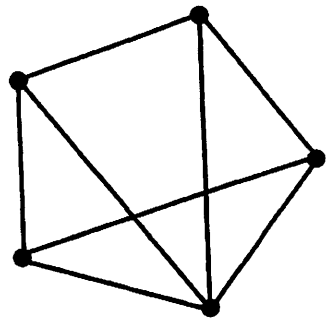


Figure 5: See remark 4.4

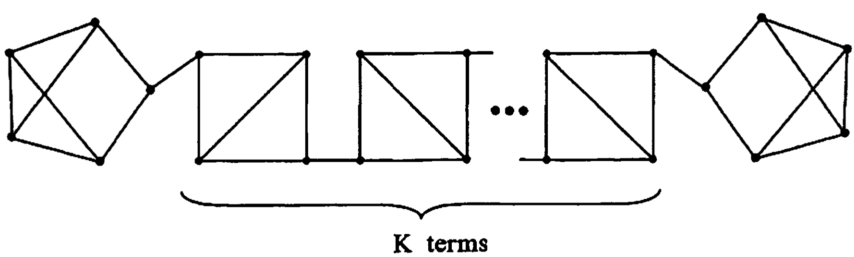


Figure 6: A family of graphs, parameterized by the number k of C_4^* e-components.

denote by V^* the set of vertices of T which are not leaves. Thus, we have for the tree T

$$l = \sum_{v \in V^*} (d(v) - 2) + 2. \tag{6}$$

(This expression for the number of leaves is valid for every tree with at least two vertices). Notice that since $\delta(G) \geq 3$, the leaves of T cannot correspond to single vertices of G and therefore they must correspond to 2-edge connected subgraphs of G.

Now in order to prove part (2) we begin first with a special case, namely, all vertices in V^* are single vertices of G. Then, since $\delta(G) \geq 3$, we have $d(v) \geq 3$ for every vertex $v \in V^*$ and from (6) we get:

$$l \geq m + 2. \tag{7}$$

Now, every leaf L of T is a non trivial 2-edge connected subgraph of G, and since $\delta(G) \geq 3$ we get $d(v) \geq 3$ for every vertex $v \in L$, except (maybe) for a single vertex whose degree is 2. We thus get by Proposition 4.1, that every leaf L of T satisfies

$$rc(L) \leq \frac{3}{4}|V(L)| - \frac{7}{4}. \tag{8}$$

We now color G as follows:

- Every leaf L of G is rainbow colored, such that the number of colors at most $\frac{3}{4}|V(L)| - \frac{7}{4}$ (which is possible according to (8)).
- We use different colors for different leaves.
- The remaining $l + m - 1$ edges of G are colored using $l + m - 1$ new colors.

Then, we obtain a rainbow coloring of G. If we denote the leaves of T by $L_j, j = 1, 2, \ldots, l$ we get, by the arguments above that

$$\begin{aligned} rc(G) &\leq \sum_{j=1}^{l}\left(\frac{3}{4}|V(L_j)| - \frac{7}{4}\right) + l + m - 1 \\ &= \frac{3}{4}\left(\sum_{j=1}^{l}|V(L_j)| + m\right) - \frac{3}{4}l + \frac{1}{4}m - 1 \end{aligned}$$

Observe that the total number of vertices of G is $n = \sum_{j=1}^{l}|V(L_j)| + m$ and that from inequality (7) one has $m \leq l - 2$ so that,

$$rc(G) \leq \frac{3}{4}n - \frac{3}{4}l + \frac{1}{4}(l-2) - 1$$

namely:

$$rc(G) \leq \frac{3}{4}n - \frac{1}{2}l - \frac{3}{2} \tag{9}$$

as stated in the Theorem.

In the general case let α be the number of vertices of T which are e-blocks of G. We notice that in part (2) of the theorem $\alpha \geq 2$. Now, if $\alpha = 2$, then T is a path with two leaves which are 2-edge connected, and all the other vertices (if there are other vertices) are vertices of G. We then get that part (2) is a consequence of the special case we previously proved.

We continue by induction over α. Assume that $\alpha \geq 3$, and that part (2) of the theorem is valid for every graph G for which the number of vertices of T which correspond to e-blocks of G is less than α (but larger or equal to 2). We notice that since $\alpha \geq 3$, there exists in T at least one vertex that is not a leaf. If all the non-leaf vertices of T are vertices of G then the result was proved earlier. Thus, there is in T a non-leaf vertex H, which is an e-block of G. We consider two cases:

case 1: Assume that the degree of H in T is at least 3. We replace H by a single vertex, v_H and get a graph G_1, with $n(G_1) = n_1 = n - |V(H)| + 1$, and the number of e-blocks is $\alpha - 1$. Any vertex of T that was previously adjacent to H will be adjacent to v_H. Now, since the degree of H in T was at least 3, the degree of v_H in G is also at least 3, and it is easy to see that all the conditions of the theorem in case (2) hold for G_1. Therefore, by induction, we get $rc(G_1) = \frac{3}{4}n_1 - \frac{1}{2}l - \frac{3}{2}$. (We notice that the number of leaves of T is identical to the number of leaves in the 2-edge-skeleton of G_1).

Now, since H is 2-edge connected and in particular every vertex of it lies on a cycle, we get by Theorem 3.6 that $rc(H) \leq \frac{3}{4}|V(H)| - 1$. Thus, we have

$$\begin{aligned}
rc(G) &\leq rc(G_1) + rc(H) \\
&\leq \frac{3}{4}n_1 - \frac{1}{2}l - \frac{3}{2} + \frac{3}{4}|V(H)| - 1 \\
&\leq \frac{3}{4}(n_1 + |V(H)| - 1) - \frac{1}{2}l - \frac{3}{2} - \frac{1}{4} \\
&= \frac{3}{4}n - \frac{1}{2}l - \frac{7}{4} \\
&\leq \frac{3}{4}n - \frac{1}{2}l - \frac{3}{2}
\end{aligned}$$

and claim (2) of the theorem is satisfied.

case 2: The degree of H in T is 2. (It cannot have a degree 1, as H is not a leaf of T). Then H has exactly two neighbors in T. We now omit H from T and connect its two neighbors by an edge. We denote the graph obtained by G_1. If one of the neighbors of H is a 2-edge connected subgraph of G (or possibly, both neighbors), then the edge added will connect the vertices which were connected to H. Now, G_1 satisfies condition (2) of the theorem and $n(G_1) = n - |V(H)|$. Furthermore, the number of leaves in the 2-edge-skeleton tree of G_1 is also l, and the number of non-leaf e-blocks in G_1 is $\alpha - 1$. Therefore we obtain from the induction's hypothesis:

$$rc(G_1) \leq \frac{3}{4}(n - |V(H)|) - \frac{1}{2}l - \frac{3}{2}. \tag{10}$$

Thus,

$$\begin{aligned} rc(G) &\leq rc(G_1) + rc(H) + 1 \\ &\leq \frac{3}{4}(n - |V(H)|) - \frac{1}{2}l - \frac{3}{2} + \frac{3}{4}|V(H)| - 1 + 1 \\ &= \frac{3}{4}n - \frac{1}{2}l - \frac{3}{2} \end{aligned}$$

as required.

■

5 appendix

Let G be a 2-edge connected graph, with $n(G) = 10$. We assume that there is at most one vertex $v \in V(G)$ for which $d(v) = 2$ and $d(u) \geq 3$ for each $u \in V(G) \backslash \{v\}$. We need to show that $rc(G) \leq 5$ (so that we shall have $rc(G) \leq \frac{3}{4}n - \frac{7}{4}$.

We decompose G into its e-blocks, obtaining the 2-edge-skeleton T induced by this decomposition. Evidently, T cannot have a leaf which corresponds to an e-block which is C_3 (otherwise the constraint on the degrees is violated). We analyze G according to the number of vertices in its largest e-block, G^*

- $n(G^*) = 4$. In this case since it is impossible that any leaf of the 2-edge-skeleton T corresponds to a C_3 e-block, the only possible structure of T is a 3 chain, where each vertex in the chain is an e-block with 4 vertices. By the degree constraints at least one of the e-blocks must be K_4. Thus we get $rc(G) = 2 + 2 + 1 = 5$.

- $n(G^*) = 5$. The only possibility in this case is that T is constructed of e-blocks of sizes 5,4 and 3. Since the C_3 component cannot be a leaf, it has to be the middle e-block in the 3-chain T. Then, by the degree constrain the 4-node e-block is K_4, and since for the G^* e-block less than 3 colors are required, we finally get $rc(G) \leq 1 + 1 + 3 = 5$.
- $n(G^*) = 6$. The only possibility in this case is that T has two vertices that correspond to e-blocks with 6 and 5 vertices. Then using proposition 4.1 for 6 and 5-vertex e-blocks we get $rc(G) \leq 2 + 3 = 5$.
- $n(G^*) = 7$. The only possibility in this case is that T has two vertices that correspond to e-blocks with 7 and 4 vertices. For the 7 vertex 2-edge connected subgraph G_i we have $rc(G_i) \leq \frac{3}{4}7 - \frac{7}{4} = 3\frac{1}{2}$, namely $rc(G_i) \leq 3$. The 4 vertex e-block requires no more than 2 colors, so we get $rc(G) \leq 3 + 2 = 5$.
- $n(G^*) = 10$, namely that G has a single e-block. In other words G is a 2-connected graph. Using Theorem 2.1 we get $rc(G) \leq 5$ as required.

References

[1] B. Bollobas, Extremal Graph Theory, *Academic Press, London , (1978)*

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza and R. Yuster, *On rainbow connection* , Electronic Journal of Combinatorics 15(2008), R57.

[3] G. Chartrand, G.L. Jones , K.A. McKeon and P.Zhang, *Rainbow connection in graphs* , Math. Bohem. 133(2008)85-98.

[4] M. Krievlevitch and R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory, 63(2010), 185-191.

[5] Xueliang Li, Yongtang Shi, Yuefang Sun *Rainbow Connections of Graphs: A Survey*, Graphs and Combinatorics, 29(2013), 1-38.

[6] X. Li, S. Liu, L.S. Chandran, R. Mathew, D. Rajendraprasad, *Rainbow connection number and connectivity*, Electron. J. Combin. 19(2012), P20.

[7] I. Schiermeyer, *Rainbow connection in graphs with minimum degree three*, Lecture Notes in Computer Science, 5874(2009), 432-437.

[8] D. West, *Introduction to Graph Theory*, Simon and Schuster A Viacom Company, Upper Saddle River, NJ 07458, (1996).