On the minimum spectral radius of trees with given matching number*

Weiling Sun[1]†, Tianmei Song[1], Ji-Ming Guo[2]‡, Shangwang Tan[1]
1. College of Science, China University of Petroleum,
Qingdao, Shandong, 266580, China.
2. College of Science, East China University of Science
and Technology, Shanghai, 200237, China

Abstract

Let $\Re_\beta$ denote the set of trees on $n = k\beta + 1$ $(k \geq 2)$ vertices with matching number β. In this paper, the trees with minimal spectral radius among $\Re_\beta$ $(2 \leq \beta \leq 4)$ are determined, respectively.

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1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For $v_i \in V$, the degree of v_i, written by $d(v_i)$, is the number of edges incident with v_i. The set of vertices adjacent to v_i is denoted by $N(v_i)$. The maximum degree of G

*This research is supported by NSFC (No. 11371372).
†Email addresses: sunweiling-ling@163.com(W.Sun), jimingguo@hotmail.com(J.-M. Guo), songtianmei2007@163.com(T.Song), tswang@sina.com(S.Tan)
‡Corresponding author

is denoted by $\Delta(G)$. Let $A(G) = (a_{ij})$ be the adjacency matrix of G with $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$ otherwise. It is well known that $A(G)$ is real and symmetric, so its eigenvalues are real. The spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. Since G is connected, by the Perron-Frobenius theory of nonnegative matrices, $\rho(G)$ has multiplicity one and there exists an unique unit positive eigenvector, which is called Perron vector of G, corresponding to $\rho(G)$. The characteristic polynomial of $A(G)$ is called the characteristic polynomial of G, denoted by $\Phi(G)$.

The diameter of G, denoted by $d(G)$, is the maximum distance between any two vertices of G. A set of pairwise independent edges in G is called a matching in G. While a matching of maximum cardinality is a maximum matching in G. The matching number β of G is the cardinality of a maximum matching of G. It is well known that $\beta \leq \frac{n}{2}$, with equality if and only if G has a perfect matching.

Belardo et. al [6] determined the minimal spectral radius of trees with diameter at most four. Feng et. al [5] determined the minimal Laplacian spectral radius among $\Re_\beta$ $(2 \leq \beta \leq 4)$, where $\Re_\beta$ denotes the set of trees on $n = k\beta + 1 (k \geq 2)$ vertices with matching number β. In this paper, the trees with minimal spectral radius among $\Re_\beta$ for $2 \leq \beta \leq 4$ are determined, respectively.

2 Preliminaries

Lemma 2.1 ([1]) *Let G be a connected graph with maximum degree Δ. Then $\rho(G) \geq \sqrt{\Delta}$, with equality if and only if G is a star with $1 + \Delta$ vertices.*

Lemma 2.2 ([4]) *Let G be a connected graph and G' be a proper subgraph of G. Then $\rho(G') < \rho(G)$.*

An internal path of a graph G is a sequence of vertices $v_1, v_2, \cdots, v_k$ with $k \geq 2$ such that:

(1). The vertices in the sequence are distinct (except possibly $v_1 = v_k$);

(2). v_i is adjacent to v_{i+1}, $(i = 1, 2, \cdots, k-1)$;

(3). The vertex degrees $d(v_i)$ satisfy $d(v_1) \geq 3, d(v_2) = \cdots = d(v_{k-1}) = 2$ (unless $k = 2$) and $d(v_k) \geq 3$.

Let W_n be the tree on n vertices obtained from a path P_{n-4} (of length $n-5$) by attaching two new pendant edges to each end vertex of P_{n-4}, respectively.

Lemma 2.3 ([2]) *Let G be a connected graph that is not isomorphic to W_n, G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\rho(G_{uv}) < \rho(G)$.*

Lemma 2.4 ([3]) *Let u, v be two vertices of the connected graph G. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d(v))$ are some vertices of $N(v) \setminus \{N(u) \cup u\}$ and $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G, where x_i corresponds to $v_i (1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting vv_i and adding $uv_i (1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 2.5 ([1]) *Let $e = uv$ be an edge and v be a vertex of the tree T, respectively. Then*

$$\Phi(T) = \Phi(T-e) - \Phi(T-u-v) \text{ and } \Phi(T) = x\Phi(T-v) - \sum_{wv \in E} \Phi(T-w-v).$$

Lemma 2.6 ([4]) *Let F, H be two graphs. If $\Phi(F, x) < \Phi(H, x)$ for any $x \geq \rho(H)$, then $\rho(F) > \rho(H)$.*

Lemma 2.7 ([6]) *Let T be a tree on $n = 2k+1$ vertices with diameter $d(T) = 3$. Then*

$$\rho(T) \geq \rho(H_2(k, k-1)),$$

with equality if and only if $T = H_2(k, k-1)$, where $H_2(k, k-1)$ is the tree obtained from an edge uv by attaching k and $k-1$ pendant vertices at u and v, respectively.

3 Main results

Let $T_2(s,t)$ be the tree with $n = s+t+5$ vertices obtained from the path $v_1v_2v_3$ by attaching $s+1$ and $t+1$ pendant vertices at v_1 and v_3, respectively. Let $\mathcal{T}_2 = \{T_2(s,t) \mid s+t+5 = 2k+1\}$, especially $T_2^* = T_2(k-2,k-2)$. Let $T_3(s,r,t)$ be the tree with $n = s+t+r+7$ vertices obtained from the path $v_1v_2v_3v_4v_5$ by attaching $s+1$, r and $t+1$ pendant vertices at v_1, v_3, v_5, respectively. Let $\mathcal{T}_3 = \{T_3(s,r,t) \mid s+r+t+7 = 3k+1\}$, especially $T_3^* = T_3(k-2,k-2,k-2)$. Let $T_4(s_1,s_2,s_3,s_4)$ be the tree obtained from the path $v_1v_2v_3v_4v_5v_6v_7$ by attaching s_1+1, s_2, s_3, s_4+1 pendant vertices at v_1, v_3, v_5, v_7, respectively. By symmetry, we have $T_4(s_1,s_2,s_3,s_4) = T_4(s_4,s_3,s_2,s_1)$. Let $\mathcal{T}_4 = \{T_4(s_1,s_2,s_3,s_4) \mid s_1+s_2+s_3+s_4+9 = 4k+1\}$, especially, $T_4^{(1)} = T_4(k-2,k-2,k-2,k-2)$, $T_4^{(2)} = T_4(k-1,k-3,k-2,k-2)$, $T_4^{(3)} = T_4(k-1,k-3,k-3,k-1)$.

Lemma 3.1 *Under the above notations, we have* $\rho(T_2^*) = \sqrt{k+1}$, $\rho(T_3^*) = \sqrt{k+\sqrt{2}}$ *and* $\rho(T_4^{(1)}) = \rho(T_4^{(2)}) = \rho(T_4^{(3)}) = \sqrt{k+\frac{1+\sqrt{5}}{2}}$.

Proof. From Lemma 2.5, we have
$\Phi(T_2^*) = x^{2k-3}(x^2-k-1)(x^2-k+1)$;
$\Phi(T_3^*) = x^{3k-5}(x^2-k)(x^4-2kx^2+k^2-2)$;
$\Phi(T_4^{(1)}) = x^{4k-7}(x^4-(2k-1)x^2+k^2-k-1)(x^4-(2k+1)x^2+k^2+k-1)$;
$\Phi(T_4^{(2)}) = x^{4k-7}(x^4-(2k-1)x^2+k^2-k-2)(x^4-(2k+1)x^2+k^2+k-1)$;
$\Phi(T_4^{(3)}) = x^{4k-7}(x^4-(2k-1)x^2+k^2-k-3)(x^4-(2k+1)x^2+k^2+k-1)$.

By direct calculation, the result follows. □

If $k = 2$, it is well known that T_2^*, T_3^* and $T_4^{(1)}$ have the minimal spectral radius in $\Re_2$, $\Re_3$ and $\Re_4$, respectively [1]. In the following, we prove that T_2^*, T_3^*, $T_4^{(i)}$ $(i = 1,2,3)$ have the minimal spectral radius in $\Re_2$, $\Re_3$, $\Re_4$, respectively, when $k \geq 3$.

3.1 The minimal spectral radius of trees in $\Re_2$

Theorem 3.2 *If $T \in \Re_2$, then $\rho(T) \geq \rho(T_2^*)$, with equality if and only if $T = T_2^*$.*

Proof. Since $T \in \Re_2$, we have $3 \leq d(T) \leq 4$. If $d(T) = 4$, then $T \in \mathcal{T}_2$. If $T \neq T_2^*$, then from Lemma 2.4, we have $\rho(T) > \rho(T_2^*)$. If $d(T) = 3$, then T_2^* can be obtained from $H_2(k, k-1)$ by subdividing the nonpendant edge and deleting a suitable pendant vertex. From Lemmas 2.2, 2.3 and 2.7, we have $\rho(T) \geq \rho(H_2(k, k-1)) > \rho(T_2^*)$. □

3.2 The minimal spectral radius of trees in $\Re_3$

Lemma 3.3 *Fix r, let $T_3(s,r,t) \in \mathcal{T}_3$. Then $\rho(T_3(s,r,t)) \geq \rho(T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil))$ with equality if and only if $T_3(s,r,t) = T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil)$.*

Proof. If $|s-t| \leq 1$, then $T_3(s,r,t) = T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil) = T_3(\lceil \frac{s+t}{2} \rceil, r, \lfloor \frac{s+t}{2} \rfloor)$; if $|s-t| > 1$, applying Lemma 2.4 to $T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil)$, we have either $\rho(T_3(s,r,t)) > \rho(T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil))$ or $\rho(T_3(t,r,s)) > \rho(T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil))$. Note that $\rho(T_3(s,r,t)) = \rho(T_3(t,r,s))$. The result follows. □

Lemma 3.4 *If $T \in \Re_3, d(T) = 6$, then $\rho(T) \geq \rho(T_3^*)$, with equality if and only if $T = T_3^*$.*

Proof. Since $T \in \Re_3$, $d(T) = 6$, we have $T = T_3(s,r,t) \in \mathcal{T}_3$, where $s + r + t + 7 = 3k + 1$. From Lemma 3.3, we have $\rho(T) \geq \rho(T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil))$, with equality if and only if $T = T_3(\lfloor \frac{s+t}{2} \rfloor, r, \lceil \frac{s+t}{2} \rceil)$. We distinguish the following three cases.

Case 1. $k-r-2$ is an even number. If $k-r-2 = 0$, then $\lfloor \frac{s+t}{2} \rfloor = \lceil \frac{s+t}{2} \rceil = k-2$. The result follows. If $k-r-2 \neq 0$, then $\lfloor \frac{s+t}{2} \rfloor = \lceil \frac{s+t}{2} \rceil = \frac{3k-6-r}{2}$. From Lemma 2.5, we have

$$\Phi(T_3(\tfrac{3k-6-r}{2}, r, \tfrac{3k-6-r}{2}))$$
$$= x^{3k-5}(x^2 - \tfrac{3k-r-2}{2})(x^4 - \tfrac{3k+r+2}{2}x^2 + \tfrac{3kr-r^2}{2} + 3k - 2r - 4).$$

So, $\rho(T) \geq \rho(T_3(\frac{3k-6-r}{2}, r, \frac{3k-6-r}{2})) = \sqrt{k + \frac{\sqrt{9(\frac{k-2-r}{2})^2+8} - \frac{k-2-r}{2}}{2}} > \sqrt{k+\sqrt{2}} = \rho(T_3^*)$

Case 2. $k-2-r$ is an odd number and $k-2-r \geq 1$. Then $\lfloor \frac{s+t}{2} \rfloor = \frac{3k-7-r}{2}$, $\lceil \frac{s+t}{2} \rceil = \frac{3k-5-r}{2}$. By Lemmas 2.1 and 3.3, we have

$$\rho(T) \geq \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) > \sqrt{\frac{3k-1-r}{2}}. \tag{3.1}$$

If $\rho(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})) \leq \sqrt{\frac{3k-1-r}{2}}$, then from Eq. (3.1), we have

$$\begin{aligned} \rho(T) &\geq \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) \\ &> \rho(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})). \end{aligned} \tag{3.2}$$

If $\rho(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})) > \sqrt{\frac{3k-1-r}{2}}$, then from Lemma 2.5, we have for $x > \sqrt{\frac{3k-1-r}{2}}$,

$$\begin{aligned} &\Phi(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) - \Phi(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})) \\ &= x^{3k-5}(\frac{3r-3k+7}{2}x^2 + \frac{(3k-3r-7)(3k-r-3)}{4} + 1) \\ &< x^{3k-5}(\frac{3r-3k+9}{2}) \leq 0. \end{aligned}$$

From Lemma 2.6 and Eq. (3.1), we have

$$\begin{aligned} \rho(T) &\geq \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) \\ &> \rho(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})). \end{aligned} \tag{3.3}$$

Since $k-2-r$ is an odd number, $k-2-(r+1)$ is an even number. Thus from Eqs. (3.2), (3.3) and Case 1, we have $\rho(T) > \rho(T_3(\frac{3k-7-r}{2}, r+1, \frac{3k-7-r}{2})) \geq \rho(T_3^*)$.

Case 3. $k-2-r$ is an odd number and $k-2-r \leq -1$. Then $\lfloor \frac{s+t}{2} \rfloor = \frac{3k-7-r}{2}$, $\lceil \frac{s+t}{2} \rceil = \frac{3k-5-r}{2}$. From Lemmas 2.1 and 3.3, we have

$$\rho(T) \geq \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) > \sqrt{r+2}. \tag{3.4}$$

If $\rho(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2})) \le \sqrt{r+2}$, from Eq. (3.4), we have

$$\begin{aligned}\rho(T) &\ge \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) \\ &> \rho(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2})). \end{aligned} \tag{3.5}$$

If $\rho(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2})) > \sqrt{r+2}$, then from Lemma 2.5, we have for $x > \sqrt{r+2}$,

$$\begin{aligned}&\Phi(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) - \Phi(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2})) \\ &= x^{3k-5}(-\frac{3r-3k+5}{2}x^2 - \frac{(3k-r-1)(3k-3r-5)}{4} - 1) \\ &< x^{3k-5}(-(\frac{3r-3k+5}{2})^2 - 1) < 0.\end{aligned}$$

Form Lemma 2.6 and Eq. (3.4), we have

$$\begin{aligned}\rho(T) &\ge \rho(T_3(\frac{3k-7-r}{2}, r, \frac{3k-5-r}{2})) \\ &> \rho(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2})). \end{aligned} \tag{3.6}$$

Since $k-2-r$ is an odd number, $k-2-(r-1)$ is an even number. From Eqs. (3.5), (3.6) and Case 1, we have $\rho(T) > \rho(T_3(\frac{3k-5-r}{2}, r-1, \frac{3k-5-r}{2}) \ge \rho(T_3^*)$. □

Let $H_3(s_1, s_2, s_3), F_3(s_1, s_2, s_3), L_3(s_1, s_2, s_3), L_3^{(a)}(s_1, s_2, s_3)$ be the graphs as Fig. 1. Let $\mathcal{H}_3 = \{H_3(s_1, s_2, s_3) \mid s_1, s_3 \ge 1, s_2 \ge 0, \sum_{i=1}^{3} s_i + 4 = 3k+1, k \ge 3\}$, $\mathcal{F}_3 = \{F_3(s_1, s_2, s_3) \mid s_1, s_2, s_3 \ge 1, \sum_{i=1}^{3} s_i + 3 = 3k+1, k \ge 3\}$, $\mathcal{L}_3 = \{L_3(s_1, s_2, s_3) \mid s_1, s_2, s_3 \ge 1, \sum_{i=1}^{3} s_i + 4 = 3k+1, k \ge 3\}$, $\mathcal{L}_3^{(a)} = \{L_3^{(a)}(s_1, s_2, s_3) \mid s_1, s_2, s_3 \ge 1, \sum_{i=1}^{3} s_i + 5 = 3k+1, k \ge 3\}$.

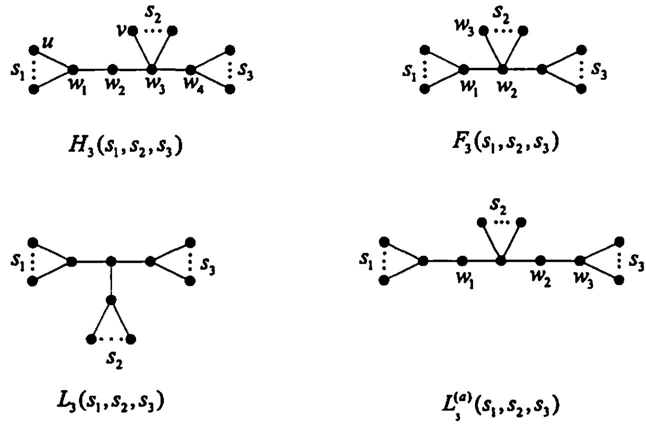


Figure 1: Trees $H_3(s_1,s_2,s_3), F_3(s_1,s_2,s_3), L_3(s_1,s_2,s_3), L_3^{(a)}(s_1,s_2,s_3)$.

Theorem 3.5 *If $T \in \Re_3$, then $\rho(T) \geq \rho(T_3^*)$, with equality if and only if $T = T_3^*$.*

Proof. Since $T \in \Re_3$, we have $4 \leq d(T) \leq 6$. Without loss of generality, we distinguish the following three cases.

Case 1. If $d(T) = 6$, then from Lemma 3.4, the result follows.

Case 2. If $d(T) = 5$, then $T \in \mathcal{H}_3$. Thus $T = H_3(s_1,s_2,s_3)$, where $\sum_{i=1}^{3} s_i + 4 = 3k+1$. We consider the following two subcases.

Subcase 2.1. If $s_2 = 0$, then $s_1 + s_3 = 3k-3$ $(k \geq 3)$. Suppose that $s_1 \geq s_3$. If $s_3 = 1$, then $s_1 = 3k-4$, $\Delta(T) = s_1 + 1 = 3k-3 > k+2$. From Lemmas 2.1 and 3.1, we have $\rho(T) = \rho(H_3(s_1,0,1)) > \rho(T_3^*)$. If $s_3 \geq 2$, then we obtain $T_3(s_1-2,0,s_3-1) \in \mathcal{T}_3$ from $H_3(s_1,0,s_3)$ by subdividing the edge w_3w_4 and deleting the vertex u. From Lemmas 2.2, 2.3 and 3.4, we have $\rho(T) = \rho(H_3(s_1,0,s_3)) > \rho(T_3(s_1-2,0,s_3-1)) \geq \rho(T_3^*)$.

Subcase 2.2. $s_2 \geq 1$. If $s_3 = 1$, then $s_1 + s_2 = 3k-4$. Thus $\Delta(T) \geq 2k > k+2$ $(k \geq 3)$. From Lemmas 2.2 and 3.1, we have $\rho(T) = \rho(H_3(s_1,s_2,1)) > \rho(T_3^*)$. If $s_3 \geq 2$, then we obtain $T_3(s_1,s_2-1,s_3-1) \in \mathcal{T}_3$ from $H_3(s_1,s_2,s_3)$ by subdividing the edge w_3w_4 and deleting the vertex v. From Lemmas 2.2, 2.3 and 3.4, we have $\rho(T) = \rho(H_3(s_1, s_2, s_3)) > \rho(T_3(s_1-1, s_2-1, s_3-1)) \geq \rho(T_3^*)$.

Case 3. If $d(T) = 4$, then $T \in \mathcal{F}_3 \bigcup \mathcal{L}_3$. We distinguish the following two subcases.

Subcase 3.1. If $T \in \mathcal{F}_3$, then $T = F_3(s_1, s_2, s_3)$, where $\sum_{i=1}^{3} s_i + 3 = 3k+1$. If $s_1 = s_3 = 1$, then $s_2 = 3k-4$, $\Delta(T) = s_2 + 2 = 3k - 4 \geq k + 2$ $(k \geq 3)$. From Lemmas 2.1 and 3.1, we have $\rho(T) = \rho(F_3(1, s_2, 1)) > \rho(T_3^*)$. If $s_1 \geq 2$ or $s_2 \geq 2$, then we can suppose $s_1 \geq 2$ by symmetry. Furthermore, the graph $H_3(s_1, s_2 - 1, s_3) \in \mathcal{H}_3$ can be obtained from $F_3(s_1, s_2, s_3)$ by subdividing the edge w_1w_2 and deleting the vertex w_3. From Lammas 2.2, 2.3 and the proof of Case 2, we have $\rho(T) = \rho(F_3(s_1, s_2, s_3)) > \rho(H_3(s_1, s_2 - 1, s_3)) > \rho(T_3^*)$.

Subcase 3.2. If $T \in \mathcal{L}_3$, then $T = L_3(s_1, s_2, s_3)$, where $\sum_{i=1}^{3} s_i + 4 = 3k+1$. Let $L_3^{(a)}(s_1, s_2-1, s_3) \in \mathcal{L}_3^{(a)}$, then $L_3^{(a)}(s_1, s_2-1, s_3) \in \mathcal{T}_3$ and $L_3^{(a)}(s_1, s_2-1, s_3) - w_2w_3 + w_1w_3 = L_3(s_1, s_2, s_3) \in \mathcal{L}_3$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $L_3(s_1, s_2, s_3)$ and suppose $x_{w_1} \geq x_{w_2}$. From Lemmas 2.4 and 3.4, we have $\rho(T) = \rho(L_3(s_1, s_2, s_3)) = \rho(L_3^{(a)}(s_1, s_2 - 1, s_3) - w_2w_3 + w_1w_3) > \rho(L_3^{(a)}(s_1, s_2 - 1, s_3)) \geq \rho(T_3^*)$. □

3.3 The minimal spectral radius of trees in $\Re_4$

Lemma 3.6 *If $T \in \Re_4$, then $\Delta(T) \geq k$. Moreover, if T has the minimal spectral radius, then $k \leq \Delta(T) \leq k+1$.*

Proof. From [5], it is easy to see that $\Delta(T) \geq k$. If T has the minimal spectral radius and $\Delta(T) \geq k+2$, from Lemma 2.1, we have $\rho(T) > \sqrt{\Delta(T)} \geq \sqrt{k+2} > \sqrt{k + \frac{1+\sqrt{5}}{2}} = \rho(T_4^{(1)})$, a contradiction. □

Lemma 3.7 *If $T \in \Re_4$, $d(T) = 8$, then $\rho(T) \geq \rho(T_4^{(1)})$, with equality if and only if $T \in \{T_4^{(1)}, T_4^{(2)}, T_4^{(3)}\}$.*

Proof. Since $T \in \Re_4$, $d(T) = 8$, we have $T \in \mathcal{T}_4$. So, we can suppose that $T = T_4(s_1, s_2, s_3, s_4)$, where $\sum_{i=1}^{4} s_i + 9 = 4k + 1$. From Lemma 2.5, we have

$$\Phi(T) = x^{\sum_{i=1}^{4} s_i + 1}[x^8 - (\sum_{i=1}^{4} s_i + 8)x^6 + (\frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{4} s_i s_j + 6\sum_{i=1}^{4} s_i + 21)x^4$$

$$- (\frac{1}{6}\sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{4} s_i s_j s_k + 2\sum_{\substack{i,j=1 \\ i \neq j}}^{4} s_i s_j + 10\sum_{i=1}^{4} s_i + s_2 + s_3 + 20)x^2$$

$$+ \prod_{i=1}^{4} s_i + \frac{1}{3}\sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{4} s_i s_j s_k + 2\sum_{\substack{i,j=1 \\ i \neq j}}^{4} s_i s_j - s_1 s_2 - s_1 s_4 - s_3 s_4$$

$$+ 6\sum_{i=1}^{4} s_i - 2s_1 - 2s_4 + 5]. \tag{3.7}$$

From the proof of Lemma 3.6, we suppose $k \leq \Delta(T) \leq k+1$. If $\Delta(T) = k$, then $T = T_4^{(1)}$. If $\Delta(T) = k + 1$, then $s_i \leq k - 1 (1 \leq i \leq 4)$, $\sum_{i=1}^{4} s_i = 4k - 8$. Thus, at least one and at most three of the s_i's are $k - 1$. If there are three of the s_i's are $k - 1$, then another one is $k - 5$. Thus $T \in \{T_4(k-1, k-1, k-5, k-1), T_4(k-1, k-1, k-1, k-5)\}$. If there are two of the s_i's are $k - 1$, then another two are $k - 2, k - 4$ or $k - 3, k - 3$. Thus T must be one of the following trees:

$T_4(k-1, k-1, k-2, k-4)$, $T_4(k-1, k-1, k-4, k-2)$, $T_4(k-1, k-2, k-1, k-4)$, $T_4(k-1, k-4, k-1, k-2)$, $T_4(k-1, k-2, k-4, k-1)$, $T_4(k-2, k-1, k-1, k-4)$, $T_4(k-1, k-1, k-3, k-3)$, $T_4(k-1, k-3, k-1, k-3)$, $T_4(k-1, k-3, k-3, k-1) = T_4^{(3)}$, $T_4(k-3, k-1, k-1, k-3)$.

If there is only one of the s_i's is $k-1$, then another three are $k-2, k-2, k-3$. Thus T must be one of the following trees:

$T_4(k-1, k-2, k-2, k-3)$, $T_4(k-1, k-2, k-3, k-2)$, $T_4(k-1, k-3, k-2, k-2) = T_4^{(2)}$, $T_4(k-2, k-1, k-2, k-3)$, $T_4(k-2, k-1, k-3, k-2)$, $T_4(k-3, k-1, k-2, k-2)$.

When $x \geq \rho(T_4^{(1)}) = \sqrt{k + \frac{1+\sqrt{5}}{2}}$, from Eq. (3.7), we have

$\Phi(T_4(k-1,k-1,k-5,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-6x^4 + (12k+10)x^2 - 6k^2 - 10k - 2] \le x^{4k-7}(2\sqrt{5} - 6) < 0;$

$\Phi(T_4(k-1,k-1,k-1,k-5)) - \Phi(T_4^{(1)}) = x^{4k-7}[-6x^4 + (12k+6)x^2 - 6k^2 - 6k + 2] \le x^{4k-7}(-4) < 0;$

$\Phi(T_4(k-1,k-1,k-2,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k+1)x^2 - 3k^2 - 3k + 1] \le x^{4k-7}(-3 - \sqrt{5}) < 0;$

$\Phi(T_4(k-1,k-1,k-4,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k+3)x^2 - 3k^2 - k - 1] \le x^{4k-7}(-4) < 0;$

$\Phi(T_4(k-1,k-2,k-1,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k+1)x^2 - 3k^2 - k + 4] \le x^{4k-7}(-\sqrt{5}) < 0;$

$\Phi(T_4(k-1,k-4,k-1,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k+3)x^2 - 3k^2 - 3k + 2] \le x^{4k-7}(-1) < 0;$

$\Phi(T_4(k-1,k-2,k-4,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k+4)x^2 - 3k^2 - 4k + 1] \le x^{4k-7}(\frac{\sqrt{5}-3}{2}) < 0;$

$\Phi(T_4(k-2,k-1,k-1,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}(-3x^4 + 6kx^2 - 3k^2 + 2) \le x^{4k-7}(\frac{-3\sqrt{5}-5}{2}) < 0;$

$\Phi(T_4(k-1,k-1,k-3,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-2x^4 + 4kx^2 - 2k^2) \le x^{4k-7}(-\sqrt{5} - 3) < 0;$

$\Phi(T_4(k-1,k-3,k-1,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-2x^4 + 4kx^2 - 2k^2 + 4) \le x^{4k-7}(-\sqrt{5} + 1) < 0;$

$\Phi(T_4(k-3,k-1,k-1,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}[-2x^4 + (4k-2)x^2 - 2k^2 + 2k + 2] \le x^{4k-7}(-2 - 2\sqrt{5}) < 0;$

$\Phi(T_4(k-1,k-2,k-2,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-x^4 + 2kx^2 - k^2 + 1) \le x^{4k-7}(\frac{-\sqrt{5}-1}{2}) < 0;$

$\Phi(T_4(k-1,k-2,k-3,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-x^4 + (2k+1)x^2 - k^2 - k] \le x^{4k-7}(-1) < 0;$

$\Phi(T_4(k-2,k-1,k-2,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}[-x^4 + (2k-1)x^2 - k^2 + k] \le x^{4k-7}(-2 - \sqrt{5}) < 0;$

$\Phi(T_4(k-2,k-1,k-3,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-x^4 + 2kx^2 - k^2] \le x^{4k-7}(\frac{-\sqrt{5}-3}{2}) < 0;$

$\Phi(T_4(k-3,k-1,k-2,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-x^4 + (2k-1)x^2 - k^2 + k + 1] \leq x^{4k-7}(-\sqrt{5}-1) < 0$.

By Lemmas 2.6 and 3.1, we have $\rho(T) \geq \rho(T_4^{(1)})$, with equality if and only if $T \in \{T_4^{(1)}, T_4^{(2)}, T_4^{(3)}\}$. □

Let $F_4(s_1,s_2,s_3,s_4)$ and $H_4(s_1,s_2,s_3,s_4)$ be trees defined as Fig. 2. Let $\mathcal{F}_4 = \{F_4(s_1,s_2,s_3,s_4) \mid s_1,s_4 \geq 1, s_2,s_3 \geq 0, \sum_{i=1}^{4} s_i + 6 = 4k+1, k \geq 3\}$, $\mathcal{H}_4 = \{F_4(s_1,s_2,s_3,s_4) \mid s_i \geq 1, \sum_{i=1}^{4} s_i + 6 = 4k+1, 1 \leq i \leq 4, k \geq 3\}$.

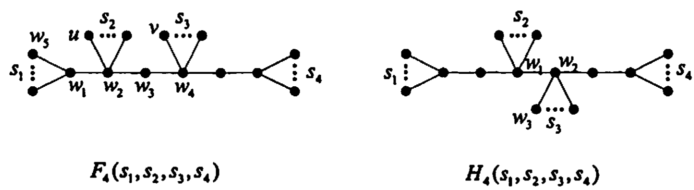


Figure 2: Trees $F_4(s_1,s_2,s_3,s_4), H_4(s_1,s_2,s_3,s_4)$.

Lemma 3.8 *If $T \in \Re_4$ and $d(T) = 7$, then $\rho(T) > \rho(T_4^{(1)})$.*

Proof. Since $T \in \Re_4$, $d(T) = 7$, we have $T \in \{\mathcal{F}_4, \mathcal{H}_4\}$. We separate into the following two cases.

Case 1. $T \in \mathcal{F}_4$, then $T = F_4(s_1,s_2,s_3,s_4)$, where $\sum_{i=1}^{4} s_i + 6 = 4k + 1$, $s_1, s_4 \geq 1, s_2, s_3 \geq 0$. We consider the following four subcases.

Subcase 1.1. If $s_2 = s_3 = 0$, then $s_1 + s_4 = 4k-5$. So we suppose that $s_1 > s_4$. If $s_4 = 1$, then $s_1 = 4k-6$, $\Delta(T) = s_1 + 1 = 4k-5 > k+1$ $(k \geq 3)$. From Lemma 3.6, we have $\rho(T) = \rho(F_4(s_1,0,0,1)) > \rho(T_4^{(1)})$. If $s_4 \geq 2$, then $s_1 > s_4 \geq 2$. Thus we obtain $T_4(s_1-2,0,0,s_4-1) \in \mathcal{T}_4$ from $F_4(s_1,0,0,s_4)$ by subdividing the edge w_1w_2 and deleting the vertex w_5. From Lemma 3.7, we have $\rho(T_4(s_1-2,0,0,s_4-1)) \geq \rho(T_4^{(1)})$. Therefore $\rho(T) = \rho(F_4(s_1,0,0,s_4)) > \rho(T_4^{(1)})$.

Subcase 1.2. If $s_2 = 0, s_3 \geq 1$, then $s_1 + s_3 + s_4 = 4k-5$. If $s_1 = 1$, then

$s_3 + s_4 = 4k - 6, \Delta(T) \geq 2k - 1 \geq k + 2$ $(k \geq 3)$. From Lemma 3.6, we have $\rho(T) = \rho(F_4(1, 0, s_3, s_4)) > \rho(T_4^{(1)})$. If $s_1 \geq 2$, then we obtain $T_4(s_1 - 1, 0, s_3 - 1, s_4 - 1) \in \mathcal{T}_4$ from $F_4(s_1, 0, s_3, s_4)$ by subdividing the edge w_1w_2 and deleting the vertex v. From Lemmas 2.2 and 2.3, we have $\rho(F_4(s_1, 0, s_3, s_4)) > \rho(T_4(s_1 - 1, 0, s_3 - 1, s_4 - 1))$. From Lemma 3.7, we have $\rho(T_4(s_1 - 1, 0, s_3 - 1, s_4 - 1)) \geq \rho(T_4^{(1)})$. Therefore $\rho(T) = \rho(F_4(s_1, 0, s_3, s_4)) > \rho(T_4^{(1)})$.

Subcase 1.3. If $s_2 \geq 1, s_3 = 0$, then $s_1 + s_2 + s_4 = 4k - 5$. If $s_1 = 1$, then $s_2 + s_4 = 4k - 6$, $\Delta(T) \geq 2k - 1 \geq k + 2$ $(k \geq 3)$. From Lemma 3.6, we have $\rho(T) = \rho(F_4(1, s_2, 0, s_4)) > \rho(T_4^{(1)})$. If $s_1 \geq 2$, then we obtain $T_4(s_1 - 1, s_2 - 1, 0, s_4 - 1) \in \mathcal{T}_4$ from $F_4(s_1, s_2, 0, s_4)$ by subdividing w_1w_2 and deleting u. From Lemmas 2.2 and 2.3, we have $\rho(F_4(s_1, s_2, 0, s_4)) > \rho(T_4(s_1 - 1, s_2 - 1, 0, s_4 - 1))$. From Lemma 3.7, we have $\rho(T_4(s_1 - 1, s_2 - 1, 0, s_4 - 1)) \geq \rho(T_4^{(1)})$. Therefore $\rho(T) = \rho(F_4(s_1, s_2, 0, s_4)) > \rho(T_4^{(1)})$.

Subcase 1.4. $s_2, s_3 \geq 1$. If $s_1 = 1$, then we obtain $F_4(s_2+1, 0, s_3, s_4) \in \mathcal{F}_4$ from $F_4(1, s_2, s_3, s_4)$ by subdividing the edge w_2w_3 and deleting the vertex w_5. From Lemmas 2.2, 2.3 and Subcase 1.2, we have $\rho(F_4(1, s_2, s_3, s_4)) > \rho(F_4(s_2 + 1, 0, s_3, s_4)) > \rho(T_4^{(1)})$. If $s_2 \geq 2$, then we get $T_4(s_1 - 1, s_2 - 1, s_3, s_4 - 1) \in \mathcal{T}_4$ by subdividing w_1w_2 and deleting u of $F_4(s_1, s_2, s_3, s_4)$. From Lemmas 2.2, 2.3 and 3.7, we have $\rho(F_4(s_1, s_2, s_3, s_4)) > \rho(T_4(s_1 - 1, s_2 - 1, s_3, s_4 - 1)) \geq \rho(T_4^{(1)})$. Thus $\rho(T) = \rho(F_4(s_1, s_2, s_3, s_4)) > \rho(T_4^{(1)})$.

Case 2. If $T \in \mathcal{H}_4$, then $T = H_4(s_1, s_2, s_3, s_4)$, where $s_i \geq 1, \sum_{i=1}^{4} s_i + 6 = 4k + 1, 1 \leq i \leq 4$. We obtain $T_4(s_1 - 1, s_2, s_3 - 1, s_4 - 1) \in \mathcal{T}_4$ from $H_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_1w_2 and deleting the vertex w_3. From Lemmas 2.2, 2.3 and 3.7, we have $\rho(H_4(s_1, s_2, s_3, s_4)) > \rho(T_4(s_1 - 1, s_2, s_3 - 1, s_4 - 1)) \geq \rho(T_4^{(1)})$. Therefore $\rho(T) = \rho(H_4(s_1, s_2, s_3, s_4)) > \rho(T_4^{(1)})$. □

Let $I_4(s_1, s_2, s_3, s_4)$, $J_4(s_1, s_2, s_3, s_4)$, $L_4(s_1, s_2, s_3, s_4)$, $M_4(s_1, s_2, s_3, s_4)$, $N_4(s_1, s_2, s_3, s_4)$ be trees defined as Fig. 3. Let $\mathcal{I}_4 = \{I_4(s_1, s_2, s_3, s_4) \mid$

$s_1, s_2, s_4 \geq 1, s_3 \geq 0, \sum_{i=1}^{4} s_i + 5 = 4k+1, k \geq 3\}$, $\mathcal{J}_4 = \{J_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \geq 1, \sum_{i=1}^{4} s_i + 5 = 4k+1, k \geq 3\}$, $\mathcal{L}_4 = \{L_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3 \geq 1, s_4 \geq 0, \sum_{i=1}^{4} s_i + 6 = 4k+1, k \geq 3\}$, $\mathcal{M}_4 = \{M_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3 \geq 1, s_4 \geq 0, \sum_{i=1}^{4} s_i + 7 = 4k+1, k \geq 3\}$, $\mathcal{N}_4 = \{N_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3 \geq 1, s_4 \geq 0, \sum_{i=1}^{4} s_i + 6 = 4k+1, k \geq 3\}$.

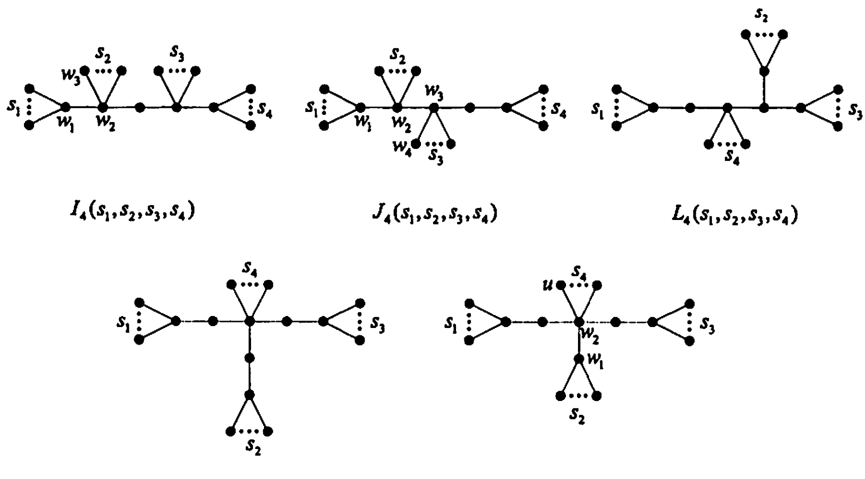


Figure 3: Trees $I_4(s_1, s_2, s_3, s_4)$, $J_4(s_1, s_2, s_3, s_4)$, $L_4(s_1, s_2, s_3, s_4)$, $M_4(s_1, s_2, s_3, s_4), N_4(s_1, s_2, s_3, s_4)$.

Lemma 3.9 *If $T \in \Re_4$ and $d(T) = 6$, then $\rho(T) > \rho(T_4^{(1)})$.*

Proof. Since $T \in \Re_4$, $d(T) = 6$, we have $T \in \{\mathcal{I}_4, \mathcal{J}_4, \mathcal{L}_4, \mathcal{M}_4, \mathcal{N}_4\}$. We distinguish the following five cases.

Case 1. If $T \in \mathcal{I}_4$, then $T = I_4(s_1, s_2, s_3, s_4), s_1, s_2, s_4 \geq 1,\ s_3 \geq 0, \sum_{i=1}^{4} s_i + 5 = 4k+1$. We consider the following two subcases.

Subcase 1.1. $s_3 = 0$. If $s_1 = 1$, then $s_2 + s_4 = 4k-5, \Delta(T) \geq 2k-1 \geq k + 2$ $(k \geq 3)$. From the proof of Lemma 3.6, we have $\rho(T) > \rho(T_4^{(1)})$. If $s_1 \geq 2$, we obtain $F_4(s_4, 0, s_2-1, s_1) \in \mathcal{F}_4$ from $I_4(s_1, s_2, 0, s_4)$ by subdividing the edge w_1w_2 and deleting the vertex w_3. From Lemmas 2.2, 2.3 and 3.8, we have $\rho(I_4(s_1, s_2, 0, s_4)) > \rho(F_4(s_4, 0, s_2-1, s_1)) > \rho(T_4^{(1)})$. Therefore, $\rho(T) = \rho(I_4(s_1, s_2, 0, s_4)) > \rho(T_4^{(1)})$.

Subcase 1.2. $s_3 \geq 1$. If $s_1 = s_4 = 1$, then $s_2+s_3 = 4k-6$, $\Delta(T) \geq 2k-2 \geq k+2$ $(k \geq 3)$. From the proof of Lemma 3.6, we have $\rho(T) > \rho(T_4^{(1)})$. If $s_1 \geq 2$, then we obtain $F_4(s_4, s_3, s_2-1, s_1) \in \mathcal{F}_4$ from $I_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_1w_2 and deleting the vertex w_3. From Lemmas 2.2, 2.3 and 3.8, we have $\rho(I_4(s_1, s_2, s_3, s_4)) > \rho(F_4(s_4, s_3, s_2-1, s_1)) > \rho(T_4^{(1)})$. Therefore, $\rho(T) > \rho(T_4^{(1)})$. If $s_4 \geq 2$, we get the result by the similar method.

Case 2. If $T \in \mathcal{J}_4$, then $T = J_4(s_1, s_2, s_3, s_4)$, where $s_1, s_2, s_3, s_4 \geq 1$, $\sum_{i=1}^{4} s_i+5 = 4k+1$. We obtain $F_4(s_1, s_2, s_3-1, s_4) \in \mathcal{F}_4$ from $J_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_2w_3 and deleting the vertex w_4. From Lemmas 2.2, 2.3 and 3.8, we have $\rho(J_4(s_1, s_2, s_3, s_4)) > \rho(F_4(s_1, s_2, s_3-1, s_4)) > \rho(T_4^{(1)})$. Therefore $\rho(T) = \rho(J_4(s_1, s_2, s_3, s_4)) > \rho(T_4^{(1)})$.

Case 3. If $T \in \mathcal{L}_4$, then $T = L_4(s_1, s_2, s_3, s_4)$, where $s_1, s_2, s_3 \geq 1, s_4 \geq 0$, $\sum_{i=1}^{4} s_i + 6 = 4k+1$. From Lemma 2.5, we have

$$\begin{aligned}
&\Phi(L_4(s_1, s_2, s_3, s_4) \\
&= x^{\sum_{i=1}^{4} s_i - 2}[x^8 - (\sum_{i=1}^{4} s_i + 5)x^6 + (\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j + 4\sum_{i=1}^{4} s_i - s_4 + 5)x^4 \\
&\quad - (\frac{1}{6}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4} s_is_js_k + \sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3} s_is_j + 2\sum_{i=1}^{4} s_i + s_2 + s_3)x^2 \\
&\quad + \prod_{i=1}^{4} s_i + \frac{1}{6}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4} s_is_js_k + \prod_{i=1}^{3} s_i + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j - s_1s_4]. \qquad (3.8)
\end{aligned}$$

We consider the following two subcases.

Subcase 3.1. If $s_4 = 0$, then $s_1 + s_2 + s_3 = 4k-5$. If $k > 5$, then $\Delta(T) \geq k+2$, from the proof of Lemma 3.6, we have $\rho(T) > \rho(T_4^{(1)})$. If

$k=3$, then $s_1+s_2+s_3=7$, $s_i \geq 1$ $(i=1,2,3)$, $3 \leq \Delta(T) \leq 4$. Therefore T must be one of the following trees:

$L_4(1,3,3,0)$, $L_4(3,1,3,0)$, $L_4(3,2,2,0)$, $L_4(2,2,3,0)$.

If $k=4$, then $s_1+s_2+s_3=11$, $s_i \geq 1$ $(i=1,2,3)$, $4 \leq \Delta(T) \leq 5$. Therefore $T \in \{L_4(3,4,4,0), L_4(4,4,3,0)\}$. If $k=5$, then $s_1+s_2+s_3=15$, $s_i \geq 1$ $(i=1,2,3)$, $5 \leq \Delta(T) \leq 6$. Therefore $T = L_4(5,5,5,0)$. By direct calculation, we have $\rho(T) > \rho(T_4^{(1)})$, $3 \leq k \leq 5$.

Subcase 3.2. If $s_4 \geq 1$, then $\sum_{i=1}^{4} s_i = 4k-5$. From Lemma 3.6, we can suppose that $k \leq \Delta(T) \leq k+1$. If $\Delta(T)=k$, then $s_1, s_2, s_3 \leq k-1$, $s_4 \leq k-2$, $\sum_{i=1}^{4} s_i = 4k-5$. Therefore $T = L_4(k-1,k-1,k-1,k-2)$. If $\Delta(T)=k+1$, then $s_1, s_2, s_3 \leq k$, $s_4 \leq k-1$, $\sum_{i=1}^{4} s_i = 4k-5$. Thus, at most three of the s_i's are k. If there are three of the s_i's are k, then $T = L_4(k,k,k,k-5)$. If there are two of the s_i's are k, then another two must be $k-1, k-4$ or $k-2, k-3$. Therefore T must be one of the following trees:

$L_4(k,k,k-1,k-4)$, $L_4(k-1,k,k,k-4)$, $L_4(k,k,k-4,k-1)$, $L_4(k-4,k,k,k-1)$, $L_4(k,k,k-2,k-3)$, $L_4(k-2,k,k,k-3)$, $L_4(k,k,k-3,k-2)$, $L_4(k-3,k,k,k-2)$.

If there is only one of the s_i's is k, then another three must be $k-1, k-1, k-3$ or $k-1, k-2, k-2$. Therefore T must be one of the following trees:

$L_4(k,k-1,k-1,k-3)$, $L_4(k-1,k,k-1,k-3)$, $L_4(k,k-1,k-3,k-1)$, $L_4(k-1,k,k-3,k-1)$, $L_4(k-3,k,k-1,k-1)$, $L_4(k,k-1,k-2,k-2)$, $L_4(k-1,k,k-2,k-2)$, $L_4(k-2,k,k-1,k-2)$, $L_4(k,k-2,k-2,k-1)$, $L_4(k-2,k,k-2,k-1)$.

If $s_1, s_2, s_3, s_4 \leq k$, then $s_4 = k-1$. Therefore $T \in \{L_4(k-1,k-1,k-2,k-1), L_4(k-2,k-1,k-1,k-1)\}$.

When $x \geq \rho(T_4^{(1)}) = \sqrt{k+\frac{1+\sqrt{5}}{2}}$, from Eqs. (3.7) and (3.8), we have

$\Phi(L_4(k-1,k-1,k-1,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-x^4+(2k-2)x^2-k^2+2k] \leq x^{4k-7}(\frac{-5-3\sqrt{5}}{2}) < 0$;

$\Phi(L_4(k,k,k,k-5)) - \Phi(T_4^{(1)}) = x^{4k-7}[-7x^4 + (14k+10)x^2 - 7k^2 - 10k - 1] \leq x^{4k-7}(\frac{-13+3\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k,k,k-1,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k+3)x^2 - 4k^2 - 3k + 3] \leq x^{4k-7}(\frac{-3-\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-1,k,k,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k+2)x^2 - 4k^2 - 2k - 1] \leq x^{4k-7}(-6-\sqrt{5}) < 0;$

$\Phi(L_4(k,k,k-4,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-7x^4 + (14k+6)x^2 - 7k^2 - 6k + 3] \leq x^{4k-7}(\frac{-9-\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-4,k,k,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-7x^4 + (14k+2)x^2 - 7k^2 - 2k - 1] \leq x^{4k-7}(\frac{-21-5\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k,k,k-2,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-3x^4 + 6kx^2 - 3k^2 + 5) \leq x^{4k-7}(\frac{1-3\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-2,k,k,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}[-3x^4 + (6k-2)x^2 - 3k^2 + 2k - 1] \leq x^{4k-7}(\frac{-13-5\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k,k,k-3,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k+1)x^2 - 4k^2 - k + 5] \leq x^{4k-7}(\frac{-1-3\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-3,k,k,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k-2)x^2 - 4k^2 + 2k - 1] \leq x^{4k-7}(-8-3\sqrt{5}) < 0;$

$\Phi(L_4(k,k-1,k-1,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-2x^4 + 4kx^2 - 2k^2 + 3) \leq x^{4k-7}(-\sqrt{5}) < 0;$

$\Phi(L_4(k-1,k,k-1,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}[-2x^4 + (4k-1)x^2 - 2k^2 + k] \leq x^{4k-7}(\frac{-7-3\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k,k-1,k-3,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}(-4x^4 + 8kx^2 - 4k^2 + 3) \leq x^{4k-7}(-3-2\sqrt{5}) < 0;$

$\Phi(L_4(k-1,k,k-3,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k-1)x^2 - 4k^2 + k + 2] \leq x^{4k-7}(\frac{-9-5\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-3,k,k-1,k-1)) - \Phi(T_4^{(1)}) = x^{4k-7}[-4x^4 + (8k-3)x^2 - 4k^2 + 3k] \leq x^{4k-7}(\frac{-15-7\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k,k-1,k-2,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}[-2x^4 + (4k-1)x^2 - 2k^2 + k + 3] \leq x^{4k-7}(\frac{-1-3\sqrt{5}}{2}) < 0;$

$\Phi(L_4(k-1,k,k-2,k-2))-\Phi(T_4^{(1)}) = x^{4k-7}[-2x^4+(4k-2)x^2-2k^2+2k+1] \le x^{4k-7}(-3-2\sqrt{5}) < 0$;

$\Phi(L_4(k-2,k,k-1,k-2))-\Phi(T_4^{(1)}) = x^{4k-7}[-2x^4+(4k-3)x^2-2k^2+3k-1] \le x^{4k-7}(\frac{-11-5\sqrt{5}}{2}) < 0$;

$\Phi(L_4(k,k-2,k-2,k-1))-\Phi(T_4^{(1)}) = x^{4k-7}[-3x^4+(6k-2)x^2-3k^2+2k+3] \le x^{4k-7}(\frac{-5-5\sqrt{5}}{2}) < 0$;

$\Phi(L_4(k-2,k,k-2,k-1))-\Phi(T_4^{(1)}) = x^{4k-7}[-3x^4+(6k-4)x^2-3k^2+4k+1] \le x^{4k-7}(\frac{-11-7\sqrt{5}}{2}) < 0$;

$\Phi(L_4(k-1,k-1,k-2,k-1))-\Phi(T_4^{(1)}) = x^{4k-7}[-2x^4+(4k-3)x^2-2k^2+3k] \le x^{4k-7}(\frac{-9-5\sqrt{5}}{2}) < 0$;

$\Phi(L_4(k-2,k-1,k-1,k-1))-\Phi(T_4^{(1)}) = x^{4k-7}[-2x^4+(4k-4)x^2-2k^2+4k-1] \le x^{4k-7}(-6-3\sqrt{5}) < 0$.

From Lemma 2.6, we have $\rho(T) > \rho(T_4^{(1)})$.

Case 4. If $T \in \mathcal{M}_4$, then $T = M_4(s_1,s_2,s_3,s_4)$, where $s_1,s_2,\ s_3 \ge 1$, $s_4 \ge 0$, $\sum_{i=1}^{4} s_i = 4k-6$. From Lemma 2.5, we have

$$\begin{aligned}
&\Phi(M_4(s_1,s_2,s_3,s_4)) \\
&= x^{\sum_{i=1}^{4} s_i - 1}[x^8 - (\sum_{i=1}^{4} s_i + 6)x^6 + (\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j + 5\sum_{i=1}^{4} s_i - 2s_4 + 9)x^4 \\
&- (\frac{1}{6}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4} s_is_js_k + \sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j + \sum_{\substack{i,j=1\\ i\neq j}}^{3} s_is_j + 5\sum_{i=1}^{4} s_i - 2s_4 + 4)x^2 + \prod_{i=1}^{4} s_i \\
&+ \frac{1}{6}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4} s_is_js_k + 2\prod_{i=1}^{3} s_i + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{4} s_is_j + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3} s_is_j + \sum_{i=1}^{4} s_i].
\end{aligned} \tag{3.9}$$

We distinguish the following two subcases.

Subcase 4.1. If $s_4 = 0$, then $s_1+s_2+s_3 = 4k-6$. If $k > 6$, then $\Delta(T) \ge k+2$. From the proof of Lemma 3.6, we have $\rho(T) = \rho(M_4(s_1,s_2,s_3,0)) > \rho(T_4^{(1)})$. If $k = 3$, then $T \in \{M_4(1,2,3,0), M_4(2,2,2,0)\}$; if $k = 4$, then $T \in \{M_4(2,4,4,0), M_4(3,3,4,0)\}$; if $k = 5$, then $T = M_4(4,5,5,0)$; if $k =$

6, then $T = M_4(6,6,6,0)$. We have $\rho(T) > \rho(T_4^{(1)})$ by direct calculation for $3 \le k \le 6$.

Subcase 4.2. If $s_4 \ge 1$, then $\sum_{i=1}^{4} s_i = 4k-6$. From Lemma 3.6, we suppose that $k \le \Delta(T) \le k+1$. If $\Delta(T) = k$, it is easy to see that $s_1, s_2, s_3 \le k-1$, $s_4 \le k-3$. Therefore $T = M_4(k-1, k-1, k-1, k-3)$. If $\Delta(T) = k+1$, then $s_1, s_2, s_3 \le k$, $s_4 \le k-2$. If $s_1 = s_2 = s_3 = k$, and for $\sum_{i=1}^{4} s_i = 4k-6$, then $s_4 = k-6$. So $T = M_4(k,k,k,k-6)$. If there are two with value of k, we just discuss $s_1 = s_2 = k$ for symmetry. For $s_1 = s_2 = k$, and $\sum_{i=1}^{4} s_i = 4k-6$, we have $s_3 + s_4 = 2k-6$ and $s_3, s_4 \in \{k-2, k-3, k-4\}$. Thus T must be one of the following trees:

$M_4(k,k,k-1,k-5)$, $M_4(k,k,k-2,k-4)$, $M_4(k,k,k-4,k-2)$, $M_4(k,k,k-3,k-3)$.

If there is only one with value of k, we just discuss $s_1 = k$ for symmetry. For $s_1 = k$ and $\sum_{i=1}^{4} s_i = 4k-6$, then $s_2 + s_3 + s_4 = 3k-6$ and $s_2, s_3 \in \{k-1, k-2, k-3, k-4\}$, $s_4 \in \{k-2, k-3, k-4\}$. Thus T must be one of the following trees:

$M_4(k,k-1,k-1,k-4)$, $M_4(k,k-1,k-2,k-3)$, $M_4(k,k-1,k-3,k-2)$, $M_4(k,k-2,k-2,k-2)$.

If $s_i \le k-1$ $(i = 1,2,3)$, for $\sum_{i=1}^{4} s_i = 4k-6$ and $\Delta(T) = k+1$, then $s_4 = k-2$. Thus $T = M_4(k-1,k-1,k-2,k-2)$. When $x \ge \rho(T_4^{(1)}) = \sqrt{k + \frac{1+\sqrt{5}}{2}}$, from lemma 3.7 and 3.9, we have

$\Phi(M_4(k-1,k-1,k-1,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-1) < 0$;

$\Phi(M_4(k,k,k,k-6)) - \Phi(T_4^{(1)}) = x^{4k-7}(-6x^4 + (12k+14)x^2 - 6k^2 - 14k - 7) \le x^{4k-7}(4\sqrt{5} - 9) < 0$;

$\Phi(M_4(k,k,k-1,k-5)) - \Phi(T_4^{(1)}) = x^{4k-7}(-3x^4 + (6k+6)x^2 - 3k^2 - 6k - 2) \le x^{4k-7}(\frac{3\sqrt{5}-7}{2}) < 0$;

$\Phi(M_4(k,k,k-2,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}(-2x^4 + (4k+2)x^2 - 2k^2 - 2k + 1) \le x^{4k-7}(-1) < 0$;

$\Phi(M_4(k,k,k-4,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}(-6x^4+(12k+6)x^2-6k^2-6k+1) \le x^{4k-7}(-5) < 0$;

$\Phi(M_4(k,k,k-3,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-3x^4+(6k+2)x^2-3k^2-2k+2) \le x^{4k-7}(\frac{-3-\sqrt{5}}{2}) < 0$;

$\Phi(M_4(k,k-1,k-1,k-4)) - \Phi(T_4^{(1)}) = x^{4k-7}(-x^4+(2k+2)x^2-k^2-2k-1) \le x^{4k-7}(\frac{-3+\sqrt{5}}{2}) < 0$;

$\Phi(M_4(k,k-1,k-2,k-3)) - \Phi(T_4^{(1)}) = x^{4k-7}(-x^4+2kx^2-k^2) \le x^{4k-7}(\frac{-3-\sqrt{5}}{2}) < 0$;

$\Phi(M_4(k,k-1,k-3,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}(-3x^4+6kx^2-3k^2+1) \le x^{4k-7}(\frac{-3\sqrt{5}-7}{2}) < 0$;

$\Phi(M_4(k,k-2,k-2,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}(-2x^4+(4k-2)x^2-2k^2+2k+1) \le x^{4k-7}(-3-2\sqrt{5}) < 0$;

$\Phi(M_4(k-1,k-1,k-2,k-2)) - \Phi(T_4^{(1)}) = x^{4k-7}(-x^4+(2k-2)x^2-k^2+2k-1) \le x^{4k-7}(\frac{-3\sqrt{5}-7}{2}) < 0$.

From Lemma 2.6, we have $\rho(T) = \rho(M_4(s_1,s_2,s_3,s_4) > \rho(T_4^{(1)})$.

Case 5. If $T \in \mathcal{N}_4$, then $T = N_4(s_1,s_2,s_3,s_4)$, where $s_1,s_2,s_3 \ge 1$, $s_4 \ge 0$, $\sum_{i=1}^{4} s_i = 4k-5$. We consider the following two subcases.

Subcase 5.1 If $s_4 = 0$, then $s_1+s_2+s_3 = 4k-5$. If $k > 5$, $\Delta(T) \ge k+2$, then from the proof of Lemma 3.6, we have $\rho(T) = \rho(N_4(s_1,s_2,s_3,0)) > \rho(T_4^{(1)})$. From Lemma 3.6, we can suppose that $k \le \Delta(T) \le k+1$. Then $s_1,s_2,s_3 \le k$. If $k=3$, then T must be one of the following trees: $N_4(1,3,3,0), N_4(3,1,3,0), N_4(2,2,3,0)$, and $N_4(2,3,2,0)$. If $k=4$, then $T \in \{N_4(3,4,4,0),\ N_4(4,3,4,0)\}$. If $k=5$, then $T = N_4(5,5,5,0)$. So we have $\rho(T) > \rho(T_4^{(1)})$, $3 \le k \le 5$.

Subcase 5.2. If $s_4 \ge 1$, then $T = N_4(s_1,s_2,s_3,s_4)$. If $s_2 = 1$, then $s_1+s_3+s_4 = 4k-6$, $s_1 \le k$, $s_3 \le k$, $s_4 \le k-2$. If $k > 4$, then $s_1+s_3+s_4 = 4k-6 > 3k-2$ and $\Delta(T) \ge k+2$. Therefore from the proof of Lemma 3.6, we have $\rho(T) = \rho(N_4(s_1,1,s_3,s_4)) > \rho(T_4^{(1)})$ $(k > 4)$. Suppose $k \le \Delta(T) \le k+1$ from Lemma 3.6. If $k=3$, then $s_1+s_3+s_4 = 6$, $s_1,s_3 \le 3$, $s_4 = 1$. So $T = N_4(3,1,2,1)$. If $k=4$, then $s_1 + s_3 + s_4 = 10$, $s_1, s_3 \leq 4$, $s_4 \leq 2$. So $T = N_4(4,1,4,2)$. Thus, we have $\rho(T) = \rho(N_4(s_1, 1, s_3, s-4)) > \rho(T_4^{(1)})$, $3 \leq k \leq 4$. If $s_2 \geq 2$, then $M_4(s_1, s_2, s_3, s_4) \in \mathcal{M}_4$ is the tree obtained from $N_4(s_1, s_2, s_3, s_4)$ by subdividing w_1w_2 and deleting u. From Lemmas 2.2, 2.3 and Case 4, we have $\rho(T) = \rho(N_4(s_1, s_2, s_3, s_4)) > \rho(M_4(s_1, s_2, s_3, s_4 - 1)) > \rho(T_4^{(1)})$. □

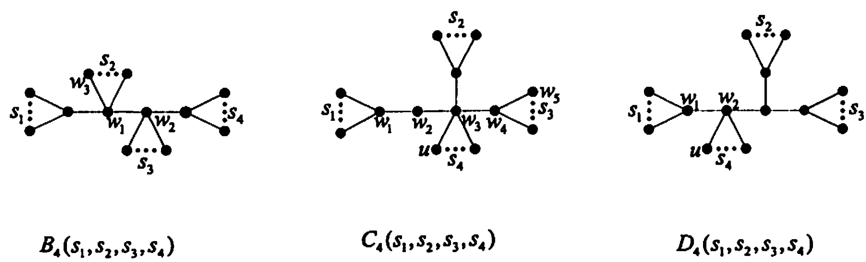


Figure 4: Trees $B_4(s_1, s_2, s_3, s_4), C_4(s_1, s_2, s_3, s_4), D_4(s_1, s_2, s_3, s_4)$.

Let $B_4(s_1, s_2, s_3, s_4)$, $C_4(s_1, s_2, s_3, s_4)$, $D_4(s_1, s_2, s_3, s_4)$ be trees defined as Fig. 4. Let $\mathcal{B}_4 = \{B_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \geq 1, \sum_{i=1}^{4} s_i + 4 = 4k+1, k \geq 3\}$, $\mathcal{C}_4 = \{C_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3 \geq 1, s_4 \geq 0, \sum_{i=1}^{4} s_i + 5 = 4k+1, k \geq 3\}$, $\mathcal{D}_4 = \{D_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \geq 1, \sum_{i=1}^{4} s_i + 5 = 4k+1, k \geq 3\}$.

Lemma 3.10 *If $T \in \Re_4$ and $d(T) = 5$, then $\rho(T) > \rho(T_4^{(1)})$.*

Proof. If $T \in \Re_4$, $d(T) = 5$, then $T \in \{\mathcal{B}_4, \mathcal{C}_4, \mathcal{D}_4\}$. We distinguish the following three cases.

Case 1. If $T \in \mathcal{B}_4$, then $T = B_4(s_1, s_2, s_3, s_4)$. Therefore, $I_4(s_4, s_3, s_2 - 1, s_1) \in \mathcal{I}_4$ is the tree obtained from $B_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_1w_2 and deleting the vertex w_3. From Lemmas 2.2, 2.3 and 3.9, we have $\rho(T) = \rho(B_4(s_1, s_2, s_3, s_4)) > \rho(I_4(s_4, s_3, s_2 - 1, s_1)) > \rho(T_4^{(1)})$.

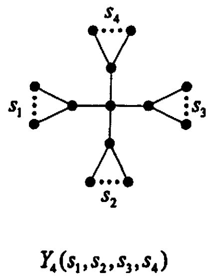
Case 2. If $T \in \mathcal{C}_4$, then $T = C_4(s_1, s_2, s_3, s_4)$, $s_1, s_2, s_3 \geq 1$, $s_4 \geq 0$, $\sum_{i=1}^{4} s_i = 4k - 4$. We consider the following two subcases.

Subcase 2.1. If $s_4 = 0$, then $s_1 + s_2 + s_3 = 4k - 4$. If $k > 4$, then $s_1 + s_2 + s_3 = 4k - 4 > 3k$. So $\Delta(T) \geq k+2$. From the proof of Lemma 3.6, we have $\rho(T) = \rho(C_4(s_1, s_2, s_3, 0)) > \rho(T_4^{(1)})$, $k > 4$. From Lemma 3.6, we suppose $k \leq \Delta(T) \leq k+1$. If $k = 3$, then $s_1, s_2, s_3 \leq 3$, $s_1 + s_2 + s_3 = 8$. So $T \in \{C_4(2,3,3,0), C_4(3,3,2,0)\}$. If $k = 4$, then $s_1, s_2, s_3 \leq 4$, $s_1 + s_2 + s_3 = 12$. So $T = C_4(4,4,4,0)$. We have $\rho(T) = \rho(C_4(s_1, s_2, s_3, 0)) > \rho(T_4^{(1)})$ by direct calculation for $3 \leq k \leq 4$.

Subcase 2.2. $s_4 \geq 1$. If $s_2 = s_3 = 1$, then $s_1 + s_4 = 4k - 6$ and $\Delta(T) \geq 2k - 1 \geq k + 2$ $(k \geq 3)$. From the proof of Lemma 3.6, we have $\rho(T) = \rho(C_4(s_1, 1, 1, s_4)) > \rho(T_4^{(1)})$. If $s_2 \geq 2$ or $s_3 \geq 2$, then we suppose $s_3 \geq 2$ by symmetry. So $N_4(s_1, s_2, s_3, s_4 - 1) \in N_4$ is the tree obtained from $C_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_3w_4 and deleting the vertex u. From Lemmas 2.2, 2.3 and 3.9, we have $\rho(T) = \rho(c_4(s_1, 1, 1, s_4)) > \rho(N_4(s_1, s_2, s_3, s_4 - 1)) > \rho(T_4^{(1)})$.

Case 3. If $T \in \mathcal{D}_4$, then $T = D_4(s_1, s_2, s_3, s_4)$, where $s_1, s_2, s_3, s_4 \geq 1$, $\sum_{i=1}^{4} s_i = 4k - 4$. If $s_1 = 1$, then $s_2 + s_3 + s_4 = 4k - 5$. If $k > 4$, $s_2 + s_3 + s_4 = 4k - 5 > 3k - 1$, then $\Delta(T) \geq k + 2$. From the proof of Lemma 3.6, we have $\rho(T) = \rho(D_4(1, s_2, s_3, s_4)) > \rho(T_4^{(1)})$, $k > 4$. If $k = 3$, $s_2, s_3 \leq 3$, $s_4 \leq 2$, $s_2 + s_3 + s_4 = 7$, then $T \in \{D_4(1,3,3,1), D_4(1,3,2,2)\}$. If $k = 4$, $s_2, s_3 \leq 4$, $s_4 \leq 3$, $s_2 + s_3 + s_4 = 11$, then $T = D_4(1,4,4,3)$. We have $\rho(T) > \rho(T_4^{(1)})$ by direct calculation for $3 \leq k \leq 4$. If $s_1 \geq 2$, then $L_4(s_1, s_2, s_3, s_4 - 1) \in \mathcal{L}_4$ is the tree obtained from $D_4(s_1, s_2, s_3, s_4)$ by subdividing the edge w_1w_2 and deleting u. From Lemmas 2.2, 2.3 and 3.9, we have $\rho(T) = \rho(D_4(s_1, 1, 1, s_4)) > \rho(L_4(s_1, s_2, s_3, s_4 - 1)) > \rho(T_4^{(1)})$. □

Let $Y_4(s_1, s_2, s_3, s_4)$, $Z_4(s_1, s_2, s_3, s_4)$ be trees defined as Fig. 5. Let $\mathcal{Y}_4 = \{Y_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \geq 1, \sum_{i=1}^{4} s_i + 5 = 4k + 1, k \geq 3\}$, $\mathcal{Z}_4 = \{Z_4(s_1, s_2, s_3, s_4) \mid s_1, s_2, s_3, s_4 \geq 1, \sum_{i=1}^{4} s_i + 4 = 4k + 1, k \geq 3\}$.



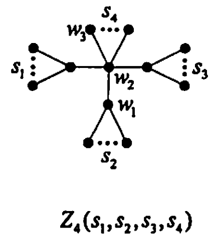


Figure 5: Trees $Y_4(s_1, s_2, s_3, s_4), Z_4(s_1, s_2, s_3, s_4)$.

Lemma 3.11 *If $T \in \Re_4$ and $d(T) = 4$, then $\rho(T) > \rho(T_4^{(1)})$.*

Proof. If $T \in \Re_4, d(T) = 4$, then $T \in \{\mathcal{Y}_4, \mathcal{Z}_4\}$. We consider the following two cases.

case 1. If $T \in \mathcal{Y}_4$, then $T = Y_4(s_1, s_2, s_3, s_4)$, where $\sum_{i=1}^{4} s_i = 4k - 4$, $s_1, s_2, s_3, s_4 \geq 1$. We suppose $k \leq \Delta(T) \leq k + 1$ from Lemma 3.6. If $\Delta(T) = k$, then $s_1, s_2, s_3, s_4 \leq k - 1$. For $\sum_{i=1}^{4} s_i = 4k - 4$, we have $T = Y_4(k-1, k-1, k-1, k-1)$. If $\Delta(T) = k + 1$, then $s_1, s_2, s_3, s_4 \leq k$. For $\sum_{i=1}^{4} s_i = 4k - 4$, at least one and at most three of the s_i's are k. If there are three of the s_i's are k, then another one is $k - 4$. So $T = Y_4(k, k, k, k - 4)$. If there are two of the s_i's are k, then another two must be $k - 1, k - 3$ or $k - 2, k - 2$. So $T \in \{Y_4(k, k, k - 1, k - 3), Y_4(k, k, k - 2, k - 2)\}$. If there is only one of the s_i's is k, then another three must be $k - 1, k - 1, k - 2$. So $T = Y_4(k, k - 1, k - 1, k - 2)$. By Lemma 2.5, we have

$$\Phi(Y_4(k-1, k-1, k-1, k-1)) = x^{4k-7}(x^2 - k + 1)^3(x^2 - k - 3).$$

Thus, $\rho(T) = \rho(Y_4(k-1, k-1, k-1, k-1)) = \sqrt{k+3} > \rho(T_4^{(1)})$. From Lemma 2.4, we have $\rho(Y_4(k, k, k, k-4)) > \rho(Y_4(k, k, k-2, k-2)) > \rho(Y_4(k, k-1, k-1, k-2)) > \rho(Y_4(k-1, k-1, k-1, k-1))$ and $\rho(Y_4(k, k, k-1, k-3)) > \rho(Y_4(k, k, k-2, k-2))$.

case 2. If $T \in \mathcal{Z}_4$, then $T = Z_4(s_1, s_2, s_3, s_4)$, where $\sum_{i=1}^{4} s_i = 4k - 3$. If $s_1 = s_2 = s_3 = 1$, then $s_4 = 4k - 6$ and $\Delta(T) = s_4 + 3 = 4k - 3 >$

$k+2$ $(k \geq 3)$. From Lemma 3.6 we have $\rho(T) = \rho(Z_4(1,1,1,4k-6)) > \rho(T_4^{(1)})$. If $s_1 \geq 2$, $s_2 \geq 2$ or $s_3 \geq 2$, we suppose $s_1 \geq 2$ by symmetry. So $C_4(s_2, s_1, s_3, s_4 - 1) \in \mathcal{C}_4$ is the tree obtained from $Z_4(s_1, s_2, s_3, s_4)$ by subdividing w_1w_2 and deleting w_3. From Lemmas 2.2, 2.3 and 3.10, we have $\rho(T) = \rho(Z_4(s_1, s_2, s_3, s_4)) > \rho(C_4(s_1, s_2, s_3, s_4 - 1)) > \rho(T_4^{(1)})$. □

Now we give the main result of the paper.

Theorem 3.12 *If $T \in \Re_4$, then $\rho(T) \geq \rho(T_4^{(1)})$, with equality if and only if $T \in \{T_4^{(1)}, T_4^{(2)}, T_4^{(3)}\}$.*

Proof. For $T \in \Re_4$, then $4 \leq d(T) \leq 8$. From Lemmas 3.7, 3.8, 3.9, 3.10 and 3.11, the result follows. □

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