Pentacyclic Graphs with Maximal Estrada Index

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Abstract

The Estrada index of a simple connected graph G of order n is defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. In this paper, we characterize all pentacyclic graphs of order n with maximal Estrada index.

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1 Introduction

In this paper we use the same techniques of [14, 20]. Let $G = (V, E)$ be a simple connected graph of order n and size m. The characteristic polynomial $\phi(G; x)$ of G is $|xI - A(G)|$, where $A(G)$ is the $(0,1)$-*adjacency* matrix of G, and I is the unit matrix. We call the eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ (for short $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) of $A(G)$ the *spectrum* of G. The *Estrada index*, put forward by Estrada [6], is defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. The concept of Estrada index in graphs has found multiple applications in a large variety of problems, see for example [7, 8, 9, 10, 11]. Several authors studied the Estrada index in graphs, see for example, [12, 13, 16, 19].

If $m = n - 1 + c$, then G is called a *c-cyclic* graph. The unique c-cyclic graphs with maximal Estrada index are determined by Ilic and Stevanovic

[15] and Zhang et al. [21] for $c = 0$, Du and Zhou [4] for $c = 1$, Wang et al. [18] for $c = 2$, Zhu, Tan and Qiu [20] for $c = 3$, and Jafari Rad et al. [14] for $c = 4$. In this paper, we consider the case $c = 5$. A c-cyclic graph with $c = 5$ is referred as a *pentacyclic graph.* We characterize all pentacyclic graphs of order n with maximal Estrada index.

For undefined graph theory notations we refer to [1]. For a vertex v, the *open neighborhood* and the *closed neighborhood* of v are denoted by $N_G(v) = \{u|uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of v is denoted by $d_G(v) = |N_G(v)|$. If $E_0 \subset E(G)$, we denote by $G - E_0$ the subgraph of G obtained by deleting the edges in E_0. If E_1 is a subset of the edge set of the complement of G, then $G + E_1$ denotes the graph obtained from G by adding the edges in E_1. Similarly, if $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If $E = \{xy\}$ and $W = \{v\}$, we write $G - xy$ and $G - v$ instead of $G - \{xy\}$ and $G - \{v\}$, respectively. We refer P_n and C_n as the path and the cycle on n vertices, respectively. For vertices u, v and w (not necessarily distinct) in G, we denote by $M_k(G; u, v)$ the number of walks in G of length k from u to v, and by $M_k(G; u, v, [w])$ the number of walks in G of length k from u to v which go through w. Denote by $W_k(G; u, v)$ a walk of length k from u to v in G, and by $\mathcal{W}_k(G; u, v)$ the set of all such walks. Clearly $M_k(G; u, v) = |\mathcal{W}_k(G; u, v)|$. Note that $M_k(G; u, v) = M_k(G; v, u)$ for any positive integer k [2]. Let G and H be two graphs with $u_1, v_1 \in V(G)$ and $u_2, v_2 \in V(H)$. If $M_k(G; u_1, v_1) \leqslant M_k(H; u_2, v_2)$ for all positive integers k, then we write $(G; u_1, v_1) \preceq (H; u_2, v_2)$. If $(G; u_1, v_1) \preceq (H; u_2, v_2)$ and there is a positive integer k_0 such that $M_{k_0}(G; u_1, v_1) < M_{k_0}(H; u_2, v_2)$, then we write $(G; u_1, v_1) \prec (H; u_2, v_2)$.

2 Preliminaries and known results

Let $M_k(G) = \sum_{i=1}^{n} \lambda_i^k$. From [2] we know that $M_k(G)$ is equal to the number of closed walks of length k in G. It is well known that $M_0(G) = n$, $M_1(G) = 0$, $M_2(G) = 2m$, $M_3(G) = 6t$, and $M_4(G) = 2\sum_{i=1}^{n} d_i^2 - 2m + 8q$, where t is the number of *triangles*, q the number of *quadrangles*, and $d_i = d_G(v_i)$ the degree of v_i in G. From the *Taylor expansion* of e^x, it can be seen that $EE(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{k!}$. Thus if for two graphs G_1 and G_2, $M_k(G_1) \geqslant M_k(G_2)$ for all $k \geqslant 0$, then $EE(G_1) \geq EE(G_2)$. Moreover, if there is at least one positive integer k_0 such that $M_{k_0}(G_1) > M_{k_0}(G_2)$, then $EE(G_1) > EE(G_2)$. We make use of the following lemmas.

Lemma 1. *[2] Let v be a vertex of a graph G, and $C(v)$ be the set of all* *cycles containing v. Then* $\phi(G;x) = x\phi(G-v;x) - \sum_{uv\in E(G)} \phi(G-u-v;x) - 2\sum_{Z\in C(v)} \phi(G-V(Z);x)$, *where* $\phi(G-u-v;x) = 1$ *if* $G = P_2$, *and* $\phi(G-V(Z);x) = 1$ *if G is a cycle.*

Lemma 2. *[3] Let H be a graph and* $u, v \in V(H)$. *Suppose that* $w_i \in V(H)$, *and* $uw_i, vw_i \notin E(H)$ *for* $i = 1, 2, \ldots, r$, *where r is a positive integer. Let* $E_u = \{uw_1, uw_2, \ldots, uw_r\}$, $E_v = \{vw_1, vw_2, \ldots, vw_r\}$, $H_u = H + E_u$ *and* $H_v = H + E_v$. *If* $(H;u) \prec (H;v)$ *and* $(H;w_i,u) \preceq (H;w_i,v)$ *for* $1 \le i \le r$, *then* $EE(H_u) < EE(H_v)$.

Lemma 3. *[5] Let* G_1 *and* G_2 *be connected graphs with* $u \in V(G_1)$ *and* $v \in V(G_2)$. *Let G be the graph obtained by joining u to v and G′ be the graph obtained by identifying u with v, and attaching a pendant vertex to the common vertex. If* $d_G(u), d_G(v) \ge 2$, *then* $EE(G) < EE(G')$.

Given two vertex-disjoint connected graphs G and H and two vertices $u \in V(G)$ and $w \in V(H)$, the *coalescence* of G and H, denoted by $G(u) \circ H(w)$, is the graph obtained by identifying the vertex u of G with the vertex w of H.

Lemma 4. *[18] Let u and v be two vertices of a connected graph* H_1, *and w be a vertex of a connected graph* H_2, *where* H_2 *is disjoint from* H_1. *Let* H_2' *be a copy of* H_2, *containing the vertex w′ corresponding to w of* H_2, *and* $G = (H_1(u) \circ H_2(w))(v) \circ H_2'(w')$.
(i) If there exists an automorphism σ of H_1 *such that it interchanges u and v, then* $(G;u,t) = (G;v,\sigma(t))$ *for any vertex t.*
(ii) If $\bar{H}_1$ *is obtained from* H_1 *by adding some edges incident with v but not u,* $\bar{H}_2$ *is obtained from* H_2' *by adding some vertices or edges such that the resulting graph is connected, and* $\bar{G}$ *is obtained from G by replacing* H_1 *with* $\bar{H}_1$ *or* H_2' *with* $\bar{H}_2$, *then* $(\bar{G};u,t) \prec (\bar{G};v,\sigma(t))$.

3 Main results

Denote by F_n the class of all *pentacyclic* graphs of order n. For a graph $G \in F_n$, the base of G, denoted by $B(G)$, is the *minimal pentacyclic* subgraph of G. Obviously, $B(G)$ is the unique *pentacyclic* subgraph of G containing no pendant vertex, and G can be obtained from $B(G)$ by planting trees to some vertices of $B(G)$. We know that *pentacyclic* graphs have the following two types of bases (as shown in Figures 1 – 2): G_i^5 $(i = 1, \ldots, 12)$ and G_i^6 $(i = 1, \ldots, 29)$.

Let $F_n^5 = \{G | B(G) \cong G_i^5, i \in \{1, \ldots, 12\}\}$ and $F_n^6 = \{G | B(G) \cong G_i^6, i \in$

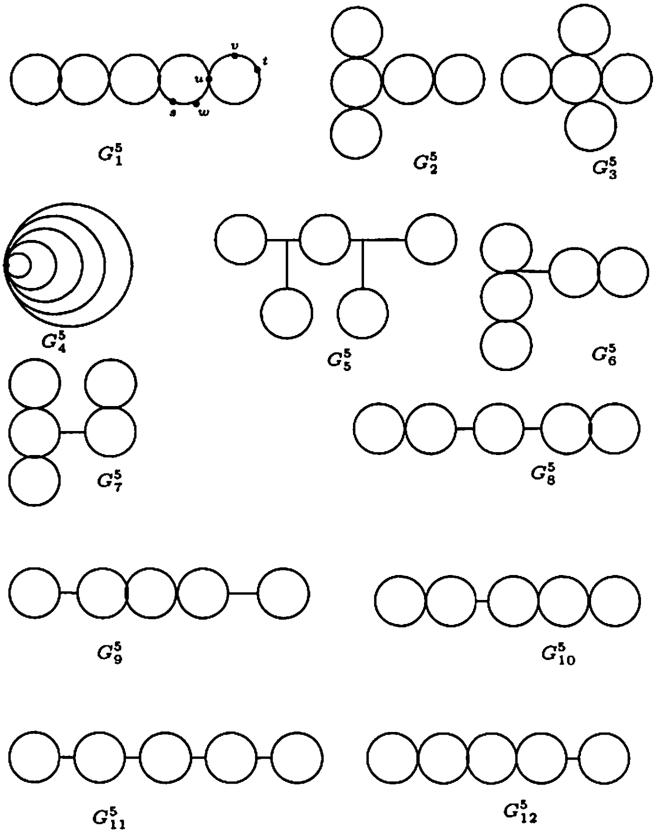

Figure 1: The graphs G_i^5 $(i = 1, \ldots, 12)$.

$\{1, \ldots, 29\}\}$. Then $F_n = F_n^5 \cup F_n^6$. The following lemmas can be obtained readily by Lemma 3, and so we do not state a proof.

Lemma 5. *If G is an extremal graph with maximal Estrada index in F_n, then G is obtained from its base by attaching some pendant vertices.*

Lemma 6. *(i) If G is an extremal graph with maximal Estrada index in F_n^5, then $B(G) \cong G_i^5$ for some $i \in \{1, 2, 3, 4\}$.*
(ii) If G is an extremal graph with maximal Estrada index in F_n^6, then $B(G) \cong G_i^6$ for some $i \in \{1, 2, \ldots, 18\}$.

Lemma 7. *If G_1 is an extremal graph with maximal Estrada index in F_n^5, then there exists a graph G_2 in F_n^6 such that $EE(G_2) > EE(G_1)$.*

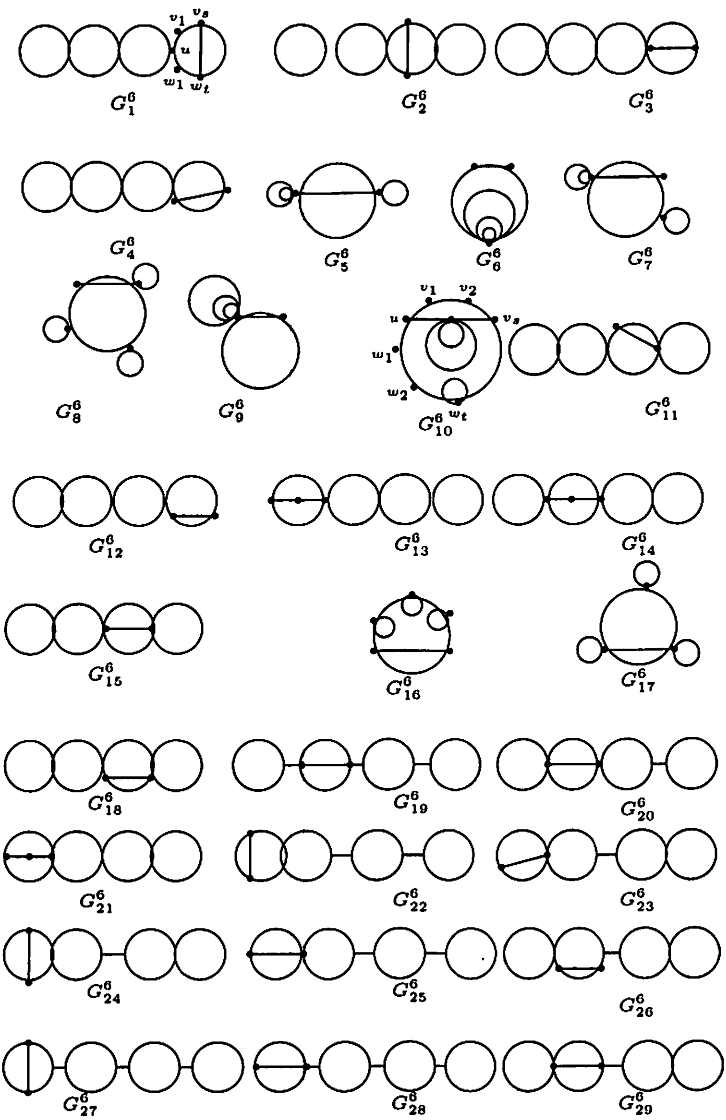

Figure 2: The graphs G_i^6 $(i = 1, \ldots, 29)$.

Proof. By Lemma 6(i), we know that $B(G_1) \cong G_i^5$ for $i \in \{1, 2, 3, 4\}$. If $B(G_1) \cong G_1^5$, then we let uv, vt, uw,$ws \in E(G_1)$ (as shown in Figure 1.). Without loss of generality, assume that $d_{G_1}(w) \geq d_{G_1}(v)$. Let H_1 be the graph obtained from G_1 by deleting ws, vt, $d_{G_1}(w) - 2$ pendant edges attached at w and $d_{G_1}(v) - 2$ pendant edges attached at v. There exists an automorphism σ of H_1 which interchange v and w, and preserves all other vertices. Let $H_2 \cong K_{1,d_{G_1}(v)-2}$ with center v' and $G_0 = (H_1(v) \circ H_2(v'))(w) \circ H_2(v')$. By Lemma 4 (i), we have $(G_0; w, x) = (G_0; v, \sigma(x))$ for any vertex $x \in V(G_0)$. Let G_3 be the graph obtained from G_0 by adding the edge ws and $d_{G_1}(w) - d_{G_1}(v)$ pendant edges attached at w. By Lemma 4 (ii), we have $(G_3; w, x) \succ (G_3; v, \sigma(x))$ for any vertex $x \in V(G_3)$. Obviously, $G_1 = G_3 + vt$. Let $G_2 = G_3 + wt$. It is obvious that $G_2 \in F_n^5$. Then by Lemma 2, we deduce that $EE(G_2) > EE(G_1)$. The proof for the cases $B(G_1) \cong G_2^5$ or G_3^5 or G_4^5 is similarly verified. □

Corollary 1. *If G is a graph with maximal Estrada index in F_n, then $B(G) \cong G_i^6$ $(i \in \{1, 2, \dots, 18\})$.*

For two vertices v_0 and v_s of degree at least three in a graph G, the internal path of G is a walk $v_0 v_1 \dots v_s$ such that the vertices $v_0, v_1, \dots, v_s$ are distinct, and $d_G(v_i) = 2$, for $0 < i < s$.

Lemma 8. *Let $G \in F_n^6$, and let P_u^i $(1 \le i \le d_B(G)(u))$ be the internal path in $B(G)$ with one end vertex u, where $d_B(G)(u) \ge 3$ $(u \in B(G))$. If there exist two paths P_u^k and P_u^l $(1 \le l, k \le d_B(G)(u))$ with $|P_u^k| \ge 1$ and $|P_u^l| \ge 3$, then there exists a graph $\hat{G} \in F_n^6$ such that $|E(B(G))| - |E(B(\hat{G}))| = 1$ and $EE(\hat{G}) > EE(G)$.*

Proof. Let $P_u^k = uv_1 \dots v_s$ and $P_u^l = uw_1 \dots w_t$, where $s \ge 2$ and $t \ge 3$. (as shown in Figure 2.) Without loss of generality assume that $d_G(w_1) \ge d_G(v_1)$. Let H_1 be the graph obtained from G by deleting the edges $w_1 w_2$, $v_1 v_2$, $d_G(v_1) - 2$ pendant edges attached at v_1 and $d_G(w_1) - 2$ pendant edges attached at w_1. There exists an automorphism σ of H_1 which interchange v_1 and w_1, and preserves all other vertices. Let $H_2 \cong K_{1, d_G(v_1)-2}$ with center v' and $G_0 = (H_1(v_1) \circ H_2(v'))(w_1) \circ H_2(v')$. By Lemma 4 (i), we have $(G_0; w_1, v) = (G_0; v_1, \sigma(v))$ for any vertex $v \in V(G_0)$. Let G_1 be the graph obtained from G_0 by adding edges $w_1 w_2$, and $d_G(w_1) - d_G(v_1)$ pendant edges attached at w_1. By Lemma 4(ii), we have $(G_1; w_1, v) \succ (G_1; v_1, \sigma(v))$ for any vertex $v \in V(G_1)$. Obviously, $G = G_1 + v_1 v_2$. Let $\hat{G} = G_1 + w_1 v_2$. Observe that $\hat{G} \in F_n^6$ and $|E(B(G))| - |E(B(\hat{G}))| = 1$. By Lemma 2, we have $EE(\hat{G}) > EE(G)$. □

Similar to the proof of Lemma 8, we have the following.

Lemma 9. *Let $G \in F_n^6$, and let $P_u^k = uv_1v_2$ and $P_u^l = uw_1w_2$ be two internal paths in $B(G)$, where $d_B(G)(u) \ge 3$, $(u \in B(G))$. If $v_2 \ne w_2$, then there exists a graph $\hat{G} \in F_n^6$ such that $|E(B(G))| - |E(B(\hat{G}))| = 1$ and $EE(\hat{G}) > EE(G)$.*

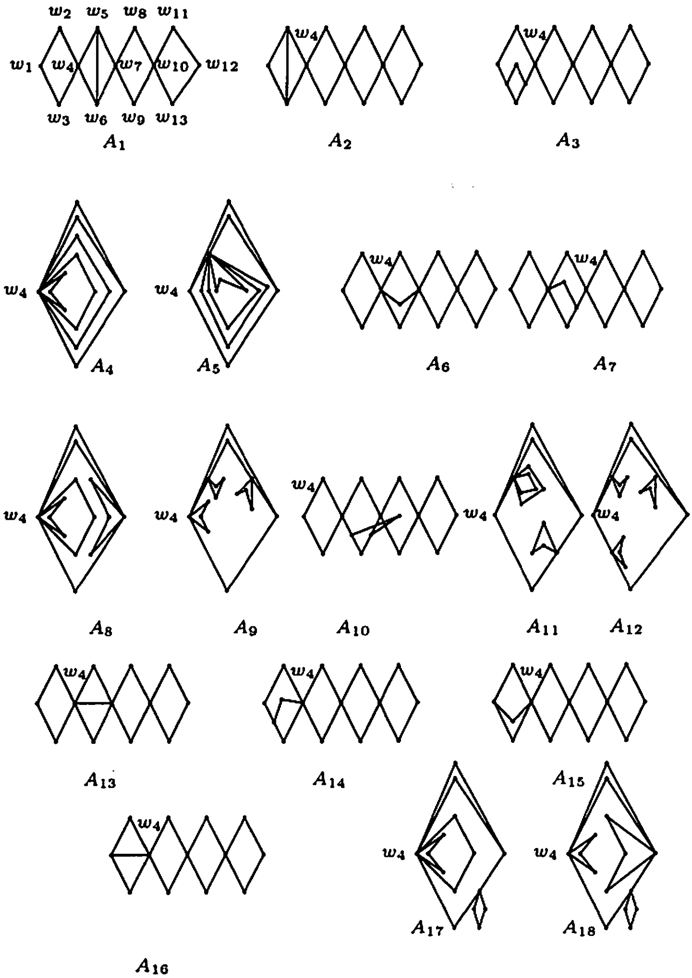

Figure 3: The graphs A_1,...,A_{18}.

As a consequence of Lemma 9 we have the following.

Corollary 2. *If G is a graph with maximal Estrada index in F_n^6, then $B(G) \cong A_i$ for some $i \in \{1, 2, \ldots, 18\}$ (as shown in Figure 3.).*

Lemma 10. *Let G be an extremal graph with maximal Estrada index and $B(G) \cong A_i$ for some $i \in \{1, 2, \ldots, 18\}$ (as shown in Figure 3.). Then G is obtained from A_i by attaching $n - |V(A_i)|$ pendant vertices at a vertex w_4 with maximum degree in A_i, $i = 1, 2, \ldots, 18$.*

Proof. For the case of $B(G) \cong A_1$, let w_i's ($i = 1, 2, \ldots, 13$) be the vertices of A_1 (as shown in Figure 3.). Assume that each w_i is attached to m_i pendant edges in G, where $m_i \geq 0$ and $\sum_{i=1}^{13} m_i = n - 13$. For convenience, denote $G = A_1(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13})$. We consider the following cases. Assume that at least nine values of m_1, m_2, m_3, m_5, m_6, m_8, m_9, m_{11}, m_{12}, m_{13}, are nonzero. Let H_1 be the graph obtained from $A_1(m_1, 0, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13})$ by deleting the pendant vertices of w_1. There exists an automorphism which interchanges w_1, w_2 and preserves all other vertices. By Lemma 4(ii), $(A_1(m_1, 0, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13}); w_1) \succ (A_1(m_1, 0, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13}); w_2)$. By Lemma 2, $(A_1(m_1+m_2, 0, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13}) \succ (A_1(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10}, m_{11}, m_{12}, m_{13}))$. This is a contradiction. So at most eight values of m_1, m_2, m_3, m_5, m_6, m_8, m_9, m_{11}, m_{12}, m_{13}, are nonzero. Without loss of generality assume that $m_2 = m_3 = 0$. With a similar argument, we obtain a contradiction. So at most seven values of m_1, m_2, m_3, m_5, m_6, m_8, m_9, m_{11}, m_{12}, m_{13}, are nonzero. Continuing this argument we obtain that all of m_2, m_3, m_5, m_6, m_8, m_9, m_{11}, m_{12}, m_{13}, are zero, i.e. $m_2 = m_3 = m_5 = m_6 = m_8 = m_9 = m_{11} = m_{12} = m_{13} = 0$. Then $G = A_1(m_1, 0, 0, m_4, 0, 0, m_7, 0, 0, m_{10}, 0, 0, 0)$. If all of m_4, m_7, m_{10} are nonzero with a similar argument, as before, we obtain a contradiction. So at least one of m_4, m_7, m_{10} are zero, say $m_7 = 0$. Then $G = A_1(m_1, 0, 0, m_4, 0, 0, 0, 0, 0, m_{10}, 0, 0, 0)$. If both m_4 and m_{10} are nonzero, with a similar argument as before, we obtain a contradiction. So at least one of m_4 and m_{10} are zero, say $m_{10} = 0$. Then $G = A_1(m_1, 0, 0, m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. If both m_1 and m_4 are nonzero, we let H_1 be the graph obtained from A_1 by deleting the edges w_4w_5 and w_3w_4. There exists an automorphism which interchanges w_1, w_4 and preserves all other vertices. By Lemma 4 (ii), $(A_1(0, 0, 0, m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0); w_4) \succ (A_1(0, 0, 0, m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0); w_1)$. By Lemma 2, $A_1(0, 0, 0, m_1 + m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0) \succ A_1(m_1, 0, 0, m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. This is a contradiction. Thus $G = A_1(0, 0, 0, m_4, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. The proof for $B(G) \cong A_i, i \in \{2, 3, \ldots, 18\}$ is similarly verified. □

Let F_i be the graph obtained from A_i by attaching $n - |V(A_i)|$ pendant vertices at a vertex with maximum degree in A_i ($i \in \{1, 2, \ldots, 18\}$). By Lemma 1, we obtain $\phi(F_i; x)$ for $i = 1, 2, ..., 5$ as follows. To see $\phi(F_i; x)$ for $i = 6, 7, \ldots, 18$ see Appendix.

$\phi(F_1; x) = x^{n-13}[x^{13} - x^{12} - (n+3)x^{11} + (n-7)x^{10} + (12n-83)x^9 - (6n-77)x^8 - (33n-331)x^7 + (9n-149)x^6 - (2n-34)x^5 - (40n-80)x^4 + (24n-$

$288)x^3] = x^{n-13}f_1(x)$;

$\phi(F_2;x) = x^{n-13}[x^{13} - (n+8)x^{11} - 4x^{10} + (17n-70)x^9 + (2n+32)x^8 - (91n-736)x^7 - (28n-118)x^6 + (183n-1849)x^5 + (98n-954)x^4 - (108n - 1212)x^3 - (72n-808)x^2] = x^{n-13}f_2(x)$;

$\phi(F_3;x) = x^{n-16}[x^{16} - (n+9)x^{14} + (21n-102)x^{12} - (157n-1476)x^{10} + (519n-6067)x^8 - (742n-9713)x^6 + (360n-4972)x^4] = x^{n-16}f_3(x)$;

$\phi(F_4;x) = x^{n-14}[x^{14} - (n+9)x^{12} + (18n-71)x^{10} - (105n-899)x^8 + (232n - 2580)x^6 + (144n-1720)x^4] = x^{n-14}f_4(x)$;

$\phi(F_5;x) = x^{n-14}[x^{14} - (n+5)x^{12} + (10n-32)x^{10} - (36n-292)x^8(56n - 632)x^6 - (32n-14)x^4] = x^{n-14}f_5(x)$.

Note that $\phi(F_i;x)$ for $i = 1,2,\ldots,18$ plays a key role in the proof of the main theorem. Also the Estrada index $EE(F_i)$ for $i = 1,2,\ldots,18$ are computed in Table 1 for $n = 13,\ldots,18$, (See Appendix). We are now ready to state the main result of this paper.

Theorem 1. *Let G be a graph in F_n. If $n \geq 13$, then $EE(G) \leq EE(F_1)$, with equality if and only if $G \cong F_1$.*

Proof. By a direct calculation, we can see that for $n \geq 11$,

$f_1(\sqrt{n-10}) = -n(n-10)^{\frac{11}{2}} + (n-10)^{\frac{13}{2}} + 12n(n-10)^{\frac{9}{2}} - 3n^5 - 33n(n-10)^{\frac{7}{2}} - 83(n-10)^{\frac{9}{2}} + 176n^4 - 2n(n-10)^{\frac{5}{2}} + 331(n-10)^{\frac{7}{2}} - 4103n^3 + 24n(n-10)^{\frac{3}{2}} + 34(n-10)^{\frac{5}{2}} + 47530n^2 - 288(n-10)^{\frac{3}{2}} - 273700n + 627000 < 0.$

This implies that $\lambda_1(F_1) > \sqrt{n-10}$. It is easy to see that the graph $F_1 - w_4$ has eigenvalues $\pm\sqrt{2}$, -2.513, 2.649, -1.638, 0.907, -0.407, 2, -1, and 0 where the multiplicity of 0 is $n-12$ and the multiplicity of others is one. By interlacing property of eigenvalues of $A(F_1 - w_4)$ and $A(F_1)$, we have $\lambda_i(F_1) \geq \lambda_i(F_1 - w_4)$ for $i = 2,3,\ldots,n-1$ (see [2]). Then

$EE(F_1) = \sum_{i=1}^{n} e^{\lambda_i(F_1)} > e^{\lambda_1(F_1)} + \sum_{i=2}^{n-1} e^{\lambda_i(F_1-w_4)} > e^{\sqrt{n-10}} + (n-12) + e^{-1} + e^{-2.513} + e^{2.649} + e^{-1.638} + e^{0.907} + e^{-0.405} + e^2 + e^{\sqrt{2}} + e^{-\sqrt{2}} = H_1.$

We prove that $EE(F_1) > EE(F_i)$ for $i = 2,\ldots,18$. We only prove for $i = 2,3,4,5$. The other cases are similarly verified.

Case 1. $i = 2$. By a direct calculation, the graph $F_2 - w_4$ has eigenvalues ± 2.548, ± 0.629, ± 1.763, 2, and -1, where the multiplicity of -1 is two (and the multiplicity of others is one). For $14 \leq n \leq 26$, we have

$f_2(\sqrt{n-11}) = -(n+8)(n-11)^{\frac{11}{2}} + (17n-70)(n-11)^{\frac{9}{2}} + (n-11)^{\frac{13}{2}} +$

$(2n+32)(n-11)^4-(91n-736)(n-11)^{\frac{7}{2}}-4(n-11)^5+(183n-1849)(n-11)^{\frac{5}{2}}+(98n-954)(n-11)^2-(108n-212)(n-11)^{\frac{3}{2}}+90(n-11)^3>0.$

For $n\geq 19$, $f_2(1)=100n-1814>0$. By interlacing property of eigenvalues of F_2-w_4 and F_2 we have $\lambda_2(F_2)\leq\lambda_1(F_2-w_4)=2.548$. Further, since $f_2(1)=100n-1814>0$, we have $2.548<\lambda_1(F_2)<\sqrt{n-11}$. Similarly, by the fact that $\lambda_i(F_2)\leq\lambda_{i-1}(F_2-w_4)$ for $i=2,3,\ldots,n$, we obtain the following.

$EE(F_2)=\sum_{i=1}^{n}e^{\lambda_i(F_2)}\leq e^{\lambda_1(F_2)}+\sum_{i=1}^{n-1}e^{\lambda_i(F_2-w_4)}<e^{\sqrt{n-11}}+2e^{-1}+e^{2.548}+e^{-2.548}+e^{0.629}+e^{-0.629}+e^{1.763}+e^{-1.763}+e^2=H_2.$

$H_1-H_2=e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{-2.513}+e^{2.649}+e^{-1.638}+e^{0.907}+e^{-0.405}+e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-1}-e^{2.548}-e^{-2.548}-e^{0.629}-e^{-0.629}-e^{1.763}-e^{-1.763}>e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{2.649}+e^{0.907}+e^{\sqrt{2}}-e^{2.548}-e^{0.629}-e^{1.763}.$

Note that $e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{2.649}+e^{0.907}+e^{\sqrt{2}}-e^{2.548}-e^{0.629}-e^{1.763}$ for $n\geq 13$. Then $H_1-H_2>0$. So $EE(F_1)>EE(F_2)$.

Case 2. $i=3$. By a direct calculation, the graph F_3-w_4 has eigenvalues ±2.548, ±0.629, ±1.763, ±2.175, and ±1.126 with multiplicity one. For $n\geq 25$, we have

$f_3(\sqrt{n-11})=n^7-105n^6+4657n^5-113245n^4+1632386n^3-13961046n^2+65648792n-13102364>0$. For $0\leq n\leq 13$, $f_3(1)=-360n+5012>0$. By interlacing property of eigenvalues of F_3-w_4 and F_3, we obtain that $\lambda_2(F_3)\leq\lambda_1(F_3-w_4)=2.548$. Further, since $f_3(1)=-360n+5012>0$ we obtain that $2.548<\lambda_1(F_3)<\sqrt{n-11}$. Similarly, by the fact that $\lambda_i(F_3)\leq\lambda_{i-1}(F_3-w_4)$ for $i=2,3,\ldots,n$, we find that

$EE(F_3)=\sum_{i=1}^{n}e^{\lambda_i(F_3)}\leq e^{\lambda_1(F_3)}+\sum_{i=1}^{n-1}e^{\lambda_i(F_3-w_4)}<e^{\sqrt{n-11}}+e^{2.548}+e^{-2.548}+e^{0.629}+e^{-0.629}+e^{1.763}+e^{-1.763}+e^{2.175}+e^{-2.175}+e^{1.126}+e^{-1.126}=H_3.$

$H_1-H_3=e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{-2.513}+e^{-1}+e^2+e^{2.649}+e^{-1.638}+e^{0.907}+e^{-0.405}+e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{2.548}-e^{-2.548}-e^{0.629}-e^{-0.629}-e^{1.763}-e^{-1.763}-e^{2.175}-e^{-2.175}-e^{1.126}-e^{-1.126}>e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{2.649}+e^2+e^{0.907}+e^{\sqrt{2}}-e^{2.548}-e^{0.629}-e^{1.763}-e^{2.175}-e^{1.126}.$

Note that $e^{\sqrt{n-10}}-e^{\sqrt{n-11}}+(n-12)+e^{2.649}+e^2+e^{0.907}+e^{\sqrt{2}}-e^{2.548}-e^{0.629}-e^{1.763}-e^{2.175}-e^{1.126}>0$, for $n\geq 18$. Then $H_1-H_3>0$. So $EE(F_1)>EE(F_3)$.

Case 3. $i=4$. By a direct calculation, the graph F_4-w_4 has eigenvalues

± 2.548, ± 0.629, ± 1.763, and $\pm\sqrt{3}$ with multiplicity one. For $13 \le n \le 21$, we have

$f_4(\sqrt{n-11}) = -2n^6 + 154n^5 - 486n^4 + 81032n^3 - 753438n^2 + 7315998n - 7608480 > 0.$

For $n \ge 13$, $f_4(1) = 288n - 3480 > 0$. By interlacing property of eigenvalues of $F_4 - w_4$ and F_4, we have $\lambda_2(F_4) \le \lambda_1(F_4 - w_4) = 2.548$. Further, it can be seen that $f_4(1) = 288n - 3480 > 0$. Thus we have $2.548 < \lambda_1(F_4) < \sqrt{n-11}$. Similarly, by the fact that $\lambda_i(F_4) \le \lambda_{i-1}(F_4 - w_4)$ for $i = 2, 3, \ldots, n$, we have

$EE(F_4) = \sum_{i=1}^{n} e^{\lambda_i(F_4)} \le e^{\lambda_1(F_4)} + \sum_{i=1}^{n-1} e^{\lambda_i(F_4 - w_4)} < e^{\sqrt{n-11}} + e^{2.548} + e^{-2.548} + e^{0.629} + e^{-0.629} + e^{1.763} + e^{-1.763} + e^{\sqrt{3}} + e^{-\sqrt{3}} = H_4.$

$H_1 - H_4 = e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + (n-12) + e^{-2.513} + e^{-1} + e^2 + e^{2.649} + e^{-1.638} + e^{0.907} + e^{-0.405} + e^{\sqrt{2}} + e^{-\sqrt{2}} - e^{2.548} - e^{-2.548} - e^{0.629} - e^{-0.629} - e^{1.763} - e^{-1.763} - e^{\sqrt{3}} - e^{-\sqrt{3}} > e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + (n-12) + e^2 + e^{2.649} + e^{0.907} + e^{\sqrt{2}} - e^{2.548} - e^{0.629} - e^{1.763} - e^{\sqrt{3}}.$

Note that $e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + (n-12) + e^2 + e^{2.649} + e^{0.907} + e^{\sqrt{2}} - e^{2.548} - e^{0.629} - e^{1.763} - e^{\sqrt{3}}$, for $n \ge 13$. Then $H_1 - H_4 > 0$. So $EE(F_1) > EE(F_4)$.

Case 4. $i = 5$. By a direct calculation, the graph $F_5 - w_4$ has eigenvalues $\pm\sqrt{2}$, $\pm\sqrt{3}$, and 0, where has multiplicity of 0 is $n - 11$ and multiplicity $\sqrt{2}$ and $-\sqrt{2}$ are three. (and the multiplicity of others is one). For $n = 10, 12, 13, \ldots, 16$, we have

$f_5(\sqrt{n-11}) = -6n^6 + 438n^5 - 13248n^4 + 212604n^3 - 1909878n^2 + 9108462n - 18020772 > 0$. For $0 \le n \le 23$, $f_5(1) = -3n + 72 > 0$. By interlacing property of eigenvalues of $F_5 - w_4$ and F_5, $\lambda_2(F_5) \le \lambda_1(F_5 - w_4) = \sqrt{3}$. Further, we can see that $f_5(1) = -3n + 72 > 0$, and thus $\sqrt{3} < \lambda_1(F_5) < \sqrt{n-11}$. Similarly, by the fact that $\lambda_i(F_5) \le \lambda_{i-1}(F_5 - w_4)$ for $i = 2, 3, \ldots, n$,

$EE(F_5) = \sum_{i=1}^{n} e^{\lambda_i(F_5)} \le e^{\lambda_1(F_5)} + \sum_{i=1}^{n-1} e^{\lambda_i(F_5 - w_4)} < e^{\sqrt{n-11}} + (n-11) + 3e^{\sqrt{2}} + 3e^{-\sqrt{2}} + e^{\sqrt{3}} + e^{-\sqrt{3}} = H_5.$

$H_1 - H_5 = e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + e^{-1} + e^{-2.513} + e^{2.649} + e^{-1.638} + e^{0.907} + e^{-0.405} + e^2 - 2e^{\sqrt{2}} - 2e^{-\sqrt{2}} - e^{\sqrt{3}} - e^{-\sqrt{3}} - 1 > e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + e^{2.649} + e^{0.907} + e^2 - 2e^{\sqrt{2}} - e^{\sqrt{3}} - 1.$

Note that for $n \ge 11$, $e^{\sqrt{n-10}} - e^{\sqrt{n-11}} + e^{2.649} + e^{0.907} + e^2 - 2e^{\sqrt{2}} - e^{\sqrt{3}} - 1 > 0$. So $EE(F_1) > EE(F_5)$. □

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