ON BI-PERIODIC FIBONACCI AND LUCAS NUMBERS BY MATRIX METHOD

ELIF TAN

ABSTRACT. In this paper, we define a new matrix identity for bi-periodic Fibonacci and Lucas numbers. By using the matrix method, we give simple proofs of several properties of these numbers. Moreover, we obtain a new binomial sum formula for bi-periodic Fibonacci and Lucas numbers which generalize the former results.

1. INTRODUCTION

The Fibonacci numbers F_n defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Its associated numbers known as Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. These numbers are famous for possessing interesting properties and important applications to diverse disciplines such as mathematics and computer science. See [16] and [9] for additional references and history.

There are a lot of generalizations of the Fibonacci and Lucas sequences. Horadam [5] considered a generalized sequence $\{W_n\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2},\ n \geq 2$$

with the initial conditions $W_0 = a$ and $W_1 = b$, where a, b, p, q are arbitrary integers. If we take $a = 0$, $b = 1$ in $\{W_n\}$, we get the generalized Fibonacci sequence $\{U_n\}$ and if we take $a = 2$, $b = p$ in $\{W_n\}$, we get the generalized Lucas sequence $\{V_n\}$.

It is well-known that for the matrix $M := \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}$, we have

$$M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}.$$

2000 *Mathematics Subject Classification.* 11B39, 40C05 .

Key words and phrases. Fibonacci sequence, bi-periodic sequence, matrix method.

The matrix method is extremely useful for obtaining lots of famous Fibonacci properties, such as Cassini's identity, D'Ocagne's identity, and convolution property, etc. See [4, 7, 10, 6] for detailed history of the matrix technique.

In this paper, we consider a further generalization of the Fibonacci and Lucas sequences, named as, bi-periodic Fibonacci and bi-periodic Lucas sequences. They are emerged as a generalization of the best known sequences in the literature, such as Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, etc. Edson and Yayenie [3] introduced the bi-periodic Fibonacci sequence as follows:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence. The Binet formula of the sequence $\{q_n\}$ is given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{2}$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ that is, α and β are the roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

Similar to (1), by taking initial conditions $p_0 = 2$ and $p_1 = a$, Bilgici [1] introduced the bi-periodic Lucas sequence as:

$$p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \\ bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \end{cases}, \ n \geq 2. \tag{3}$$

It gives the classical Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$. The Binet formula of the sequence $\{p_n\}$ is given by

$$p_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^n + \beta^n) \tag{4}$$

where α and β are defined in (2). Let $\Delta := a^2b^2+4ab \neq 0$. Note that $\alpha+\beta = ab$, $\alpha - \beta = \sqrt{\Delta}$ and $\alpha\beta = -ab$. For some properties of these sequences see [3, 17, 1, 12, 15], and see also [11, 13] for another generalization of these sequences.

Recently in [14], Tan and Ekin gave some matrix identities for the special cases of the sequences in [11], and obtained some properties for the even indices of these sequences. In [2], by using the determinant sum property of matrices, Choo derived some general identities which reduce some known results in [3, 1] as a special case. Also in [8], Jang and Jun gave a linearization of the bi-periodic Fibonacci sequences and used this result to obtain a matrix identity for this sequence.

Here, we defined a new matrix identity for the bi-periodic Fibonacci sequence and the bi-periodic Lucas sequences. By using the matrix method we give simple proofs of several identities for these numbers. Moreover, we obtain a new binomial sum formula for the bi-periodic Fibonacci and Lucas numbers which generalize the former results. The results of this paper not only give alternative proofs of known identities but also include new results.

2. A MATRIX IDENTITY FOR THE BI-PERIODIC FIBONACCI NUMBERS

Let $S := \begin{pmatrix} ab & ab \\ 1 & 0 \end{pmatrix}$, then we have

$$S^n = (ab)^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} b^{\xi(n)} q_{n+1} & a^{\xi(n)} b q_n \\ a^{-\xi(n+1)} q_n & b^{\xi(n)} q_{n-1} \end{pmatrix} \tag{5}$$

which can be easily proven by induction and by the help of (1).

We use the matrix identity (5) to get the following theorem easily which is given in [3, Theorem 3,5,6].

Theorem 2.1. *For any nonnegative integers m and n, we have*

i) $(-1)^n a = a^{1-\xi(n)} b^{\xi(n)} q_{n+1} q_{n-1} - a^{\xi(n)} b^{1-\xi(n)} q_n^2$

ii) $(-1)^n a^{\xi(m-n)} q_{m-n}$
$= a^{\xi(mn+m)} b^{\xi(mn+n)} q_m q_{n+1} - a^{\xi(mn+n)} b^{\xi(mn+m)} q_{m+1} q_n, m \geq n$

iii) $a^{\xi(n)} q_n = a^{\xi(mn)} b^{\xi(mn+n)} q_{n-m+1} q_m + a^{\xi(mn+n)} b^{\xi(mn)} q_{n-m} q_{m-1}, n \geq m$

iv) $a^{\xi(m+n)} q_{m+n} = a^{\xi(mn+m)} b^{\xi(mn+n)} q_{n+1} q_m + a^{\xi(mn+n)} b^{\xi(mn+m)} q_{m-1} q_n.$

Proof. By taking the determinant of both sides of (5), we get the result (*i*) which is the *Cassini's identity* for the sequence $\{q_n\}$.

Since matrix S is invertible,

$$S^{-n} = \frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{(-ab)^n} \begin{pmatrix} b^{\xi(n)} q_{n-1} & -a^{\xi(n)} b q_n \\ -a^{-\xi(n+1)} q_n & b^{\xi(n)} q_{n+1} \end{pmatrix}.$$

Considering $S^{n-m} = S^n S^{-m}$, by using the matrix identity (5) and by equating the $(1,2)$ entries of the matrices on both sides and by using the parity function property $\xi(m-n) + \xi(m) - \xi(n) = 2\xi(mn+m)$, we get the identity (*ii*) which is the *d'Ocagne's identity* for the sequence $\{q_n\}$.

Also, considering $S^{n-1} = S^{n-m} S^{m-1}$, by using the matrix identity (5) and by equating the $(1,1)$ entries of the matrices on both sides, we get the result (*iii*) which is the *convolution property* for the sequence $\{q_n\}$.

Similarly, since $S^{n+m} = S^n S^m$, by using the matrix identity (5) and by equating $(1,2)$ entries of the matrices on both sides and also by using the parity function property $\xi(m+n) + \xi(m) - \xi(n) = 2\xi(mn+m)$, we get the result (iv). □

The following theorem gives the relation between bi-periodic Fibonacci and Lucas numbers which can be seen in [1, Theorem 3], but now we give this identity by using the matrix method.

Theorem 2.2. *For every integer n, we have*

$$p_n = q_{n+1} + q_{n-1}.$$

Proof. Let α and β denote the roots of $p(x) = x^2 - abx - ab$ that is the characteristic polynomial of S. The eigenvalues and corresponding eigenvectors of the matrix S are $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and $\nu_1 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$, $\nu_2 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}$. Therefore; $P^{-1}SP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}$. It follows $S^n = P\begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix}P^{-1}$.

If we consider the above matrix equality and the matrix identity (5), the trace of the matrix S^n can be obtained easily as:

$$tr(S^n) = \alpha^n + \beta^n = (ab)^{\lfloor \frac{n}{2} \rfloor} b^{\xi(n)} (q_{n+1} + q_{n-1}).$$

Then, by using the Binet formula of bi-periodic Lucas sequence we obtain the desired result. □

3. A matrix identity for the bi-periodic Lucas numbers

By using the matrix S, we obtain a similar matrix identity for the bi-periodic Lucas numbers as:

$$\begin{pmatrix} ab & 2ab \\ 2 & -ab \end{pmatrix} \begin{pmatrix} ab & ab \\ 1 & 0 \end{pmatrix}^n = (ab)^{\lfloor \frac{n+1}{2} \rfloor} \begin{pmatrix} b^{\xi(n+1)} p_{n+1} & a^{\xi(n+1)} b p_n \\ a^{-\xi(n)} p_n & b^{\xi(n+1)} p_{n-1} \end{pmatrix} \tag{6}$$

which can be easily proven by induction and by the help of (3).

The following theorem gives the Cassini's identity for the bi-periodic Lucas sequence $\{p_n\}$ which can be easily obtained by taking the determinant of both sides of (6).

Theorem 3.1. [1, Corollary 2] *For any positive integer n, we have*

$$\left(\frac{b}{a}\right)^{\xi(n+1)} p_{n+1}p_{n-1} - \left(\frac{b}{a}\right)^{\xi(n)} p_n^2 = (-1)^{n+1}(ab+4).$$

4. A MATRIX IDENTITY INVOLVING BI-PERIODIC FIBONACCI AND LUCAS NUMBERS

Now, we find another matrix identity to prove some relations between the bi-periodic Fibonacci and the bi-periodic Lucas numbers.

Setting $T := \frac{1}{2}\begin{pmatrix} ab & \Delta \\ 1 & ab \end{pmatrix}$, we have

$$T^n = \frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{2}\begin{pmatrix} b^{\xi(n)}p_n & \frac{\Delta}{a^{1-\xi(n)}}q_n \\ \frac{1}{a^{1-\xi(n)}}q_n & b^{\xi(n)}p_n \end{pmatrix} \tag{7}$$

which can be easily proven by induction and by the help of (1) and (3).

By using the matrix identity (7), we get the following theorem which can be found in [1, Theorem 5 and Corollary 3,4,6].

Theorem 4.1. *For any positive integers m and n, we have*

i) $4(-1)^n = \left(\frac{b}{a}\right)^{\xi(n)} p_n^2 - \frac{\Delta}{a^2}\left(\frac{a}{b}\right)^{\xi(n)} q_n^2$

ii) $p_{n+m} = \frac{1}{2}\left(\left(\frac{b}{a}\right)^{\xi(n)\xi(m)} p_np_m + \frac{\Delta}{a^2}\left(\frac{a}{b}\right)^{1-\xi(n+1)\xi(m+1)} q_nq_m\right)$

iii) $q_{n+m} = \frac{1}{2}\left(\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} q_np_m + \left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} q_mp_n\right)$

iv) $p_{n-m} = \frac{(-1)^m}{2}\left(\left(\frac{b}{a}\right)^{\xi(n)\xi(m)} p_np_m - \frac{\Delta}{a^2}\left(\frac{a}{b}\right)^{1-\xi(n+1)\xi(m+1)} q_nq_m\right), n \geq m$

v) $q_{n-m} = \frac{(-1)^m}{2}\left(\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} q_np_m - \left(\frac{b}{a}\right)^{\xi(m+1)\xi(n)} q_mp_n\right), n \geq m$

vi) $\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} q_np_m = q_{n+m} + (-1)^m q_{n-m},\ n \geq m$

vii) $\left(\frac{b}{a}\right)^{\xi(n)\xi(m)} p_np_m = p_{n+m} + (-1)^m p_{n-m},\ n \geq m.$

Proof. By taking the determinant of both sides of (7), we get the desired result (i).

Considering $T^{n+m} = T^nT^m$, by using the matrix identity (7) and by equating the corresponding entries of the matrices on both sides and using the parity function properties

$$\xi(n) + \xi(m) - \xi(n+m) = 2\xi(n)\xi(m)$$

and

$$\xi(n) + \xi(m) + \xi(n+m) = 2(1 - \xi(n+1)\xi(m+1))$$

we get (ii) and by using the property

$$\xi(n) - \xi(m) + \xi(n+m) = 2\xi(m+1)\xi(n)$$

we get (iii).

Similarly, let consider $T^{n-m} = T^n T^{-m} = T^n (T^m)^{-1}$ and by using the matrix identity (7), we get the results (iv) and (v).

Finally, consider the equation

$$T^{n+m} + (-1)^m T^{n-m} = T^n T^m + (-1)^m T^n (T^m)^{-1}$$

then we get the identities (vi) and (vii). □

Now, by using the matrix T we give the following binomial sum formula for the bi-periodic Fibonacci and Lucas sequences. For this purpose we need the following identity:

$$T^n = (ab)^{\lfloor \frac{n}{2} \rfloor} \left(a^{-\xi(n+1)} q_n T + b^{\xi(n)} q_{n-1} I \right) \tag{8}$$

where I is a 2×2 unit matrix.

Theorem 4.2. *For any nonnegative integers n, r and m with $m > 1$. Then we have*

$$q_{mn+r} = \frac{a^{1-\xi(mn+r)}}{(ab)^{\lfloor \frac{mn+r}{2} \rfloor}} \sum_{k=0}^{n} \binom{n}{k} q_m^k q_{m-1}^{n-k} q_{k+r} \delta[m,n,r,k] \tag{9}$$

$$p_{mn+r} = \frac{b^{-\xi(mn+r)}}{(ab)^{\lfloor \frac{mn+r}{2} \rfloor}} \sum_{k=0}^{n} \binom{n}{k} q_m^k q_{m-1}^{n-k} p_{k+r} \gamma[m,n,r,k] \tag{10}$$

where

$$\delta[m,n,r,k] \; : = (ab)^{\lfloor \frac{k+r}{2} \rfloor + n \lfloor \frac{m}{2} \rfloor} a^{-\xi(m+1)k-1+\xi(k+r)} b^{\xi(m)(n-k)}$$

$$\gamma[m,n,r,k] \; : = (ab)^{\lfloor \frac{k+r}{2} \rfloor + n \lfloor \frac{m}{2} \rfloor} a^{-\xi(m+1)k} b^{\xi(m)(n-k)+\xi(k+r)}.$$

Proof. By considering the matrix identity (7)

$$T^{mn+r} = \frac{(ab)^{\lfloor \frac{mn+r}{2} \rfloor}}{2} \begin{pmatrix} b^{\xi(mn+r)} p_{mn+r} & \frac{\Delta}{a^{1-\xi(mn+r)}} q_{mn+r} \\ \frac{1}{a^{1-\xi(mn+r)}} q_{mn+r} & b^{\xi(mn+r)} p_{mn+r} \end{pmatrix}$$

and the identity (8) we have

$$\begin{aligned} T^{mn+r} &= \left((ab)^{\lfloor \frac{m}{2} \rfloor} \left(a^{-\xi(m+1)} q_m T + b^{\xi(m)} q_{m-1} I \right) \right)^n T^r \\ &= \sum_{k=0}^{n} \binom{n}{k} (ab)^{n \lfloor \frac{m}{2} \rfloor} a^{-\xi(m+1)k} b^{\xi(m)(n-k)} q_m^k q_{m-1}^{n-k} T^{k+r} \end{aligned}$$

By equating the corresponding entries we obtain the desired results. □

Note that, the above theorem generalizes the results

$$a^{\xi(r)} q_{2n+r} = \sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+r)} (ab)^{\lfloor \frac{k}{2} \rfloor + \xi(k)\xi(r)} q_{k+r}$$

and

$$b^{\xi(r)}p_{2n+r}=\sum_{k=0}^{n}\binom{n}{k}b^{\xi(k+r)}(ab)^{\lfloor\frac{k}{2}\rfloor+\xi(k)\xi(r)}p_{k+r}$$

which can be found in [3, Remark 1] and [1, Theorem 7].

5. Acknowledgement

We thank the referee for helpful suggestions and comments.

References

[1] Bilgici, G. Two generalizations of Lucas sequence. Appl. Math. Comput. 245, 526–538 (2014)

[2] Choo, Y. On the Generalizations of Fibonacci Identities. Results. Math. (2015) doi:10.1007/s00025-015-0515-6

[3] Edson, M., Yayenie, O. A new generalizations of Fibonacci sequences and extended Binet's Formula. Integers. 9 (A48), 639-654 (2009)

[4] Gould, H.W. A history of the Fibonacci Q-matrix and a higher-dimensional problem. Fibonacci Quart. 19 (3), 250-257 (1981)

[5] Horadam, A.F. A generalized Fibonacci sequence. Amer. Math. Monthly. 68 (5), 455-459 (1961)

[6] Johnson, R.C. Fibonacci Numbers and Matrices. http://www.maths.dur.ac.uk/~dma0rcj/PED/fib.pdf (2009) Accessed 15 June 2009

[7] Kalman, D., Mena, R. The Fibonacci Numbers-Exposed. Mathematics Magazine. 76, 167-181 (2003)

[8] Jang, H.Y., Jun, P. S. Linearlization of Generalized Fibonacci Sequeces. Korean J. Math. 22 (3), 443-454 (2014)

[9] Koshy, T. Fibonacci and Lucas Numbers with Applications. Wiley, New York (2001)

[10] Melham, R. S., Shannon, A. G. Some summation identities using generalized Q-matrices. The Fibonaci Quaterly. 33 (1), 64-73 (1995)

[11] Panario, D., Sahin, M., Wang, Q. A Family of Fibonacci-like Conditional Sequences. Integers. 13, A78 (2013).

[12] Ramìrez, J. Bi-periodic incomplete Fibonacci sequence, Ann. Math. Inform. 42, 83-92 (2013)

[13] Sahin, M. The Gelin-Cesaro identity in some conditional sequences, Hacet. J. Math. Stat. 40(6), 855-861 (2011)

[14] Tan, E., Ekin, A.B. Some Identities On Conditional Sequences By Using Matrix Method. Miskolc Mathematical Notes. accepted (2016)

[15] Tan, E., Ekin, A.B. Bi-Periodic Incomplete Lucas Sequences. Ars Combinatoria. 123, 371-380 (2015)

[16] Vajda S. Fibonacci & Lucas Numbers and The Golden Section, Theory and Applications. Ellis Horwood Ltd. Chishester (1989)

[17] Yayenie, O. A note on generalized Fibonacci sequence, Appl. Math. Comput. 217, 5603-5611 (2011)

Department of Mathematics, Ankara University, Ankara, Turkey
E-mail address: etan@ankara.edu.tr