Decompositions of λK_n into LOE and OLE Graphs

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ABSTRACT. Hein and Sarvate show how to decompose λ copies of a complete graph K_n for some minimal value of λ into so called LOE and OLE graphs. In this paper, we will show that for all possible values of λ, the necessary conditions are sufficient for the LOE and OLE decompositions.

1. Introduction

A *graph* G is an ordered pair (V, E) where V is an n–set (the set of *points*), and E is a subset of the set of the $\binom{n}{2}$ pairs of distinct elements of V (the set of *edges*). This definition can be generalized to that of a *multigraph* by allowing E to be a multiset, where edges can occur with *frequencies* greater than or equal to 1. A complete multigraph λK_n ($\lambda \geq 1$) is a graph on n points with λ edges between every pair of distinct points. Decomposition of a λK_n into subgraphs is a well known classical problem. For an excellent survey on graph decompositions, see **[1]**. Recently several people including Chan, Hein, El-Zanati and Lapchinda have worked on decomposing a λK_n into multigraphs. In fact, similar decompositions have been attempted earlier in various papers. For example, see Priesler and Tarsi **[8]**. Ternary designs also provide such decompositions. For a survey on ternary designs, see Billington **[2, 3]**. A well studied combinatorial design, BIBD, which can be used to find graph decompositions is defined below. On the other hand BIBD itself can be considered as a decomposition of λK_v in complete graphs K_k.

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DEFINITION 1. Given a finite set V of v elements (also called points) and integers k and $\lambda \geq 1$, a *balanced incomplete block design* (BIBD), denoted as BIBD(v, k, λ), is a pair (V, B) where B is a collection of subsets (also called blocks) of V such that every block contains exactly $k < v$ points and every pair of distinct elements is contained in exactly λ blocks.

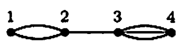
1.1. LOE and OLE Graphs. Following definitions and examples are from Hein and Sarvate [**7**].

DEFINITION 2. *Let $V = \{a, b, c, d\}$. An* LOE graph $\langle a, b, c, d\rangle$ *on V is a graph where the frequencies of edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are 1, 2 and 3 (respectively).*

EXAMPLE 1. *Consider $G_1 = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1,2\}, \{2,3\}, \{2,3\}, \{3,4\}, \{3,4\}, \{3,4\}\}$. Then G_1 is an LOE graph, denoted $\langle 1, 2, 3, 4\rangle$:*

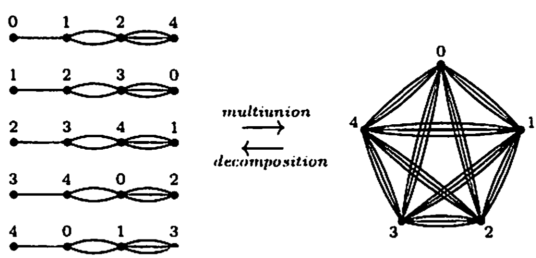
DEFINITION 3. *Let $V = \{a, b, c, d\}$. An* OLE graph $[a, b, c, d]$ *on V is a graph where the frequencies of edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are 2, 1 and 3 (respectively).*

EXAMPLE 2. *Consider $G_2 = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1,2\}, \{1,2\}, \{2,3\}, \{3,4\}, \{3,4\}, \{3,4\}\}$. Then G_2 is an OLE graph, denoted $[1, 2, 3, 4]$:*



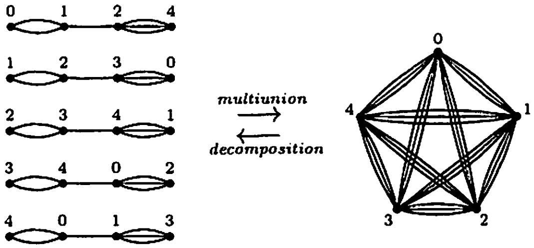
DEFINITION 4. *For any positive integers $n \geq 4$ and $\lambda \geq 3$, an* LOE–decomposition *of λK_n denoted as $LOE(n, \lambda)$ is a collection of LOE graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n.*

EXAMPLE 3. *Considering the set of points to be $V = \mathbb{Z}_5$, the LOE base graph $\langle 0, 1, 2, 4\rangle$ (when developed modulo 5) constitutes an $LOE(5, 3)$.*



DEFINITION 5. *For any positive integers $n \geq 4$ and $\lambda \geq 3$, an* OLE–decomposition *of λK_n denoted as $OLE(n, \lambda)$ is a collection of OLE graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n.*

EXAMPLE 4. *Considering the set of points to be $V = \mathbb{Z}_5$, the OLE base graph $[0, 1, 2, 4]$ (when developed modulo 5) constitutes an $OLE(5, 3)$.*



THEOREM 1 ([7]). *An LOE-decomposition and an OLE-decomposition of a λK_n exist for the minimum value of λ. The minimum value of λ is*

a) $\lambda = 3$, *when* $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$
b) $\lambda = 4$, *when* $n \equiv 3, 6, 7, 10 \pmod{12}$
c) $\lambda = 6$, *when* $n \equiv 2, 11 \pmod{12}$

THEOREM 2 ([7]). *An $LOE(n, 4)$ exists and an $OLE(n, 4)$ exists for $n \equiv 0, 1 \pmod 3$ and $n \geq 4$.*

2. The Necessary Conditions

Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n and six edges in an LOE or an OLE graph, in order for an LOE(n, λ) or an OLE(n, λ) to exist, we must have $\lambda n(n-1) \equiv 0 \pmod{12}$ (where $\lambda \geq 3$ and $n \geq 4$). Specifically, for different λ values, the necessary conditions for n are as follows.

1) If $\lambda \equiv 0 \pmod 6$, there is no condition for n.
2) If $\lambda \equiv 1, 5 \pmod 6$, a necessary condition is $n \equiv 0, 1, 4, 9 \pmod{12}$.
3) If $\lambda \equiv 2, 4 \pmod 6$, a necessary condition is $n \equiv 0, 1, 3, 4 \pmod 6$, i.e. $n \equiv 0, 1 \pmod 3$.
4) If $\lambda \equiv 3 \pmod 6$, a necessary condition is $n \equiv 0, 1 \pmod 4$.

If $\lambda \equiv 3 \pmod 6$, i.e. $\lambda = 6t + 3 = 3(2t + 1)$ $(t \geq 0)$, by taking $2t + 1$ copies of an LOE- and an OLE-decomposition from Theorem 1 part (a), we have the following result.

THEOREM 3. *If $\lambda \equiv 3 \pmod 6$, the above necessary condition for n is sufficient.*

Since case 4 of the necessary conditions is proven to be sufficient for both LOE- and OLE-decompositions in Theorem 3, we will focus on the other three cases in the following sections.

3. LOE Decompositions

3.1. $\lambda \equiv 0, 2, 4 \pmod 6$. One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*, for example, see Stinson [9] for details. We have modified this technique to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non-∞ elements are developed, while ∞ is simply rewritten each time. We further note that the multiplicity of the edges is fixed by position, as per the LOE graph.

THEOREM 4. *An LOE(n, λ) exists for $\lambda \equiv 0 \pmod 6$.*

Proof: From case 1 of the necessary conditions in Section 2, there is no condition for n if $\lambda \equiv 0 \pmod 6$. We first prove there exists an LOE$(n, 6)$.

Let $n = 2t+1 (t \geq 2)$ and $\lambda = 6$. We consider the set V as Z_{2t+1}. The number of graphs required for an LOE$(n, 6)$ is $\frac{6\times(2t+1)\times 2t}{2\times 6} = t(2t+1)$. Thus, we need t base graphs (modulo $2t+1$). The differences we must achieve (modulo $2t + 1$) are $1, 2, \dots, t$. One family of the base graphs is $\{\langle t+1, t, 0, 2t-1\rangle, \langle t+1, t-1, 0, 2t-2\rangle, \langle t+1, t-2, 0, 2t-3\rangle, \dots, \langle t+1, 3, 0, t+2\rangle, \langle t+1, 2, 0, t\rangle, \langle t+1, 1, 0, 2t\rangle\}$.

Let $n = 2t (t \geq 2)$ and $\lambda = 6$. We consider the set V as $Z_{2t-1} \cup \{\infty\}$. The number of graphs required for an LOE$(n, 6)$ is $\frac{6\times 2t\times(2t-1)}{2\times 6} = t(2t-1)$. Thus, we need t base graphs (modulo $2t-1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \dots, t-1$. One family of the base graphs is $\{\langle t-1, t-2, 0, 2t-3\rangle, \langle t-1, t-3, 0, 2t-4\rangle, \dots, \langle t-1, 3, 0, t+2\rangle, \langle t-1, 2, 0, t+1\rangle, \langle t-1, 1, 0, 2t-2\rangle, \langle t-1, 0, \infty, 1\rangle, \langle \infty, t, 0, t-1\rangle\}$. Thus, an LOE$(n, 6)$ exists.

Let $\lambda \equiv 0 \pmod 6 = 6s$. By taking s copies of an LOE$(n, 6)$, we have an LOE(n, λ). □

THEOREM 5. *An LOE(n, λ) exists for $n \equiv 0, 1 \pmod 3$ and $\lambda \equiv 2, 4 \pmod 6$.*

Proof: Let $n \equiv 0, 1 \pmod 3$ and $\lambda \equiv 2 \pmod 6 = 6t + 2 = 6(t-1) + 8$ where $t \geq 1$. Combining an LOE$(n, 6(t-1))$ (by Theorem 4) and two copies of an LOE$(n, 4)$ (by Theorem 2), we have an LOE$(n, 6t + 2)$.

Let $n \equiv 0, 1 \pmod 3$ and $\lambda \equiv 4 \pmod 6 = 6t + 4$. Combining an LOE$(n, 6t)$ (by Theorem 4) and an LOE$(n, 4)$ (by Theorem 2), we have an LOE$(n, 6t + 4)$. □

3.2. $\lambda \equiv 1 \pmod 6$.

THEOREM 6. *An LOE(n, λ) exists for $n \equiv 0, 1, 4, 9 \pmod{12}$ and $\lambda \equiv 1 \pmod 6$.*

Proof: Let $n \equiv 0, 1, 4, 9 \pmod{12}$ and $\lambda \equiv 1 \pmod 6 = 6t + 1$. Combining an LOE$(n, 3)$ (by Theorem 1) and an LOE$(n, 4)$ (by Theorem 2), we have an LOE$(n, 7)$. Combining an LOE$(n, 6(t-1))$ (by Theorem 4) and an LOE$(n, 7)$, we have an LOE$(n, 6(t-1) + 7 = 6t + 1)$. □

3.3. $\lambda \equiv 5 \pmod 6$.

THEOREM 7. *An LOE(n, λ) exists for $n \equiv 0, 1 \pmod{12}$ and $\lambda \equiv 5 \pmod 6$.*

Proof: Let $n \equiv 0 \pmod{12}$ and $\lambda = 5$. We consider the set V as $Z_{12t-1} \cup \{\infty\}$. The number of graphs required for an LOE$(n, 5)$ is $\frac{5 \times 12t \times (12t-1)}{2 \times 6} = 5t(12t-1)$. Thus, we need $5t$ base graphs (modulo $12t-1$). The differences we must achieve (modulo $12t-1$) are $1, 2, \ldots, 6t-1$. For an LOE$(12, 5)$, one family of the base graphs is $\{\langle 5, 0, \infty, 1\rangle, \langle 5, 0, 1, 3\rangle, \langle 5, 0, 2, 10\rangle, \langle 5, 0, 3, 7\rangle, \langle 5, 0, 4, 3\rangle\}$. For an LOE$(12t, 5)$ where $t \geq 2$, one family of the base graphs is $\{\langle 1, s+1, 0, s+2\rangle, \langle 2, s+2, 0, s+3\rangle, \langle 3, s+3, 0, s+4\rangle, \langle 4, s+4, 0, s+5\rangle, \langle 5, s+5, 0, s+1\rangle, \langle 6t-1, 0, \infty, 1\rangle, \langle 6t-1, 0, 6t-5, 12t-9\rangle, \langle 6t-1, 0, 6t-4, 12t-2\rangle, \langle 6t-1, 0, 6t-3, 12t-5\rangle, \langle 6t-1, 0, 6t-2, 12t-7\rangle\}$ where $s = 6i + 1$ for $i = 0, 1, \ldots, t-2$. Thus, an LOE$(n, 5)$ exists for $n \equiv 0 \pmod{12}$.

Similarly, Let $n \equiv 1 \pmod{12}$ and $\lambda = 5$. We consider the set V as Z_{12t+1}. The number of graphs required for an LOE$(n, 5)$ is

$\frac{5\times(12t+1)\times 12t}{2\times 6} = 5t(12t+1)$. Thus, we need $5t$ base graphs (modulo $12t+1$). The differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. One family of the base graphs is $\{\langle 1, s+1, 0, s+2\rangle, \langle 2, s+2, 0, s+3\rangle, \langle 3, s+3, 0, s+4\rangle, \langle 4, s+4, 0, s+5\rangle, \langle 5, s+5, 0, s+1\rangle\}$ where $s = 6i+1$ for $i = 0, 1, \ldots, t-1$. Thus, an LOE$(n, 5)$ exists for $n \equiv 1 \pmod{12}$.

Let $n \equiv 0, 1 \pmod{12}$ and $\lambda \equiv 5 \pmod 6 = 6t+5$. Combining an LOE$(n, 6t)$ (by Theorem 4) and an LOE$(n, 5)$, we have an LOE$(n, 6t+5)$. □

The following lemma shows that an LOE-decomposition may not exist even if the necessary condition is satisfied.

LEMMA 1. *An LOE*$(4, 5)$ *does not exist.*

Proof: Let $V = \{1, 2, 3, 4\}$ and $\lambda = 5$. Notice that no edge say $\{1, 2\}$ can occur singly (appear as a single edge in an LOE graph) in a decomposition two or more times as then edge $\{3, 4\}$ will occur more than five times. Also, no edge can occur triply (appear as triple edges in an LOE graph) in a decomposition two or more times since $\lambda = 5$. For $n = 4$, there are six pairs of edges and five LOE graphs are required in an LOE-decomposition, thus five of the six pairs should appear singly once and five of the six pairs should appear triply once. This implies that at least four pairs appear both singly once and triply once. Since each of these four pairs can not appear singly one more time, but $\lambda = 5$, an LOE$(4, 5)$ does not exist. □

EXAMPLE 5. *An LOE*$(9, 5)$ *exists. Let* $V = \{1, \ldots, 9\}$. *The number of graphs required for an LOE*$(9, 5)$ *is* $\frac{5\times 9\times 8}{2\times 6} = 30$. *A BIBD*$(9, 3, 1)$ *has the following 12 blocks:* $\{1, 2, 3\}$, $\{1, 4, 7\}$, $\{1, 5, 9\}$, $\{1, 6, 8\}$, $\{4, 5, 6\}$, $\{2, 5, 8\}$, $\{2, 6, 7\}$, $\{2, 4, 9\}$, $\{7, 8, 9\}$, $\{3, 6, 9\}$, $\{3, 4, 8\}$, $\{3, 5, 7\}$. *For each block* $\{a, b, c\}$, *construct three LOE graphs* $\langle *, a, b, c\rangle$, $\langle *, c, a, b\rangle$ *and* $\langle *, b, c, a\rangle$. *Notice that each of the three pairs* $\{a, b\}$, $\{a, c\}$ *and* $\{b, c\}$ *appears five times in the three LOE graphs. Since there are 12 blocks from the BIBD*$(9, 3, 1)$, *we have 36 LOE graphs. By removing six LOE graphs and replacing* $*$ *in each of the 30 remaining LOE graph with an appropriate vertex such that the edges not containing* $*$ *removed from the six LOE graphs will appear in the 30 LOE graphs with the same multiplicity. The six LOE graphs to be removed are* $\langle *, 1, 2, 3\rangle$, $\langle *, 4, 5, 6\rangle$, $\langle *, 7, 8, 9\rangle$, $\langle *, 8, 1, 6\rangle$, $\langle *, 2, 4, 9\rangle$ *and* $\langle *, 5, 7, 3\rangle$. *The 30 LOE graphs for an LOE*$(9, 5)$ *are as follows:* $\langle 7, 3, 1, 2\rangle$, $\langle 4, 2, 3, 1\rangle$, $\langle 1, 6, 4, 5\rangle$, $\langle 7, 5, 6, 4\rangle$, $\langle 4, 9, 7, 8\rangle$, $\langle 1, 8, 9, 7\rangle$,

$\langle 2,1,4,7\rangle$, $\langle 8,7,1,4\rangle$, $\langle 9,4,7,1\rangle$, $\langle 3,2,5,8\rangle$, $\langle 9,8,2,5\rangle$, $\langle 6,5,8,2\rangle$, $\langle 2,3,6,9\rangle$, $\langle 4,9,3,6\rangle$, $\langle 1,6,9,3\rangle$, $\langle 6,1,5,9\rangle$, $\langle 8,9,1,5\rangle$, $\langle 7,5,9,1\rangle$, $\langle 4,2,6,7\rangle$, $\langle 3,7,2,6\rangle$, $\langle 5,6,7,2\rangle$, $\langle 7,3,4,8\rangle$, $\langle 1,8,3,4\rangle$, $\langle 5,4,8,3\rangle$, $\langle 2,1,6,8\rangle$, $\langle 5,6,8,1\rangle$, $\langle 8,9,2,4\rangle$, $\langle 5,4,9,2\rangle$, $\langle 2,3,5,7\rangle$ *and* $\langle 8,7,3,5\rangle$.

LEMMA 2. *An LOE*(16, 5) *exists.*

Proof: Let $n = 16$ and $\lambda = 5$. The number of graphs required for an LOE(16, 5) is $\frac{5\times16\times15}{2\times6} = 100$. Let $V_1 = \{\infty_1, \ldots, \infty_9\}$ and $V_2 = \{0, \ldots, 6\}$. An LOE(9, 5) for the points in V_1 results in 30 LOE graphs. Obtain another 70 LOE graphs by developing the following 10 base graphs modulo 7: $\langle 3,0,\infty_1,4\rangle$, $\langle 3,0,\infty_2,4\rangle$, $\langle 3,0,\infty_3,4\rangle$, $\langle 3,0,\infty_4,4\rangle$, $\langle 2,0,\infty_5,4\rangle$, $\langle 2,0,\infty_6,4\rangle$, $\langle 2,0,\infty_7,4\rangle$, $\langle 1,0,\infty_8,4\rangle$, $\langle 1,0,\infty_9,4\rangle$ and $\langle 5,2,0,1\rangle$. The total number of LOE graphs is $30 + 70 = 100$ as required for an LOE(16, 5). □

THEOREM 8. *An LOE*(n, λ) *exists for* $n \equiv 4, 9 \pmod{12}$ *and* $\lambda \equiv 5 \pmod 6$.

Proof: Let $n \equiv 9 \pmod{12} = 12t + 9$ and $\lambda = 5$. The number of graphs required for an LOE$(n, 5)$ is $\frac{5\times(12t+9)\times(12t+8)}{2\times6} = 5(4t+3)(3t+2) = 60t^2 + 85t + 30$. Let $V_1 = \{\infty_1, \ldots, \infty_{12t}\}$ and $V_2 = \{0, \ldots, 8\}$. Since an LOE$(12t, 5)$ exists by Theorem 7 and an LOE(9, 5) exists, obtain an LOE$(12t, 5)$ for the points in V_1 and an LOE(9, 5) for the points in V_2. As a result, there are $5t(12t-1)+30 = 60t^2-5t+30$ LOE graphs. Obtain another $90t$ LOE graphs by developing the following $10t$ base graphs modulo 9: $\langle \infty_{s+2}, 0, \infty_{s+1}, 1\rangle$, $\langle \infty_{s+2}, 0, \infty_{s+3}, 1\rangle$, $\langle 0, \infty_{s+2}, 1, \infty_{s+4}\rangle$, $\langle \infty_{s+4}, 0, \infty_{s+5}, 1\rangle$ and $\langle \infty_{s+4}, 0, \infty_{s+6}, 1\rangle$ where $s = 6i$ for $i = 0, \ldots 2t - 1$. The total number of LOE graphs is $60t^2-5t+30+90t = 60t^2+85t+30$ as required for an LOE$(12t+9, 5)$.

Let $n \equiv 4 \pmod{12} = 12m+4 = 12(m-1)+16 = 12t+16 (t \geq 1)$ and $\lambda = 5$. The number of graphs required for an LOE$(n, 5)$ is $\frac{5\times(12t+16)\times(12t+15)}{2\times6} = 5(3t + 4)(4t + 5) = 60t^2 + 155t + 100$. Let $V_1 = \{\infty_1, \ldots, \infty_{12t}\}$ and $V_2 = \{0, \ldots, 15\}$. An LOE$(12t, 5)$ for the points in V_1 results in $5t(12t - 1) = 60t^2 - 5t$ LOE graphs (by Theorem 7) and an LOE(16, 5) for the points in V_2 results in 100 LOE graphs (by Lemma 2). Obtain another $160t$ LOE graphs by developing the $10t$ base graphs modulo 16 (use the same $10t$ base graphs as in the previous paragraph). The total number of LOE graphs is $60t^2-5t+30+90t = 60t^2-5t+100+160t = 60t^2+155t+100$ as required for an LOE$(12t + 16, 5)$.

Let $n \equiv 4, 9 \pmod{12}$ and $\lambda \equiv 5 \pmod 6 = 6k + 5$. Combining an LOE$(n, 6k)$ (by Theorem 4) and an LOE$(n, 5)$, we have an LOE$(n, 6k + 5)$. □

4. OLE Decompositions

In this section we will show that the first three cases of the necessary conditions for OLE-decompositions in Section 2 are sufficient. Note that the base graphs used in this section are corresponding to OLE graphs, and the multiplicity of the edges is fixed by position, as per the OLE graph.

4.1. $\lambda \equiv 0, 2, 4 \pmod 6$. Apply the same base graphs in Theorem 4 to an OLE$(n, 6)$ and use the same arguments in the proof of the theorem (and replace LOE with OLE), we have the following result.

THEOREM 9. *An OLE(n, λ) exists for* $\lambda \equiv 0 \pmod 6$.

Use the same arguments in the proof of Theorem 5 (and replace LOE with OLE), we have the following theorem.

THEOREM 10. *An OLE(n, λ) exists for* $n \equiv 0, 1 \pmod 3$ *and* $\lambda \equiv 2, 4 \pmod 6$.

4.2. $\lambda \equiv 1 \pmod 6$. Use the same arguments in the proof of Theorem 6 (and replace LOE with OLE), we have the following theorem.

THEOREM 11. *An OLE(n, λ) exists for* $n \equiv 0, 1, 4, 9 \pmod{12}$ *and* $\lambda \equiv 1 \pmod 6$.

4.3. $\lambda \equiv 5 \pmod 6$.

LEMMA 3. *An OLE(n, λ) exists for* $n \equiv 0, 1 \pmod{12}$ *and* $\lambda \equiv 5 \pmod 6$.

Proof: Let $n \equiv 0 \pmod{12}$ and $\lambda = 5$. We consider the set V as $Z_{12t-1} \cup \{\infty\}$. The number of graphs required for an OLE$(n, 5)$ is $\frac{5 \times 12t \times (12t-1)}{2 \times 6} = 5t(12t - 1)$. Thus, we need $5t$ base graphs (modulo $12t - 1$). The differences we must achieve (modulo $12t - 1$) are $1, 2, \ldots, 6t - 1$. For an OLE$(12, 5)$, one family of the base graphs is $\{[5, 0, 4, \infty], [\infty, 5, 0, 4], [5, 0, 4, 1], [5, 3, 0, 1], [2, 3, 0, 9]\}$. For an OLE $(12t, 5)$ where $t \geq 2$, one family of the base graphs is $\{[2s+1, s, 0, s+2], [2s+2, s, 0, s+3], [2s+3, s, 0, s+4], [2s+4, s, 0, s+5], [2s+5, s, 0, s+1], [6t-5, 0, 6t-1, \infty], [\infty, 6t-1, 0, 6t-5], [6t-4, 0, 6t-1, 12t-4], [6t-$

$3, 0, 6t-1, 12t-3], [6t-2, 0, 6t-1, 12t-5]\}$ where $s = 6i+1$ for $i = 0, 1, \ldots, t-2$. Thus, an OLE$(n, 5)$ exists for $n \equiv 0 \pmod{12}$.

Similarly, Let $n \equiv 1 \pmod{12}$ and $\lambda = 5$. We consider the set V as Z_{12t+1}. The number of graphs required for an OLE$(n, 5)$ is $\frac{5\times(12t+1)\times 12t}{2\times 6} = 5t(12t+1)$. Thus, we need $5t$ base graphs (modulo $12t+1$). The differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. One family of the base graphs is $\{[2s+1, s, 0, s+2], [2s+2, s, 0, s+3], [2s+3, s, 0, s+4], [2s+4, s, 0, s+5], [2s+5, s, 0, s+1]\}$ where $s = 6i+1$ for $i = 0, 1, \ldots, t-1$. Thus, an OLE$(n, 5)$ exists for $n \equiv 1 \pmod{12}$.

Let $n \equiv 0, 1 \pmod{12}$ and $\lambda \equiv 5 \pmod 6 = 6t+5$. Combining an OLE$(n, 6t)$ (by Theorem 9) and an OLE$(n, 5)$, we have an OLE$(n, 6t+5)$. □

LEMMA 4. *An OLE*$(4, 5)$ *does not exist.*

Proof: Let $V = \{1, 2, 3, 4\}$ and $\lambda = 5$. Notice that no edge say $\{1, 2\}$ can occur doubly (appear as double edges in an OLE graph) in a decomposition two or more times as then edge $\{3, 4\}$ will occur more than five times. Also, no edge can occur triply (appear as triple edges in an OLE graph) in a decomposition two or more times since $\lambda = 5$. For $n = 4$, there are six pairs of edges and five OLE graphs are required in an OLE-decomposition, thus five of the six pairs should appear doubly once and five of the six pairs should appear triply once. This implies that at least four pairs appear both doubly once and triply once. Also, it is impossible to have five pairs appear both doubly once and triply once since this would imply that the sixth pair appears singly five time (i.e. appears as a single edge in each of the five OLE graphs in the decomposition).

Thus, if a decomposition exists, it must be the case that exactly four pairs appear both doubly once and triply once, and the fifth pair (say e_5) appears doubly once and singly three times, and the sixth pair (say e_6) appears singly two times and triply once. Since e_6 appears triply once, it has to appear in one of the OLE graphs which contains e_5 as the single edge in the decomposition since e_6 can not appear triply once and singly once in the same OLE graph. This implies that e_5 and e_6 share a common vertex. Without loss of generality, let $e_5 = \{1, 2\}$ and $e_6 = \{1, 3\}$. One of the OLE graphs in the decomposition must be $[4, 2, 1, 3]$. Notice that edge $\{4, 2\}$ appears doubly in the OLE graph, which implies that it must appear triply once in one of the other four OLE graphs, and that OLE graph must be $[3,1,2,4]$. This is contradictory to the assumption that $e_6 = \{1,3\}$ appears singly two times and triply once. Therefore, an OLE$(4,5)$ does not exist. □

EXAMPLE 6. *An OLE$(9,5)$ exists. Let $V = \{1,\ldots,9\}$. The number of graphs required for an OLE$(9,5)$ is $\frac{5\times 9\times 8}{2\times 6} = 30$. A K_9 can be decomposed into 12 P_4 paths. Obtain a P_4 decomposition on V such that the middle edges of six P_4s appear as single edges and double edges in six OLE graphs and the middle edges of the other six P_4s appear as triple edges in the same six OLE graphs. The remaining 24 OLE graphs can be constructed as follows: For each P_4 (a,b,c,d) from the path decomposition, construct two OLE graphs $[a,b,c,d]$ and $[d,c,b,a]$. Notice that edges $\{a,b\}$ and $\{c,d\}$ appear five times in these two OLE graphs, and edge $\{b,c\}$ appears two times in these two OLE graphs. Since edge $\{b,c\}$ is the middle edge of the path (a,b,c,d), it appears three times in the six OLE graphs constructed earlier. The 30 OLE graphs for an OLE$(9,5)$ are as follows:* $[9,4,1,2]$, $[2,1,4,9]$, $[5,4,2,8]$, $[8,2,4,5]$, $[7,4,3,9]$, $[9,3,4,7]$, $[1,8,5,6]$, $[6,5,8,1]$, $[4,8,6,9]$, $[9,6,8,4]$, $[9,8,7,1]$, $[1,7,8,9]$, $[3,1,6,4]$, $[4,6,1,3]$, $[9,2,7,5]$, $[5,7,2,9]$, $[8,3,5,9]$, $[9,5,3,8]$, $[2,5,1,9]$, $[9,1,5,2$, $[7,6,2,3]$, $[3,2,6,7]$, $[9,7,3,6]$, $[6,3,7,9]$, $[1,4,2,7]$, $[2,4,3,5]$, $[3,4,1,6]$, $[5,8,6,2]$, $[6,8,7,3]$ *and* $[7,8,5,1]$.

LEMMA 5. *An OLE$(16,5)$ exists.*

Proof: Let $n = 16$ and $\lambda = 5$. The number of graphs required for an OLE$(16,5)$ is $\frac{5\times 16\times 15}{2\times 6} = 100$. Let $V_1 = \{\infty_1,\ldots,\infty_9\}$ and $V_2 = \{0,\ldots,6\}$. An OLE$(9,5)$ for the points in V_1 results in 30 OLE graphs. Obtain 7 OLE graphs by developing the base graph $[0,2,3,6]$ modulo 7. Obtain the remaining 63 OLE graphs by developing the following 9 base graphs modulo 7: $[\infty_1,0,1,\infty_2]$, $[\infty_2,0,1,\infty_3]$, $[\infty_3,0,1,\infty_4]$, $[\infty_4,0,1,\infty_5]$, $[\infty_5,0,2,\infty_6]$, $[\infty_6,0,2,\infty_7]$, $[\infty_7,0,2,\infty_8]$, $[\infty_8,0,3,\infty_9]$, and $[\infty_9,0,3,\infty_1]$. The total number of OLE graphs is $30+7+63 = 100$ as required for an OLE$(16,5)$. □

LEMMA 6. *An OLE$(21,5)$ exists.*

Proof: Let $n = 21$ and $\lambda = 5$. The number of graphs required for an OLE$(21,5)$ is $\frac{5\times 21\times 20}{2\times 6} = 175$. Let $V_1 = \{(I,0),\ldots,(I,8)\}$ and $V_2 = \{(II,0),\ldots,(II,11)\}$. An OLE$(9,5)$ for the points in V_1 results in 30 OLE graphs, and an OLE$(12,5)$ for the points in V_2 results in 55 OLE graphs. Notice that there are 108 pairs between a vertex in V_1 and a vertex in V_2. Define the difference of each pair $((I,i),(II,j))$ to be $(j-i)$ modulo 12 where $i=0,\ldots,8$ and $j=0,\ldots,11$. There are exactly 9 pairs for each difference from 0 to 11: 9 pairs $((I,i),(II,i))$ where $i=0,\ldots 8$ have difference 0, 9 pairs $((I,i),(II,i+1))$ where $i=0,\ldots 8$ have difference 1, $\ldots$, and 9 pairs $((I,i),(II,i+11))$ where $i=0,\ldots 8$ have difference 11 (note that $i+11$ sums modulo 12). Developing the following 10 base graphs such that each base graph is used to develop eight more OLE graphs (so nine OLE graphs total including the base graph), where the vertex in V_1 is developed modulo 9 (only develop the second component of a vertex) and the vertex in V_2 is developed modulo 12 (only develop the second component of a vertex and it's partially developed since only eight more OLE graphs are developed from a base graph): $[(II,8),(I,0),(II,11),(I,2)]$, $[(II,6),(I,0),(II,11),(I,4)]$, $[(II,4),(I,0),(II,11),(I,6)]$, $[(II,2),(I,0),(II,11),(I,8)]$, $[(II,2),(I,2),(II,1),(I,0)]$, $[(I,2),(II,11),(I,1),(II,9)]$, $[(I,4),(II,11),(I,1),(II,7)]$, $[(I,6),(II,11),(I,1),(II,5)]$, $[(I,8),(II,11),(I,1),(II,3)]$, and $[(I,0),(II,1),(I,3),(II,3)]$. As a result, we have 90 OLE graphs. The total number of OLE graphs is $20+55+90=175$ as required for an OLE(21, 5). □

LEMMA 7. *An OLE*(28, 5) *exists.*

Proof: Let $n=28$ and $\lambda=5$. The number of graphs required for an OLE(28, 5) is $\frac{5\times 28\times 27}{2\times 6}=315$. Let $V_1=\{\infty_1,\ldots,\infty_9\}$ and $V_2=\{0,\ldots,18)\}$. An OLE(9, 5) for the points in V_1 results in 30 OLE graphs. To obtain the remaining 285 OLE graphs, develop the following 15 base graphs modulo 19: $[3,0,1,4]$, $[4,0,1,5]$, $[5,0,1,6]$, $[6,0,1,7]$, $[7,0,1,8]$, $[8,0,2,10]$, $[\infty_1,0,2,\infty_2]$, $[\infty_2,0,2,\infty_3]$, $[\infty_3,0,2,\infty_4]$, $[\infty_4,0,2,\infty_5]$, $[\infty_5,0,9,\infty_6]$, $[\infty_6,0,9,\infty_7]$, $[\infty_7,0,9,\infty_8]$, $[\infty_8,0,9,\infty_9]$, and $[\infty_9,0,9,\infty_1]$. The total number of OLE graphs is $30+15\times 19=315$ as required for an OLE(28, 5). □

THEOREM 12. *An OLE*(n,λ) *exists for* $n\equiv 4,9 \pmod{12}$ *and* $\lambda\equiv 5 \pmod 6$.

Proof: We use the same idea as in Lemma 6. Let $n\equiv 9 \pmod{12} = 12t+9$ and $\lambda=5$. The number of graphs required for an OLE$(n,5)$ is $\frac{5\times(12t+9)\times(12t+8)}{2\times 6}=5(4t+3)(3t+2)=60t^2+85t+30$. Let $V_1=\{(I,0),\ldots,(I,8)\}$ and $V_2=\{(II,0),\ldots,(II,12t-1)\}$. Since an OLE$(12t,5)$ exists by Lemma 3 and an OLE(9, 5) exists, obtain an OLE(9, 5) for the points in V_1 and an OLE$(12t,5)$ for the points in

V_2. As a result, there are $5t(12t-1)+30 = 60t^2-5t+30$ OLE graphs. Obtain another $90t$ OLE graphs by developing the following $10t$ base graphs such that each base graph is used to develop eight more OLE graphs (so nine OLE graphs total including the base graph), where the vertex in V_1 is developed modulo 9 and the vertex in V_2 is developed modulo $12t$: $[(II, 12s-4), (I, 0), (II, 12s-1), (I, 2)]$, $[(II, 12s-6), (I, 0), (II, 12s-1), (I, 4)]$, $[(II, 12s-8), (I, 0), (II, 12s-1), (I, 6)]$, $[(II, 12s-10), (I, 0), (II, 12s-1), (I, 8)]$, $[(II, 12s-10), (I, 2), (II, 12s+1), (I, 0)]$, $[(I, 2), (II, 12s-1), (I, 1), (II, 12s-3)]$, $[(I, 4), (II, 12s-1), (I, 1), (II, 12s-5)]$, $[(I, 6), (II, 12s-1), (I, 1), (II, 12s-7)]$, $[(I, 8), (II, 12s-1), (I, 1), (II, 12s-9)]$, and $[(I, 0), (II, 12s+1), (I, 3), (II, 12s-9)]$ where $s = 1, \dots t$ (note that when $s = t$, the third point $(II, 12s+1)$ in the fifth and the tenth base graphs should be replaced by $(II, 12s-11)$). As a result, we have $90t$ OLE graphs. The total number of OLE graphs is $60t^2-5t+30+90t = 60t^2+85t+30$ as required for an OLE$(12t+9, 5)$.

Let $n \equiv 4 \pmod{12} = 12m+4 = 12(m-1)+16 = 12t+16 (t \geq 1)$ and $\lambda = 5$. The number of graphs required for an OLE$(n, 5)$ is $\frac{5\times(12t+16)\times(12t+15)}{2\times 6} = 5(3t+4)(4t+5) = 60t^2+155t+100$. Lemma 7 proves the existence of an OLE$(28, 5)$ when $t = 1$. For $t \geq 2$, let $V_1 = \{(I, 0), \dots, (I, 15)\}$ and $V_2 = \{(II, 0), \dots, (II, 12t-1)\}$. An OLE$(12t, 5)$ for the points in V_2 results in $5t(12t-1) = 60t^2-5t$ OLE graphs (by Lemma 3) and an OLE$(16, 5)$ for the points in V_1 results in 100 OLE graphs (by Lemma 5). Obtain another $160t$ OLE graphs by developing the $10t$ base graphs such that each base graph is used to develop 15 more OLE graphs (so 16 OLE graphs total including the base graph), where the vertex in V_1 is developed modulo 16 and the vertex in V_2 is developed modulo $12t$ (use the same $10t$ base graphs as in the previous paragraph). The total number of OLE graphs is $60t^2-5t+30+90t = 60t^2-5t+100+160t = 60t^2+155t+100$ as required for an OLE$(12t+16, 5)$.

Let $n \equiv 4, 9 \pmod{12}$ and $\lambda \equiv 5 \pmod 6 = 6k+5$. Combining an OLE$(n, 6k)$ (by Theorem 9) and an OLE$(n, 5)$, we have an OLE$(n, 6k+5)$. □

5. Summary

In this paper, we addressed the necessary conditions for the existence of LOE and OLE decompositions in general (for all possible λ values), and we proved that an LOE$(4, 5)$ does not exist and an

OLE(4, 5) does not exist. For all other cases, we proved that the necessary conditions established in Section 2 are sufficient for the LOE and OLE decompositions.

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