On some 2-tight sets of polar spaces

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium, E-mail: bdb@cage.ugent.be

Abstract

Let Π be a finite polar space of rank $n \geq 2$ fully embedded into a projective space Σ. In this note, we determine all tight sets of Π of the form $(\Sigma_1 \cap \mathcal{P}) \cup (\Sigma_2 \cap \mathcal{P})$, where $\mathcal{P}$ denotes the point set of Π and Σ_1, Σ_2 are two mutually disjoint subspaces of Σ. In this way, we find two families of 2-tight sets of elliptic polar spaces that were not described before in the literature.

Keywords: tight set, polar space
MSC2000: 51A50, 51E20, 05B25

1 Introduction

Let q be a prime power and $n \in \mathbb{N} \setminus \{0, 1\}$. Suppose Π is one of the following finite polar spaces of rank n:

$$W(2n-1,q),\ Q(2n,q),\ Q^+(2n-1,q),\ Q^-(2n+1,q),\ H(2n-1,q),\ H(2n,q).$$

Throughout this note, we will implicitly assume that q is a square if Π is a Hermitian polar space. Let $\mathcal{P}$ denote the point set of Π and Σ the ambient projective space of Π. With Π, there is associated a polarity ζ of Σ, which is degenerate if $\Pi = Q(2n, q)$ and q is even. If Σ' is a subspace of Σ, then $\Sigma' \cap \Pi$ denotes the pair $(Y, \mathcal{S})$, where Y [resp. $\mathcal{S}$] denotes the set of points [resp. subspaces] of Π contained in Σ'. If $\Sigma' \cap \mathcal{P}$ is not a subspace, then $\Sigma' \cap \Pi$ is a (possibly degenerate) polar space. More background information on the properties of quadrics and Hermitian varieties that we will use throughout this note can be found in the book [9]. Now, put

$$\lambda_n := \frac{q^n - 1}{q - 1} \qquad \text{and} \qquad \lambda_{n-1} := \frac{q^{n-1} - 1}{q - 1}.$$

Then λ_n is the number of points of a generator of Π. If X is a set of points of Π, then the number of ordered pairs of distinct collinear points of X is bounded above by

$$\lambda_{n-1} \cdot |X| \cdot \left(\frac{|X|}{\lambda_n} + (q-1)\right).$$

If equality holds, then $i := \frac{|X|}{\lambda_n} \in \mathbb{N}$ and X is called *i-tight*. The empty set and $\mathcal{P}$ are examples of tight sets. We will call them the *trivial tight sets*. Any other tight set will be called *nontrivial*. Tight sets were introduced by Payne [12] for generalized quadrangles and by Drudge [7] for arbitrary polar spaces. We refer to these references for proofs of the above-mentioned facts. Drudge [7] also observed that the so-called Cameron-Liebler line classes of $\mathrm{PG}(3,q)$ with parameter i (as introduced in [4]) correspond via the Klein correspondence to i-tight sets of the hyperbolic quadric $Q^+(5,q)$.

This note arose from a successive attempt of the author to determine those tight sets that can be obtained by intersecting the polar space Π with two disjoint subspaces Σ_1 and Σ_2 of Σ, and the subsequent observation that two families of tight sets that arise in this way had not yet been described in the literature. The main difficulty in determining all these tight sets seems to come from the fact that one needs to get control over the (initially numerous) possibilities for the pair (Σ_1, Σ_2). Indeed, the dimensions of Σ_1 and Σ_2 are not known in advance as well as their mutual position with respect to the polarity ζ. Also the types of the intersections $\Sigma_1 \cap \Pi$ and $\Sigma_2 \cap \Pi$ are initially unknown. Here is our complete classification:

Theorem 1.1 *Suppose X is a nontrivial tight set of Π of the form $(\Sigma_1 \cap \mathcal{P}) \cup (\Sigma_2 \cap \mathcal{P})$, where Σ_1 and Σ_2 are two disjoint subspaces of Σ. Then one of the following cases occurs:*

(1) *X is a generator. Then X is 1-tight.*

(2) *$\Pi = Q(2n,q)$ and X is the point set of a hyperbolic quadric $Q^+(2n-1,q) \subset Q(2n,q)$. Then X is $(q^{n-1}+1)$-tight.*

(3) *$\Pi = H(2n,q)$ and X is the point set of a Hermitian variety $H(2n-1,q) \subset H(2n,q)$. Then X is $(q^{n-\frac{1}{2}}+1)$-tight.*

(4) *$\Pi = Q^-(2n+1,q)$ and X is the point set of a parabolic quadric $Q(2n,q) \subset Q^-(2n+1,q)$. Then X is (q^n+1)-tight.*

(5) *$\Pi = Q^-(2n+1,q)$ and X is the point set of a hyperbolic quadric $Q^+(2n-1,q) \subset Q^-(2n+1,q)$. Then X is $(q^{n-1}+1)$-tight.*

(6) *X is the the union of two disjoint generators. Then X is 2-tight.*

(7) *$\Pi = Q^-(5,q)$ and X is the union of a line L and the point set of a hyperbolic quadric $Q^+(3,q) \subset Q^-(5,q)$ disjoint from L. Then X is $(q+2)$-tight.*

(8) $n = 2m$ *is even,* $\Pi = W(4m-1, q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$, $\Sigma_2 = \Sigma_1^\zeta$ *and* $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong W(2m-1, q)$. *Then* X *is 2-tight.*

(9) $n = 2m$ *is even,* $\Pi = Q^-(4m+1, q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n$, $\Sigma_2 = \Sigma_1^\zeta$ *and* $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q(2m, q)$. *Then* X *is 2-tight.*

(10) $n = 2m-1$ *is odd,* $\Pi = Q^-(4m-1, q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n$, $\Sigma_2 = \Sigma_1^\zeta$ *and there exists an* $i \in \{1, 2\}$ *such that* $\Sigma_i \cap \Pi \cong Q^+(2m-1, q)$ *and* $\Sigma_{3-i} \cap \Pi \cong Q^-(2m-1, q)$. *Then* X *is 2-tight.*

Since Σ_1 and Σ_2 were allowed to be empty subspaces, Theorem 1.1 includes the case of tight sets that can be obtained by intersecting $\mathcal{P}$ with only one subspace. The nontrivial tight sets that can be obtained in this way are described in (1)–(5) of Theorem 1.1. For the examples described in (6)–(10), both Σ_1 and Σ_2 need to be nonempty.

Several constructions of tight sets can be found in the papers [1, 2, 12, 13]. The straightforward examples mentioned in (1)–(7) of Theorem 1.1 are already described there (either explicitly or implicitly). The most interesting examples in Theorem 1.1 seem to be those mentioned in (8), (9) and (10). Those living in symplectic polar spaces (case (8)) were already described in [5]. Although the constructions in (9) and (10) are similar to those of (8), these examples had not yet been recognized as tight sets[1]. Via the connection with minihypers, the classification in [5] required (among other things) to look at tight sets that arise as unions of certain subspaces of Σ that are contained in Π. The examples in (8) indeed arise in this way, but not those of (9) and (10).

The examples described in (8), (9) and (10) are examples of 2-tight sets that cannot be written as unions of two disjoint generators. In the literature, there are a number of results stating that every i-tight set with $i \leq N$ where $N \in \mathbb{N} \setminus \{0, 1\}$ is some specific number should be the union of i mutually disjoint generators. Results in this direction have been obtained for the Hermitian polar space $H(2n-1, q)$ [5, Theorem 3.9] and the hyperbolic polar space[2] $Q^+(2n-1, q)$ [3, 6, 8, 10, 11]. Such a result cannot be obtained for elliptic polar spaces in view of the above-mentioned 2-tight sets in these polar spaces.

2 Basic definitions and properties

We continue with the notation of Section 1. With Π we associate a parameter ϵ which we call the *index* of Π:

[1] With exception of those living in the generalized quadrangle $Q^-(5, q)$, see [12, II.4].

[2] If n is odd, then $Q^+(2n-1, q)$ does not have three mutually disjoint generators. In this case, the result implies the nonexistence of i-tight sets with $2 < i \leq N$.

Π	$W(2n-1,q)$	$Q(2n,q)$	$Q^+(2n-1,q)$	$Q^-(2n+1,q)$	$H(2n-1,q)$	$H(2n,q)$
ϵ	1	1	0	2	$\frac{1}{2}$	$\frac{3}{2}$

With Π there is associated a polarity ζ. The polarity ζ is symplectic if $\Pi = W(2n-1,q)$ or $\Pi \in \{Q^+(2n-1,q), Q^-(2n+1,q)\}$ with q even, orthogonal if $\Pi \in \{Q(2n,q), Q^+(2n-1,q), Q^-(2n+1,q)\}$ with q odd, and Hermitian if $\Pi \in \{H(2n-1,q), H(2n,q)\}$. If $\Pi = Q(2n,q)$ with q even, then ζ is degenerate: we have $x^\zeta = \Sigma$ if x is the nucleus of $Q(2n,q)$ and x^ζ is a hyperplane of Σ for any other point x of Σ. If x is a point of Π, then $x^\perp := x^\zeta \cap \mathcal{P}$ denotes the set of points of Π collinear with (or equal to) x.

A set X of points of Π is called an *m-ovoid* for some $m \in \mathbb{N}$ if it intersects every generator of Π in precisely m points. A set X of points of Π is called *intriguing* if there exist constants $h_1, h_2 \in \mathbb{N}$ such that $|x^\perp \cap X| = h_1$ if $x \in X$ and $|x^\perp \cap X| = h_2$ if $x \in \mathcal{P} \setminus X$. If this is the case, then we say that X is *intriguing with parameters* (h_1, h_2). If $\emptyset \neq X \neq \mathcal{P}$, then the parameters (h_1, h_2) of the intriguing set X are uniquely determined. The following was proved in [1, 2].

Lemma 2.1 ([1, 2]) *The intriguing sets of points of Π are precisely the tight sets and the m-ovoids for some $m \in \mathbb{N}$. If X is an i-*tight set *of Π, then X is an intriguing set with parameters (h_1, h_2), where*

$$h_1 = i \cdot \lambda_{n-1} + q^{n-1}, \qquad h_2 = i \cdot \lambda_{n-1}.$$

If X is an m-ovoid, then X is an intriguing set with parameters (h_1, h_2), where

$$h_1 = (m-1)(q^{n-2+\epsilon} + 1) + 1, \qquad h_2 = m(q^{n-2+\epsilon} + 1).$$

Corollary 2.2 *Let X be a nonempty intriguing set of points of Π distinct from $\mathcal{P}$, and let (h_1, h_2) be the parameters of X. Then $h_1 > h_2$ if X is a tight set and $h_1 < h_2$ if X is an m-ovoid for some $m \in \mathbb{N}$.*

The following lemma is taken from Drudge [7, Theorem 9.1].

Lemma 2.3 ([7]) *The 1-tight sets of Π are precisely the generators of Π.*

More generally, the union of i mutually disjoint generators of Π is an i-tight set. This can easily be proved, but it also follows from the following property taken from [1, 12].

Lemma 2.4 ([1, 12]) *Suppose X_1 is an i-tight set of Π and X_2 is a j-tight set of Π. If $X_1 \cap X_2 = \emptyset$, then $X_1 \cup X_2$ is an $(i+j)$-tight set of Π. If $X_1 \subseteq X_2$, then $X_2 \setminus X_1$ is a $(j-i)$-tight set of Π.*

In the following lemma, we collect a well-known property of quadrics and Hermitian varieties of finite projective spaces, see [9].

Lemma 2.5 (1) *Let Q be a (possibly degenerate) quadric of the projective space* $\mathrm{PG}(m,q)$, $m \geq 2$. *Then every plane meets Q nontrivially.*

(2) *Let $\mathcal{H}$ be a (possibly degenerate) Hermitian variety of the projective space* $\mathrm{PG}(m,q)$, $m \geq 1$ *and q square. Then every line meets $\mathcal{H}$ nontrivially.*

Lemma 2.6 *Suppose X is a set of points of* $\mathrm{PG}(m,q)$, $m \geq 1$, *with the property that every hyperplane intersects it in a constant number of points. Then X is either the empty set or the whole set of points of* $\mathrm{PG}(m,q)$.

Proof. Suppose every hyperplane intersects X in precisely a points. The number of incident point-hyperplane pairs (x,π) with $x \in X$ is then equal to $|X| \cdot \frac{q^m-1}{q-1} = \frac{q^{m+1}-1}{q-1} \cdot a$. Since the greatest common divisor of $\frac{q^m-1}{q-1}$ and $\frac{q^{m+1}-1}{q-1}$ is equal to 1, we should have that $\frac{q^m-1}{q-1}$ is a divisor of a. Since $0 \leq a \leq \frac{q^m-1}{q-1}$, this implies that either $a = 0$ or $a = \frac{q^m-1}{q-1}$, respectively corresponding to the cases where X is empty or the whole set of points. ■

3 Proof of Theorem 1.1

We continue with the notation of Sections 1 and 2. This section is devoted to the proof of Theorem 1.1. So, we will suppose that $X = (\Sigma_1 \cap \mathcal{P}) \cup (\Sigma_2 \cap \mathcal{P})$ is a nontrivial tight set of Π, where Σ_1 and Σ_2 are two disjoint subspaces of Σ.

Lemma 3.1 *Suppose X' is a nonempty set of points of Π and Σ' is a subspace of Σ containing X'. Then X' is a tight set of Π if and only if one of the following two cases occurs:*

(1) *X' is a generator of Π;*

(2) *$\Pi' := \Sigma' \cap \Pi$ is a nondegenerate polar space of rank n, and X' is a tight set of Π'.*

Proof. We first observe that the point sets described in (1) and (2) are indeed examples of tight sets of Π. We already know this for the generators of Π. In the case Π' is a nondegenerate polar space of rank n, this follows from the fact that two points of Π' are collinear in Π' if and only if they are collinear in Π, and so the conditions for X' to be tight sets of Π and

Π' are the same, namely that the total number of ordered pairs of distinct collinear points of X' should be equal to $\lambda_{n-1} \cdot |X'| \cdot \left(\frac{|X'|}{\lambda_n} + (q-1)\right)$.

Suppose now that X' is a tight set of points of Π. If X' is a set of mutually collinear points, then X' is contained in some generator π and it follows that $0 < |X'| \leq |\pi| = \lambda_n$. The fact that $\frac{|X'|}{\lambda_n} \in \mathbb{N}$ then implies that $X' = \pi$. We will therefore suppose that X' contains two distinct noncollinear points. Then Π' is a (possibly degenerate) polar space.

We show that Π' is nondegenerate as a polar space. Suppose x^* is a point of Π' that is collinear with all remaining points of Π'. If $x^* \in X'$, then as x^* is collinear with all $|X'|-1$ other points of X', every point of the intriguing set X' should be collinear with all $|X'|-1$ other points of X', in contradiction with the fact that X' contains two noncollinear points. We must therefore have that $x^* \notin X'$. But then the fact that x^* is collinear with precisely $|X'|$ points of X' would imply by Corollary 2.2 that every point $y \in X'$ is collinear with more than $|X'|$ points of X', an obvious contradiction.

Suppose X' is intriguing with parameters (h_1, h_2). Then $h_1 > 1$ by Lemma 2.1. So, X' contains two distinct collinear points and the rank n' of the nondegenerate polar space Π' must be at least 2.

We show that X' is a tight set of Π'. This is obviously the case if X' coincides with the whole point set $\mathcal{P}'$ of Π'. So, we may suppose that $\emptyset \neq X' \neq \mathcal{P}'$. By Corollary 2.2, $h_1 > h_2$. Since every point of X' is collinear with h_1 points of X' and every point of $\mathcal{P}' \setminus X'$ is collinear with h_2 points of X', the set X' must be an intriguing set of points of Π'. Since $h_1 > h_2$, X' is a tight set of Π' by Lemma 2.1 and Corollary 2.2.

In order to finish the proof of the lemma, we still need to show that $n' = n$. Since X' is a tight set of Π', we have $|x^\perp \cap X'| = \lambda_{n'-1} \cdot \frac{|X'|}{\lambda_{n'}} + q^{n'-1}$ for every point $x \in X'$. Hence,

$$\lambda_{n'-1} \cdot \frac{|X'|}{\lambda_{n'}} + q^{n'-1} = \lambda_{n-1} \cdot \frac{|X'|}{\lambda_n} + q^{n-1},$$

i.e.,

$$\left(\frac{1}{q} - \frac{q-1}{q(q^{n'}-1)}\right) \cdot |X'| + q^{n'-1} = \left(\frac{1}{q} - \frac{q-1}{q(q^n-1)}\right) \cdot |X'| + q^{n-1}.$$

Since the maps $\mathbb{N} \setminus \{0,1\} \to \mathbb{Q} : m \mapsto \left(\frac{1}{q} - \frac{q-1}{q(q^m-1)}\right) \cdot |X'|$ and $\mathbb{N} \setminus \{0,1\} \to \mathbb{Q} : m \mapsto q^{m-1}$ are two increasing functions, we should have $n' = n$. ∎

In order to prove Theorem 1.1, it suffices – in view of Lemma 3.1 – to prove the following proposition.

Proposition 3.2 *Suppose Σ_1 and Σ_2 are two complementary nonempty subspaces of Σ. Let X_i, $i \in \{1,2\}$, denote the set of points of Π contained in Σ_i and suppose that $\Sigma_i = \langle X_i \rangle$. Then $X := X_1 \cup X_2$ is a tight set if and only if one of the following cases occurs:*

(1) *$\Pi = Q^-(5,q)$, one of Σ_1, Σ_2 is a line of $Q^-(5,q)$ and the other intersects Π in a $Q^+(3,q)$;*

(2) *$\Pi \in \{W(2n-1,q), Q^+(2n-1,q), H(2n-1,q)\}$ and Σ_1, Σ_2 are two disjoint generators of Π;*

(3) *$n = 2m$ is even, $\Pi = W(4m-1,q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$, $\Sigma_2 = \Sigma_1^\zeta$ and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong W(2m-1,q)$;*

(4) *$n = 2m$ is even, $\Pi = Q^-(4m+1,q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n$, $\Sigma_2 = \Sigma_1^\zeta$ and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q(2m,q)$;*

(5) *$n = 2m-1$ is odd, $\Pi = Q^-(4m-1,q)$, $\dim(\Sigma_1) = \dim(\Sigma_2) = n$, $\Sigma_2 = \Sigma_1^\zeta$ and there exists an $i \in \{1,2\}$ such that $\Sigma_i \cap \Pi \cong Q^+(2m-1,q)$ and $\Sigma_{3-i} \cap \Pi \cong Q^-(2m-1,q)$.*

In the sequel, we will suppose that Σ_1, Σ_2, X_1 and X_2 are as in Proposition 3.2. We suppose that $X = X_1 \cup X_2$ is a tight set of Π. Put $n_1 := \dim(\Sigma_1)+1$ and $n_2 := \dim(\Sigma_2)+1$. Note that X_1 and X_2 are nonempty since $\Sigma_1 = \langle X_1 \rangle$ and $\Sigma_2 = \langle X_2 \rangle$ are nonempty.

We will first prove the validity of Proposition 3.2 in the case where Π is a symplectic polar space.

Lemma 3.3 *If $\Pi = W(2n-1,q)$, then precisely one of the following two cases occurs:*

(1) *X_1 and X_2 are two disjoint generators of Π;*

(2) *$n = 2m$ is even, $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$, $\Sigma_2 = \Sigma_1^\zeta$ and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong W(2m-1,q)$.*

Proof. Clearly, $n_1 + n_2 = 2n$ and $1 \leq n_1, n_2 \leq 2n-1$. Let r_i, $i \in \{1,2\}$, denote the remainder of the division of n_i by n. Now, $|X| = \frac{q^{n_1}-1}{q-1} + \frac{q^{n_2}-1}{q-1}$ must be divisible by $\lambda_n = \frac{q^n-1}{q-1}$. So, q^n-1 is a divisor of $(q^{n_1}-1)+(q^{n_2}-1)$ and hence also of $(q^{r_1}-1)+(q^{r_2}-1)$. Since $0 \leq (q^{r_1}-1)+(q^{r_2}-1) \leq 2(q^{n-1}-1) < q^n-1$, we must have $r_1 = r_2 = 0$. It follows that $n_1 = n_2 = n$.

If x_1 is a point of $W(2n-1,q)$ not contained in $X = \Sigma_1 \cup \Sigma_2$ and x_2 is a point of X, then each of $|x_1^\perp \cap X|$, $|x_2^\perp \cap X|$ is equal to either $2\lambda_{n-1}$ or $\lambda_n + \lambda_{n-1}$. Since $|x_2^\perp \cap X| > |x_1^\perp \cap X|$ by Corollary 2.2, we should have

$|x_1^\perp \cap X| = 2\lambda_{n-1}$ and $|x_2^\perp \cap X| = \lambda_n + \lambda_{n-1}$. The latter equality implies that either $\Sigma_1 \subseteq x_2^\perp$ or $\Sigma_2 \subseteq x_2^\perp$. Let U_1 denote the set of all points $x \in \Sigma_1$ for which $\Sigma_1 \subseteq x^\perp$ and let U_2 denote the set of all points $x \in \Sigma_1$ for which $\Sigma_2 \subseteq x^\perp$. Then U_1 and U_2 are two disjoint subspaces of Σ_1 such that $\Sigma_1 = U_1 \cup U_2$. It follows that either ($U_1 = \emptyset$ and $U_2 = \Sigma_1$) or ($U_1 = \Sigma_1$ and $U_2 = \emptyset$).

Suppose $U_1 = \emptyset$ and $U_2 = \Sigma_1$. Then $\Sigma_2 \subseteq \Sigma_1^\zeta$. It follows that $\Sigma_2 = \Sigma_1^\zeta$. Since Σ_1 and Σ_2 are disjoint, $\Sigma_1 \cap \Pi$ and $\Sigma_2 \cap \Pi$ are two nondegenerate polar spaces. Hence, $n = 2m$ is even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong W(2m-1, q)$.

Suppose $U_1 = \Sigma_1$ and $U_2 = \emptyset$. Then Σ_1 is a generator of Π. So, it is no longer true that $n = 2m$ is even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong W(2m-1, q)$. By reversing the roles of Σ_1 and Σ_2, we then see that also Σ_2 should be a generator in this case. ∎

In view of Lemma 3.3, we can now make the following assumption.

Assumption 1: $\Pi \neq W(2n-1, q)$.

For every $i \in \{1, 2\}$, put $\Pi_i = \Sigma_i \cap \Pi$, $\mathcal{U}_i = \Sigma_i \cap \Sigma_{3-i}^\zeta$ and let $\mathcal{R}_i$ denote the set of all points $x \in X_i$ for which $\Sigma_i \subseteq x^\zeta$. Also, put $n_i' := \dim(\mathcal{U}_i) + 1$. Observe that if Π_i is a (possibly degenerate) polar space, then $\mathcal{R}_i$ is its radical.

Lemma 3.4 *For every* $i \in \{1, 2\}$, $\mathcal{R}_i \cap \mathcal{U}_i = \emptyset$. *Moreover, if* $\emptyset \neq \mathcal{R}_i \neq \Sigma_i$, *then* $X_i \setminus (\mathcal{R}_i \cup \mathcal{U}_i) \neq \emptyset$.

Proof. If x were a point of $\mathcal{R}_i \cap \mathcal{U}_i$, then x^ζ would contain Σ_1 and Σ_2 and hence also $\Sigma = \langle \Sigma_1, \Sigma_2 \rangle$, in contradiction with the fact that Π is a nondegenerate polar space.

Suppose $\emptyset \neq \mathcal{R}_i \neq \Sigma_i$. If $\mathcal{U}_i \cap X_i = \emptyset$, then the fact that $\langle X_i \rangle = \Sigma_i$ implies that $X_i \setminus (\mathcal{R}_i \cup \mathcal{U}_i) \neq \emptyset$. If $\mathcal{U}_i \cap X_i \neq \emptyset$, then the fact that $\mathcal{R}_i \cap \mathcal{U}_i = \emptyset$ implies that there exists a point of $X_i \setminus (\mathcal{R}_i \cup \mathcal{U}_i)$ on a line joining a point of $\mathcal{R}_i$ with a point of $\mathcal{U}_i \cap X_i$. ∎

Lemma 3.5 *If* $\mathcal{R}_i \neq \emptyset$ *for some* $i \in \{1, 2\}$, *then* $\dim(\Sigma_i) \leq \dim(\Sigma_{3-i})$.

Proof. Without loss of generality, we may suppose that $i = 2$. If X_2 is a subspace, then $\dim(\Sigma_2) \leq n-1$ and hence $\dim(\Sigma_1) \geq n-1$ since $\dim(\Sigma_1) + \dim(\Sigma_2) = \dim(\Sigma) - 1 \geq 2n-2$. In this case, we thus have $\dim(\Sigma_2) \leq \dim(\Sigma_1)$.

So, suppose that X_2 is not a subspace. Then $\emptyset \neq \mathcal{R}_2 \neq \Sigma_2$, so Π_2 is a degenerate polar space. By Lemma 3.4, there exist points $x_1 \in \mathcal{R}_2$ and $x_2 \in X_2 \setminus (\mathcal{R}_2 \cup \mathcal{U}_2)$. Then $|x_1^\perp \cap X_2| - |x_2^\perp \cap X_2| = |X_2| - |x_2^\perp \cap X_2| = q^{n_2 - 2 + \delta}$, where $\delta = 0$ if Π is a polar space of quadratic type and $\delta = \frac{1}{2}$ if Π is a

Hermitian polar space. Indeed, on each of the q^{n_2-2} lines of Σ_2 through x_2 not contained in x_2^ζ, there are q^δ points of $X_2 \setminus \{x_2\}$. Since $X = X_1 \cup X_2$ is an intriguing set, we must have that $|x_2^\perp \cap X_1| - |x_1^\perp \cap X_1| = q^{n_2-2+\delta}$. Since $x_1, x_2 \notin \mathcal{U}_2$, the subspaces $x_2^\zeta \cap \Sigma_1$ and $x_1^\zeta \cap \Sigma_1$ are two hyperplanes of Σ_1 (i.e. subspaces of dimension $n_1 - 2$) and hence $|x_2^\perp \cap X_1| - |x_1^\perp \cap X_1| \leq |(x_2^\zeta \cap \Sigma_1) \setminus (x_1^\zeta \cap \Sigma_1)| \leq q^{n_1-2}$. It follows that $n_2 \leq n_1$. ∎

Our next goal will be to treat the case where at least one of X_1, X_2 is a subspace. This goal will be achieved in Lemmas 3.6, 3.7 and 3.8.

Lemma 3.6 *If both X_1 and X_2 are subspaces of Π, then they are also generators of Π.*

Proof. We have $0 < |X_1| \leq \lambda_n$ and $0 < |X_2| \leq \lambda_n$, implying that $0 < |X| \leq 2\lambda_n$. So, X is either 1-tight or 2-tight. However, X cannot be 1-tight by Lemma 2.3. So, X is 2-tight. But then we necessarily have $|X_1| = |X_2| = \lambda_n$, implying that X_1 and X_2 are two generators. ∎

Lemma 3.7 *If X_1 is a subspace, then X_1 is a generator.*

Proof. Suppose that X_1 is not a generator. Then X_2 is not a subspace by Lemma 3.6, implying that $\Sigma_2 \cap \Pi$ is a (possibly degenerate) polar space whose radical $\mathcal{R}_2$ has co-dimension at least 2 in Σ_2. Since X_1 is not a generator, we have $n_1 \leq n - 1$ and hence $n_2 \geq n + 1$ since $n_1 + n_2 = \dim(\Sigma) + 1 \geq 2n$. By Lemma 3.5, $\mathcal{R}_2 = \emptyset$. So, $\Sigma_2 \cap \Pi$ is a nondegenerate polar space, implying that $|x^\perp \cap X_2|$ is independent of $x \in X_2$. Since $X = X_1 \cup X_2$ is an intriguing set, also $|x^\perp \cap X_1|$ is independent of $x \in X_2$. Since X_1 is a subspace, there are then two possibilities:

- $y^\perp \cap X_1 = X_1$ for every $y \in X_2$;
- $y^\perp \cap X_1$ is a hyperplane of Σ_1 for every $y \in X_2$.

The former case cannot occur, since this would imply that $z^\zeta = \Sigma$ for every $z \in \Sigma_1$, contrary to the fact that Π is nondegenerate as a polar space. Hence, $y^\perp \cap \Sigma_1$ is a hyperplane of Σ_1 for every $y \in X_2$. Since the subspace $\Sigma_1^\zeta \cap \Sigma_2$ has dimension at least $\dim(\Sigma_2) - \dim(\Sigma_1) - 1 \geq 1$ and is disjoint from X_2, Lemma 2.5 implies that Π is a polar space of quadratic type and $\dim(\Sigma_2) - \dim(\Sigma_1) = 2$. Since $\dim(\Sigma_1) \leq n-2$ and $\dim(\Sigma_2) \geq n$, we thus have $\dim(\Sigma_1) = n - 2$, $\dim(\Sigma_2) = n$ and $\dim(\Sigma) = 2n - 1$. So, $\Pi = Q^+(2n-1, q)$. If n is even, then as $\Sigma_2 \cap \Pi$ is a parabolic quadric, we must have

$$|X| = \frac{q^{n-1}-1}{q-1} + \frac{q^n - 1}{q-1},$$

which is impossible since $\frac{|X|}{\lambda_n}$ would then not be integral. If n is odd, then

$$|X| = \frac{q^{n-1}-1}{q-1} + \frac{q^n-1}{q-1} + \eta,$$

where $\eta = q^{\frac{n-1}{2}}$ if X_2 is a hyperbolic quadric of Σ_2 and $\eta = -q^{\frac{n-1}{2}}$ if X_2 is an elliptic quadric of Σ_2. Also in this case the number $|X|$ is not divisible by $\lambda_n = \frac{q^n-1}{q-1}$. So, our original assumption was wrong and X_1 should be a generator of Π. ∎

Lemma 3.8 *If X_1 or X_2 is a subspace of Π, then one of the following two cases occurs:*

(1) *X_1 and X_2 are generators;*

(2) *$\Pi = Q^-(5,q)$, one of X_1, X_2 is a line of $Q^-(5,q)$ and the other is a $Q^+(3,q) \subseteq Q^-(5,q)$.*

Proof. Without loss of generality, we may suppose that X_1 is a subspace. Then by Lemma 3.7, X_1 is a generator and hence also a tight set of Π. Since $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$ is a tight set of Π, Lemma 2.4 implies that also X_2 is a tight set of Π. By Lemma 3.1, there are now two possibilities:

(1) X_2 is a generator;

(2) $\Sigma_2 \cap \Pi$ is a nondegenerate polar space of rank n.

Suppose case (2) occurs. Since $\Sigma_2 \cap \Pi$ is a nondegenerate polar space of the same rank as Π, the co-dimension of Σ_2 in Σ should be at most 2. So, the generator $\Sigma_1 = X_1$ which is disjoint from Σ_2 should have dimension 1 and Σ_2 should have co-dimension 2 in Σ. This implies that $\Pi \cong Q^-(5,q)$ and $\Sigma_2 \cap \Pi \cong Q^+(3,q)$. ∎

In view of Lemma 3.8, we can make the following additional assumption.

Assumption 2: Neither X_1 nor X_2 is a subspace of Π.

Then Π_1 and Π_2 are (possibly degenerate) polar spaces and $\mathcal{R}_1 \neq \Sigma_1$, $\mathcal{R}_2 \neq \Sigma_2$.

Lemma 3.9 *For every $i \in \{1,2\}$, either $\mathcal{U}_i = \Sigma_i$ or $\mathcal{U}_i \cap X_i = \emptyset$.*

Proof. Assume $\mathcal{U}_i \neq \Sigma_i$ and $x \in \mathcal{U}_i \cap X_i$. If $\mathcal{R}_i = \emptyset$, then as $\langle X_i \rangle = \Sigma_i$, there exists a point $y \in X_i \setminus \mathcal{U}_i$. If $\mathcal{R}_i \neq \emptyset$, then by Lemma 3.4, there exists a point $y \in X_i \setminus (\mathcal{U}_i \cup \mathcal{R}_i)$. Since $x, y \notin \mathcal{R}_i$, we have $|x^\perp \cap X_i| = |y^\perp \cap X_i|$

and hence also $|x^\perp \cap X_{3-i}| = |y^\perp \cap X_{3-i}|$. This is however impossible as $x^\perp \cap X_{3-i} = X_{3-i}$, while $y^\perp \cap X_{3-i}$ is a proper subset of X_{3-i}. ∎

Observe that we have $\mathcal{U}_1 = \Sigma_1$ if and only if $\mathcal{U}_2 = \Sigma_2$. Indeed this happens precisely when every $x_1 \in X_1$ is collinear with every $x_2 \in X_2$ in the polar space Π. We can thus distinguish two cases:

(A) $\mathcal{U}_1 = \Sigma_1$ and $\mathcal{U}_2 = \Sigma_2$;

(B) $\mathcal{U}_1 \cap X_1 = \mathcal{U}_2 \cap X_2 = \emptyset$.

We now treat these two cases separately.

Case A: Suppose $\mathcal{U}_1 = \Sigma_1$ and $\mathcal{U}_2 = \Sigma_2$

We then have that $\mathcal{R}_1 = \mathcal{R}_2 = \emptyset$ by Lemma 3.4.

Lemma 3.10 *We have $n_1 = n_2$.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Repeating one of the arguments in the proof of Lemma 3.5, we have that $|X_1| - |x_1^\perp \cap X_1| = q^{n_1-2+\delta}$ and $|X_2| - |x_2^\perp \cap X_2| = q^{n_2-2+\delta}$, where $\delta = 0$ in case Π is a polar space of quadratic type and $\delta = \frac{1}{2}$ in case Π is a Hermitian polar space. Since $|x_1^\perp \cap X| = |X_2| + |x_1^\perp \cap X_1| = |X| - q^{n_1+2+\delta}$ and $|x_2^\perp \cap X| = |X| - q^{n_2+\delta-2}$ are equal, we should have $n_1 = n_2$. ∎

There are thus seven cases to consider:

(1) $\Pi = H(2n-1,q)$ with $n = 2m$ even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong H(2m-1,q)$;

(2) $\Pi = H(2n-1,q)$ with $n = 2m+1$ odd and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong H(2m,q)$;

(3) $\Pi = Q^+(2n-1,q)$ with $n = 2m+1$ odd and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q(2m,q)$;

(4) $\Pi = Q^+(2n-1,q)$ with $n = 2m$ even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q^+(2m-1,q)$;

(5) $\Pi = Q^+(2n-1,q)$ with $n = 2m+2 \geq 4$ even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q^-(2m+1,q)$;

(6) $\Pi = Q^-(2n+1,q)$ with $n = 2m$ even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q(2m,q)$;

(7) $\Pi = Q^-(2n+1,q)$ with $n = 2m-1$ odd, $\Sigma_1 \cap \Pi \cong Q^+(2m-1,q)$ and $\Sigma_2 \cap \Pi \cong Q^-(2m-1,q)$.

In each of these cases, we have $\Sigma_2 = \Sigma_1^\zeta$. Symmetry allowed us to omit the possibility $\Sigma_1 \cap \Pi \cong Q^-(2m-1,q)$ and $\Sigma_2 \cap \Pi \cong Q^+(2m-1,q)$ in case $\Pi = Q^-(2n+1,q)$ with $n = 2m-1$ odd. Let ϵ denote the index of the polar space Π. For every $i \in \{1,2\}$, let r_i and ϵ_i denote the rank and index of the polar space $\Pi_i := \Sigma_i \cap \Pi$. We have collected these values in the following table:

Case	n	r_1	r_2	ϵ	ϵ_1	ϵ_2
(1)	$2m$	m	m	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
(2)	$2m+1$	m	m	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
(3)	$2m+1$	m	m	0	1	1
(4)	$2m$	m	m	0	0	0
(5)	$2m+2$	m	m	0	2	2
(6)	$2m$	m	m	2	1	1
(7)	$2m-1$	m	$m-1$	2	0	2

We have $|X| = |X_1| + |X_2|$, where

$$|X_1| = \frac{q^{r_1}-1}{q-1} \cdot (q^{r_1-1+\epsilon_1}+1), \qquad |X_2| = \frac{q^{r_2}-1}{q-1} \cdot (q^{r_2-1+\epsilon_2}+1).$$

The total number N of ordered pairs of distinct collinear points is equal to

$$\frac{q^{r_1}-1}{q-1}\cdot(q^{r_1-1+\epsilon_1}+1)\cdot\Big(q\cdot\frac{q^{r_1-1}-1}{q-1}\cdot(q^{r_1-2+\epsilon_1}+1)+\frac{q^{r_2}-1}{q-1}\cdot(q^{r_2-1+\epsilon_2}+1)\Big)$$

$$+\frac{q^{r_2}-1}{q-1}\cdot(q^{r_2-1+\epsilon_2}+1)\cdot\Big(q\cdot\frac{q^{r_2-1}-1}{q-1}\cdot(q^{r_2-2+\epsilon_2}+1)+\frac{q^{r_1}-1}{q-1}\cdot(q^{r_1-1+\epsilon_1}+1)\Big).$$

The set X is then tight if and only if

$$N = \frac{q^{n-1}-1}{q-1}\cdot|X|\cdot\Big(\frac{|X|\cdot(q-1)}{q^n-1}+(q-1)\Big) = (q^{n-1}-1)\cdot|X|\cdot\Big(\frac{|X|}{q^n-1}+1\Big).$$

It is straightforward to verify that this equality holds in each of the cases (6) and (7). In cases (1)–(5), we have $r_1 = r_2$ and $\epsilon_1 = \epsilon_2$. So, we also have

$$\begin{aligned}|X| &= 2\cdot\frac{q^{r_1}-1}{q-1}\cdot(q^{r_1-1+\epsilon_1}+1),\\ N &= |X|\cdot\frac{q^{2r_1-1+\epsilon_1}+q^{2r_1-2+\epsilon_1}+2q^{r_1}-2q^{r_1-1+\epsilon_1}-q-1}{q-1}.\end{aligned}$$

From this, one deduces that X is tight if and only if

$$\frac{q^n-1}{q-1}\cdot\frac{(q^{2r_1-1+\epsilon_1}+q^{2r_1-2+\epsilon_1}+2q^{r_1}-2q^{r_1-1+\epsilon_1}-q-1)}{q-1}$$

$$= \frac{q^{n-1}-1}{q-1} \cdot \left(2 \cdot \frac{q^{r_1}-1}{q-1} \cdot (q^{r_1-1+\epsilon_1}+1) + (q^n-1)\right).$$

Since the greatest common divisor of $\frac{q^n-1}{q-1}$ and $\frac{q^{n-1}-1}{q-1}$ is equal to 1, we should have that q^n-1 is a divisor of $2(q^{r_1}-1)(q^{r_1-1+\epsilon_1}+1)$. This leads to a contradiction in each of the cases (1)–(5), except for case (4) where the possibility $(q,m) = (3,1)$ remains. However, also this possibility cannot occur since the equality $N = (q^{n-1}-1) \cdot |X| \cdot \left(\frac{|X|}{q^n-1}+1\right)$ would then not be satisfied.

So, we know that one of the following two cases occurs:

- $\Pi = Q^-(2n+1,q)$ with $n = 2m$ even and $\Sigma_1 \cap \Pi \cong \Sigma_2 \cap \Pi \cong Q(2m,q)$;
- $\Pi = Q^-(2n+1,q)$ with $n = 2m-1$ odd, $\Sigma_1 \cap \Pi \cong Q^+(2m-1,q)$ and $\Sigma_2 \cap \Pi \cong Q^-(2m-1,q)$.

Case B: $\mathcal{U}_1 \cap X_1 = \mathcal{U}_2 \cap X_2 = \emptyset$.

We will show that there are no examples here.

Lemma 3.11 *We have $n_1', n_2' \in \{0,1\}$ in case Π is a Hermitian polar space and $n_1', n_2' \in \{0,1,2\}$ in case Π is a polar space of quadratic type.*

Proof. Let $i \in \{1,2\}$. The fact that $\mathcal{U}_i \cap X_i = \emptyset$ implies by Lemma 2.5 that $\dim(\mathcal{U}_i) \leq 0$ in case Π is a Hermitian polar space and $\dim(\mathcal{U}_i) \leq 1$ in case Π is of quadratic type. ∎

Lemma 3.12 *We have $n_1 - n_1' = n_2 - n_2'$.*

Proof. Let Σ_2' denote the projective space of dimension $n_2 - n_2' - 1$ naturally defined on the hyperplanes of Σ_2 containing $\mathcal{U}_2$. Let $\mathcal{A}_1$ denote the set of all subspaces of Σ_1 of dimension $\dim(\mathcal{U}_1)+1$ through $\mathcal{U}_1$. For every $U \in \mathcal{A}_1$, the subspace $U^\phi := U^\zeta \cap \Sigma_2$ is a hyperplane of Σ_2 containing $\mathcal{U}_2$. Then ϕ defines an isomorphism between the quotient space $\Sigma_1/\mathcal{U}_1$ and a suitable subspace of Σ_2'. So, we should have $n_1 - n_1' \leq n_2 - n_2'$. By symmetry, we should also have $n_2 - n_2' \leq n_1 - n_1'$. ∎

The following is an immediate consequence of Lemmas 3.11 and 3.12.

Corollary 3.13 *We have $n_1 - n_2 \in \{-1,0,1\}$ in case Π is a Hermitian polar space and $n_1 - n_2 \in \{-2,-1,0,1,2\}$ in case Π is a polar space of quadratic type.*

Without loss of generality, we may suppose that $n_2 \geq n_1$. The cases that remain to be considered are then the following:

- $\Pi = H(2n-1,q)$ and $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$;
- $\Pi = H(2n,q)$, $\dim(\Sigma_1) = n-1$ and $\dim(\Sigma_2) = n$;
- $\Pi = Q^+(2n-1,q)$ and $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$;
- $\Pi = Q^+(2n-1,q)$, $\dim(\Sigma_1) = n-2$ and $\dim(\Sigma_2) = n$;
- $\Pi = Q(2n,q)$, $\dim(\Sigma_1) = n-1$ and $\dim(\Sigma_2) = n$;
- $\Pi = Q^-(2n+1,q)$ and $\dim(\Sigma_1) = \dim(\Sigma_2) = n$;
- $\Pi = Q^-(2n+1,q)$, $\dim(\Sigma_1) = n-1$ and $\dim(\Sigma_2) = n+1$.

Lemma 3.14 *None of the following cases occurs:*
- $\Pi = H(2n-1,q)$ *and* $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$;
- $\Pi = Q^+(2n-1,q)$ *and* $\dim(\Sigma_1) = \dim(\Sigma_2) = n-1$.

Proof. In this case, we have $0 < |X| = |X_1| + |X_2| < 2\lambda_n$. So, $|X| = \lambda_n$ and X must be a generator by Lemma 2.3, clearly a contradiction. ∎

Lemma 3.15 *None of the following cases occurs:*
- $\Pi = H(2n,q)$, $\dim(\Sigma_1) = n-1$ *and* $\dim(\Sigma_2) = n$;
- $\Pi = Q^+(2n-1,q)$, $\dim(\Sigma_1) = n-2$ *and* $\dim(\Sigma_2) = n$;
- $\Pi = Q^-(2n+1,q)$, $\dim(\Sigma_1) = n-1$ *and* $\dim(\Sigma_2) = n+1$.

Proof. Suppose $\Pi = H(2n,q)$, $\dim(\Sigma_1) = n-1$ and $\dim(\Sigma_2) = n$. Then $\mathcal{R}_2 = \emptyset$ and $\mathcal{U}_2$ is a singleton by Lemmas 3.5, 3.11 and 3.12. Since $\mathcal{R}_2 = \emptyset$, the number $|x^\perp \cap X_2|$ is independent of the point $x \in X_2$. Hence, also the number $|x^\perp \cap X_1|$ is independent of the point $x \in X_2$. Now, the map $L \mapsto L^\zeta \cap \Sigma_1$ defines a bijection between the lines of Σ_2 through $\mathcal{U}_2$ and the hyperplanes of Σ_1. Since every line L of Σ_2 through $\mathcal{U}_2$ contains a point $x \in X_2$ (by Lemma 2.5) and $L^\zeta \cap \Sigma_1 = x^\zeta \cap \Sigma_1$, we have that every hyperplane of Σ_1 intersects X_1 in a constant number of points, in contradiction with Lemma 2.6 and the fact that $\emptyset \neq X_1 \neq \Sigma_1$.

The proof of the remaining two cases are similar. In this case, we have that $\mathcal{R}_2 = \emptyset$, $\dim(\mathcal{U}_2) = 1$ and every plane of Σ_2 through $\mathcal{U}_2$ contains a point of X_2. ∎

Lemma 3.16 *The case* $\Pi = Q(2n,q)$, $\dim(\Sigma_1) = n-1$ *and* $\dim(\Sigma_2) = n$ *cannot occur.*

Proof. In this case, we have $\mathcal{R}_2 = \emptyset$ by Lemma 3.5. We have $|X_2| = \lambda_n + \delta$, where $\delta = 0$ if n is even and $\delta \in \{q^{(n-1)/2}, -q^{(n-1)/2}\}$ if n is odd. On the other hand, we have $|X_1| \leq \lambda_n - (q-1)q^{n-2}$. Indeed, if x is a point of X_1 not belonging to $\mathcal{R}_1$, then there are at most $\lambda_{n-1} = \lambda_n - q^{n-1}$ points contained in $(x^\zeta \cap \Sigma_1) \cap X_1$ and precisely 1 point of $X_1 \setminus \{x\}$ on each of the q^{n-2} lines of Σ_1 through x not contained in $x^\zeta \cap \Sigma_1$. It follows that $|X| = |X_1| + |X_2| \leq 2\lambda_n$, with equality if and only if $n = 3$, $q = 2$ and $\delta = 2$. The case $|X| = \lambda_n$ is impossible since X would otherwise be a generator by Lemma 2.3. So, we have that $n = 3$, $q = 2$ and $\delta = 2$. Then $\Pi = Q(6, q)$, X_1 is the union of two distinct lines L_1 and L_2 contained in the plane Σ_1 and X_2 is a hyperbolic quadric $Q^+(3, 2)$ contained in the 3-dimensional subspace Σ_2. The fact that every point of $L_2 \setminus L_1$ is collinear with $\lambda_{n-1} \cdot \frac{|X|}{\lambda_n} + q^{n-1} = 10$ points of $X_1 \cup X_2$ rapidly leads to a contradiction (the fact that three of these points are contained in X_1 would imply that seven of them must be contained in X_2). ■

Lemma 3.17 *The case* $\Pi = Q^-(2n+1, q)$ *and* $\dim(\Sigma_1) = \dim(\Sigma_2) = n$ *cannot occur.*

Proof. Suppose first that $\mathcal{R}_1 = \mathcal{R}_2 = \emptyset$. Then $|X_1| = |X_2| = \lambda_n$ if n is even and $|X_1|, |X_2| \in \{\lambda_n + q^{(n-1)/2}, \lambda_n - q^{(n-1)/2}\}$ if n is odd. So, one of the following cases occurs:

- n is even, X is 2-tight and X_1, X_2 are of parabolic type. Then every point of X is collinear with $\lambda_{n-1}(q+1)$ other points of X. A straightforward computation then shows that every point of X_1 must be collinear with every point of X_2. This is however in contradiction with the fact that Σ_2 is not contained in Σ_1^ζ.

- n is odd, X is 2-tight, one of X_1, X_2 is of hyperbolic type and the other is of elliptic type. Again every point of X is collinear with $\lambda_{n-1}(q+1)$ other points of X, and a straightforward argument then shows that every point of X_1 is collinear with every point of X_2. Again this is impossible.

So, without loss of generality, we may suppose that $\mathcal{R}_1 \neq \emptyset$. This excludes the possibility that $n = 2$, because otherwise we would have $|X_1| = 2q + 1$ and $|X_2| \in \{q+1, 2q+1\}$, in contradiction with the fact that $|X| = |X_1| + |X_2|$ is divisible by $q + 1$. We will now show the following:

- $\mathcal{R}_2 \neq \emptyset$;

- if $q \neq 2$, then $|X_1| = |X_2| = q^{n-1} + \lambda_n$;

- if $q = 2$, then $|X_1|, |X_2| \in \{\lambda_n, 2^{n-1} + \lambda_n\}$.

The fact that $|X| = |X_1| + |X_2|$ is divisible by λ_n then leads to a contradiction.

As before (see Lemma 3.5), we can take a point $x \in \mathcal{R}_1$ and a point $y \in X_1 \setminus (\mathcal{R}_1 \cup \mathcal{U}_1)$. Then $|x^\perp \cap X_1| - |y^\perp \cap X_1| = q^{n-1}$ and so we must have $|y^\perp \cap X_2| - |x^\perp \cap X_2| = q^{n-1}$. This implies that there exist two hyperplanes π_1 and π_2 in Σ_2 such that $|\pi_2 \cap X_1| - |\pi_1 \cap X_1| = q^{n-1}$. Since $\dim(\pi_1) = \dim(\pi_2) = n-1$, this implies that $\pi_2 \setminus (\pi_1 \cap \pi_2) \subseteq X_2$ and $(\pi_1 \setminus (\pi_1 \cap \pi_2)) \cap X_2 = \emptyset$.

If $q \geq 3$, then this implies that $\pi_2 \subseteq X_2$ and so X_2 should be the union of two distinct hyperplanes through a subspace of co-dimension 2. This subspace of co-dimension 2 is precisely $\mathcal{R}_2$ and it follows that $|X_2| = q^{n-1} + \lambda_n$. Since $\mathcal{R}_2 \neq \emptyset$, we can repeat the same reasoning as before and conclude that also $|X_1| = q^{n-1} + \lambda_n$.

Suppose now that $q = 2$ and take a point $z \in \pi_2 \setminus (\pi_1 \cap \pi_2)$. If we denote by U a hyperplane of $\pi_1 \cap \pi_2$ contained in z^ζ, then $U \subseteq X_2$ and U^ζ contains π_1 and π_2 and hence also $\Sigma_2 = \langle \pi_1, \pi_2 \rangle$. So, $U \subseteq \mathcal{R}_2$ and since $\dim(U) = n - 3 \geq 0$, we have $\mathcal{R}_2 \neq \emptyset$. Now, X_2 consists of the union of a number of $(n-2)$-dimensional subspaces through U. Since Σ_2/U is a projective plane, $X_2 \neq \Sigma_2$ and $\langle X_2 \rangle = \Sigma_2$, the set X_2 must be the union of three or five $(n-2)$-dimensional subspaces through U. It follows that $|X_2| \in \{\lambda_n, 2^{n-1} + \lambda_n\}$. Since $\mathcal{R}_2 \neq \emptyset$, we can repeat the same reasoning as before and conclude that also $|X_1| \in \{ \lambda_n, 2^{n-1} + \lambda_n\}$. ■

References

[1] J. Bamberg, S. Kelly, M. Law and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A* 114 (2007), 1293–1314.

[2] J. Bamberg, M. Law and T. Penttila. Tight sets and m-ovoids of generalised quadrangles. *Combinatorica* 29 (2009), 1–17.

[3] L. Beukemann and K. Metsch. Small tight sets of hyperbolic quadrics. *Des. Codes Cryptogr.* 68 (2013), 11–24.

[4] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.* 46 (1982), 91–102.

[5] J. De Beule, P. Govaerts, A. Hallez and L. Storme. Tight sets, weighted m-covers, weighted m-ovoids, and minihypers. *Des. Codes Cryptogr.* 50 (2009), 187–201.

[6] J. De Beule, A. Hallez and L. Storme. A non-existence result on Cameron-Liebler line classes. *J. Combin. Des.* 16 (2008), 342–349.

[7] K. Drudge. *Extremal sets in projective and polar spaces.* Ph.D. Thesis, The University of Western Ontario (Canada), 1998.

[8] P. Govaerts and L. Storme. On Cameron-Liebler line classes. *Adv. Geom.* 4 (2004), 279–286.

[9] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries.* Oxford University Press, New York, 1991.

[10] K. Metsch. The non-existence of Cameron-Liebler line classes with parameter $2 < x \leq q$. *Bull. Lond. Math. Soc.* 42 (2010), 991–996.

[11] K. Metsch. An improved bound on the existence of Cameron-Liebler line classes. *J. Combin. Theory Ser. A* 121 (2014), 89–93.

[12] S. E. Payne. Tight pointsets in finite generalized quadrangles. *Congr. Numer.* 60 (1987), 243–260.

[13] S. E. Payne. Tight pointsets in finite generalized quadrangles. II. *Congr. Numer.* 77 (1990), 31–41.