

TRUNCATION SYMMETRY TYPE GRAPHS

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ABSTRACT. There are operations that transform a map \mathcal{M} (an embedding of a graph on a surface) into another map in the same surface, modifying its structure and consequently its set of flags $\mathcal{F}(\mathcal{M})$. For instance, by truncating all the vertices of a map \mathcal{M} , each flag in $\mathcal{F}(\mathcal{M})$ is divided into three flags of the truncated map. Orbanić, Pellicer and Weiss studied the truncation of k -orbit maps for $k \leq 3$. They introduced the notion of T -compatible maps in order to give a necessary condition for a truncation of a k -orbit map to be either k -, $\frac{3k}{2}$ - or $3k$ -orbit map. Using a similar notion, by introducing an appropriate partition on the set of flags of the maps, we extend the results on truncation of k -orbit maps for $k \leq 7$ and $k = 9$.

1. INTRODUCTION.

From the topological point of view, we define a map by a cellular embedding of a connected graph on a surface. The most symmetric maps are those for which its group of automorphisms acts transitively on its elements, called flags. This type of maps are well known under the name of regular (or reflexible) maps. Highly symmetric maps, as the regular and chiral (2-orbit maps with maximum degree of symmetry by rotation) ones, have been extensively studied, [14, 15, 18]. Little is known about maps that are neither regular nor chiral. Duarte and Hubard studied in [8] and [10], respectively, all seven types of 2-orbit maps, in different contexts.

Combinatorially, a map is completely described by a cubic edge-coloured graph, as Vince describes it in [21]. Such graph is known as the *flag graph* of the map. In 1982, Lins proposed an equivalent description to a flag graph in [12]. Other interesting characteristics regarding the flag graph are described in [1].

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Furthermore, each map has associated a graph obtained as the quotient of its flag graph, under the action of the group of automorphisms of the map, determining what we call the *symmetry type graph* of the map. In [4] are described some properties of these symmetry type graphs. A strategy of how to generate symmetry type graphs is shown in [2]. Dress and Huson refer to such graphs as the Delaney-Dress symbol, [6]. Dress and Brinkmann give an application to mathematical chemistry in [7]. Orbanić, Pellicer, Pisanski and Tucker (2011), show the 14 symmetry type graphs of edge-transitive maps, [16]. Edge-transitive maps were studied in [13] by Siran, Tucker and Watkins. Such maps have either 1, 2 or 4 orbits of flags under the action of the automorphism group.

A map can be transformed into another map by applying an operation on it. The truncation by the vertices of the map is an example of such operation. Even though a map and its truncated map are different to each other, the truncated map inherits the symmetries from the original map. If the vertices of a map \mathcal{M} are truncated up to the midpoint of its edges, then the resulting map is called the medial map of \mathcal{M} . Moreover, when the truncation goes further than the midpoint on the edges of the map \mathcal{M} , then the obtained map is known as the leapfrog map of \mathcal{M} , [5]. Furthermore, if the vertices of \mathcal{M} become the faces of the truncated map, then the resulting map is the dual map of \mathcal{M} . For instance, by truncating regular polyhedra, we can obtain some of the 13 Archimedean solids as is discussed in [3] and [20].

The truncation map $\text{Tr}(\mathcal{M})$, of a map \mathcal{M} , can be described by a division of the fundamental triangles on the surface that represent the flags of \mathcal{M} , [17]. Consequently, we can define a transformation from the flag graph of \mathcal{M} to the flag graph of its truncation map. According to such transformation of the flags of $\mathcal{F}(\mathcal{M})$ to define the set of flags $\mathcal{F}(\text{Tr}(\mathcal{M}))$ of $\text{Tr}(\mathcal{M})$, we follow a local arrangement of the flags in the truncated map, in addition to certain necessary conditions on \mathcal{M} and $\text{Tr}(\mathcal{M})$, Orbanić, Pellicer and Weiss showed the results on truncation of regular, 2-orbit and 3-orbit maps, stated as Theorem 2 and Propositions 2, 3 and 4 in this paper. The goal of this paper is to extend these results on truncation of k -orbit maps for $k \leq 7$ and $k = 9$, due to space, we leave on aside the truncation of 8-orbit maps. For this extension we will use the same local arrangement of flags used in [17].

The paper is divided into five sections. In Section 2, a map and its flag graph are defined along with their properties. The monodromy group of a map and its action on the set of flags are defined as in [11]. In Section 3 the symmetry type graph of a map is defined as a quotient of its flag graph, [4]. In Section 4 are defined operations on maps such as the dual,

truncation, and the composition of these two. The Section 4.1, related to the truncation operation, is divided into 5 subsections where are analyzed the results in [17] (on truncation of k -orbit maps) in addition with the extension of these results, given in a series of Propositions (5–11). Then, we obtain a classification of the possible symmetry type graphs for the truncation of k -orbit maps, with $k \leq 7$ and $k = 9$ (Table 2). Later, in Section 4.2, are defined the two-dimensional subdivision of a map and the leapfrog map, obtaining a classification of the possible symmetry type graphs for the leapfrog of k -orbit maps, with $k \leq 7$ and $k = 9$ (Table 4).

2. MAPS.

A map \mathcal{M} is a 2-cell embedding of a connected graph on a surface. The vertices and edges of the map \mathcal{M} are those from the embedded graph, and the faces of \mathcal{M} are homeomorphic to an open disc, such that each edge of a face is incident with at most two faces of \mathcal{M} .

Consider the barycentric subdivision \mathcal{B} of the faces of \mathcal{M} . Let Φ be a triangle in the subdivision \mathcal{B} , we shall label the vertices of Φ by Φ_0 , Φ_1 and Φ_2 if these correspond to a vertex, edge and face mutually incident in \mathcal{M} . Each triangle Φ is adjacent to exactly three other that differ on either a vertex, an edge or a face of the map. There are exactly four triangles around each edge of \mathcal{M} . Define a *flag* Φ of the map \mathcal{M} as an ordered triple (Φ_0, Φ_1, Φ_2) of a vertex, edge and face mutually incident in \mathcal{M} , and denote as $\mathcal{F}(\mathcal{M})$ the set of all flags of \mathcal{M} . Hence, every flag $\Phi \in \mathcal{F}(\mathcal{M})$ is adjacent to three other flags, called *i -adjacent* flags of Φ , which corresponding triples differ on the entry Φ_i from Φ , and are denoted by Φ^i , with $i = 0, 1, 2$. In other words, the flags of \mathcal{M} are related to the faces of \mathcal{B} . The set of all flags of \mathcal{M} is denoted by $\mathcal{F}(\mathcal{M})$.

To each map \mathcal{M} corresponds a subgroup of the permutation group $Sym(\mathcal{F}(\mathcal{M}))$, generated by three permutations s_0, s_1, s_2 such that for each flag $\Phi \in \mathcal{F}(\mathcal{M})$ the generator s_i acts on the right as follows

$$\Phi^{s_i} = \Phi \cdot s_i = \Phi^i,$$

with $i = 0, 1, 2$. Note that $(\Phi^i)^i = \Phi$, and that $\Phi^{0,2} = \Phi^{2,0}$ (since each edge in \mathcal{M} has exactly four triangles around it). Then, it follows that the generators s_0, s_1, s_2 are fixed-point free involutions in $Sym(\mathcal{F}(\mathcal{M}))$, such that $s_0 s_2 = s_2 s_0$ is also a fixed-point free involution of the flags of \mathcal{M} . Moreover, the involutions s_0, s_1, s_2 generate a subgroup, $Mon(\mathcal{M})$, of $Sym(\mathcal{F}(\mathcal{M}))$ known as the *monodromy (or connection)* group of the map \mathcal{M} , [11]. Since maps are connected, the action of $Mon(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ is transitive.

Given a map \mathcal{M} we can construct a graph $\mathcal{G}_{\mathcal{M}}$ in the following way. The vertex set of $\mathcal{G}_{\mathcal{M}}$ is $\mathcal{F}(\mathcal{M})$ and its edges are labelled by the subscripts of the generators of $\text{Mon}(\mathcal{M})$ in a natural way. In other words, two vertices $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ are adjacent by an edge of colour $i = 0, 1, 2$ in $\mathcal{G}_{\mathcal{M}}$, if and only if $\Phi^{s_i} = \Psi$ in \mathcal{M} . The graph $\mathcal{G}_{\mathcal{M}}$ is known as the flag graph of the map \mathcal{M} . Moreover, if $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ are two vertices of $\mathcal{G}_{\mathcal{M}}$, then a walk along the edges i_0, i_1, \dots, i_t that starts at Φ and ends at Ψ is a word $w = s_{i_0} s_{i_1} \cdots s_{i_t} \in \text{Mon}(\mathcal{M})$, defining inductively the action of $\text{Mon}(\mathcal{M})$ on the set $\mathcal{F}(\mathcal{M})$ as follows

$$\Phi^w = (\Phi^{i_0}) \cdot s_{i_1} \cdots s_{i_t} = \Phi^{i_0 \cdot i_1 \cdots i_t} = \Psi.$$

Observe that in $\mathcal{G}_{\mathcal{M}}$ the edges of a given colour form a perfect matching (an independent set of edges containing all the vertices of the graph), and the union of two sets of edges of different colour is a subgraph whose components are even cycles; such subgraph is known as a *2-factor* of $\mathcal{G}_{\mathcal{M}}$. In particular, note that since $(s_0 s_2)^2 = 1$ and $s_0 s_2$ is fixed-point free, the cycles with edges of alternating colours 0 and 2 are all of length four and these 4-cycles define the set of edges of \mathcal{M} . In other words, the edges of \mathcal{M} can be identified with the orbits of $\mathcal{F}(\mathcal{M})$ under the action of the subgroup generated by the involutions s_0 and s_2 ; that is, $E(\mathcal{M}) = \{\Phi^{(s_0, s_2)} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$.

Similarly, we find that the vertices and faces of \mathcal{M} are identified with the respective orbits of the subgroups $\langle s_1, s_2 \rangle$ and $\langle s_0, s_1 \rangle$ on $\mathcal{F}(\mathcal{M})$. That is, $V(\mathcal{M}) = \{\Phi^{(s_1, s_2)} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$ and $F(\mathcal{M}) = \{\Phi^{(s_0, s_1)} \mid \Phi \in \mathcal{F}(\mathcal{M})\}$. Thus, the group $\langle s_0, s_1, s_2 \rangle$ induces a transitive action on the set of all vertices, edges and faces of \mathcal{M} .

Let $\Gamma(\mathcal{M})$ denote the group of automorphisms of the map \mathcal{M} . An *automorphism* γ of \mathcal{M} is a bijection between the vertices, edges and faces of \mathcal{M} that preserves the map. In fact, every $\gamma \in \Gamma(\mathcal{M})$ “commutes” with the distinguished generators s_0, s_1, s_2 of $\text{Mon}(\mathcal{M})$, i.e. $\Phi^{s_i} \gamma = (\Phi \gamma)^{s_i}$, [11]. It is easy to see that the only automorphism that fixes any flag is the identity one. Therefore, the action of $\Gamma(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ is free (semi-regular) and partitions $\mathcal{F}(\mathcal{M})$ into orbits of the same size. If there are k orbits of flags under the action of $\Gamma(\mathcal{M})$, then \mathcal{M} is said to be a *k-orbit map*. Moreover, if $\Gamma(\mathcal{M})$ acts transitively on $\mathcal{F}(\mathcal{M})$, then \mathcal{M} is called a *regular map*.

As the automorphism group $\Gamma(\mathcal{M})$ of \mathcal{M} acts on the vertices, edges and faces of \mathcal{M} preserving their incidence, every automorphism of \mathcal{M} induces a bijection between the flags of \mathcal{M} preserving their adjacencies. Consequently $\Gamma(\mathcal{M})$ is isomorphic to the edge-colour preserving automorphism group of $\mathcal{G}_{\mathcal{M}}$, which consists of all the automorphisms of $\mathcal{G}_{\mathcal{M}}$ that send two vertices

adjacent by an edge of colour i into other two vertices adjacent by an edge of the same colour i , for each $i = 0, 1, 2$.

3. SYMMETRY TYPE GRAPH.

Recall that the automorphism group $\Gamma(\mathcal{M})$ of the map \mathcal{M} partitions the set of flags $\mathcal{F}(\mathcal{M})$ of \mathcal{M} into orbits of the same size. Let $\text{Orb}(\mathcal{M}) = \{\mathcal{O}_\Phi | \Phi \in \mathcal{F}(\mathcal{M})\}$ denote the set of all the orbits of $\mathcal{F}(\mathcal{M})$ under the action of $\Gamma(\mathcal{M})$.

From the flag graph $\mathcal{G}_\mathcal{M}$ we obtain a quotient graph $T(\mathcal{M})$ given by the action of $\Gamma(\mathcal{M})$ on $\mathcal{G}_\mathcal{M}$ in the following way. The vertices of $T(\mathcal{M})$ are the orbits of $\mathcal{F}(\mathcal{M})$ under $\Gamma(\mathcal{M})$ and two orbits $\mathcal{O}_\Phi, \mathcal{O}_\Psi \in \text{Orb}(\mathcal{M})$ are adjacent by an edge of colour $i = 0, 1, 2$ if and only if there are flags $\Phi' \in \mathcal{O}_\Phi$ and $\Psi' \in \mathcal{O}_\Psi$ such that they are i -adjacent in $\mathcal{G}_\mathcal{M}$; note that this adjacency is well-defined. The graph $T(\mathcal{M})$ might have parallel edges (edges with the same end-points) or also semi-edges (edges with a single end-point), but $T(\mathcal{M})$ has no loops. We refer to the graph $T(\mathcal{M})$ as the *symmetry type graph* of \mathcal{M} , and we also say that \mathcal{M} is a k -orbit map of type T . For instance, the symmetry type graph of a regular map is a graph with a single vertex and three semi-edges, one of each colour 0, 1 and 2. If \mathcal{M} is a k -orbit map, then $T(\mathcal{M})$ is a cubic graph with exactly k vertices and the edges are properly coloured with three colours.

For every $w \in \text{Mon}(\mathcal{M})$ and $\Phi \in \mathcal{F}(\mathcal{M})$ the action of $\text{Mon}(\mathcal{M})$ on the set $\text{Orb}(\mathcal{M})$ is defined as $\mathcal{O}_\Phi \cdot w = \mathcal{O}_{\Phi \cdot w}$. This action is transitive, as is the action of $\text{Mon}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$. Since $\mathcal{G}_\mathcal{M}$ is a connected graph, then its corresponding symmetry type graph $T(\mathcal{M})$ is connected as well. The number of types of k -orbit maps is the number of connected cubic graphs with k vertices, properly three edge-coloured, where the colours 0 and 2 are as in the Figure 1.

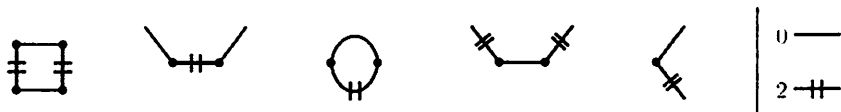


FIGURE 1. Possible quotients of 0-2 coloured 4-cycles of $\mathcal{G}_\mathcal{M}$.

In [4] are shown all symmetry type graphs for $k \leq 5$, along with more properties of these graphs, such as the following Theorem.

Theorem 1. *Let \mathcal{M} be a map with symmetry type graph $T(\mathcal{M})$. Then, the number of connected components in the 2-factor of $T(\mathcal{M})$ of colours i and j , with $i, j \in \{0, 1, 2\}$ and $i \neq j$, is the same as the number of orbits of the l -faces of \mathcal{M} , where $l \in \{0, 1, 2\}$ and $l \neq i, j$.*

The notation for the symmetry type graphs with k vertices is followed from [4] as well.

4. OPERATIONS ON MAPS.

Given a map \mathcal{M} it is possible to find other maps in the same surface as \mathcal{M} , by applying an operation to \mathcal{M} . One of the most recalled operations on maps is the dual operation. Two maps \mathcal{M} and \mathcal{N} are said to be dual of each other if there exists a bijection δ between the set of flags $\mathcal{F}(\mathcal{M})$ of \mathcal{M} and the set of flags $\mathcal{F}(\mathcal{N})$ of \mathcal{N} such that for each $\Phi \in \mathcal{F}(\mathcal{M})$ and each $i \in \{0, 1, 2\}$, $\Phi^i \delta = (\Phi \delta)^{2-i}$. The *dual map* of \mathcal{M} is denoted by \mathcal{M}^* . Note that $(\mathcal{M}^*)^* \cong \mathcal{M}$.

If there exist a bijection δ from a map \mathcal{M} to itself, then $\mathcal{M} \cong \mathcal{M}^*$ and it is called a *self-dual map*. In this case we name δ a *duality* of \mathcal{M} . Observe that for a self-dual map \mathcal{M} , if δ_1 and δ_2 are two dualities of \mathcal{M} , then the product of them is an automorphism of \mathcal{M} . In particular, the square of any duality is an automorphism, and hence the order of any duality is even.

In terms of the flag graphs, the bijection δ can be regarded as a bijection between the vertices of $\mathcal{G}_{\mathcal{M}}$ and the vertices of $\mathcal{G}_{\mathcal{M}^*}$ that sends edges of colour i of $\mathcal{G}_{\mathcal{M}}$ to edges of colour $2 - i$ of $\mathcal{G}_{\mathcal{M}^*}$, for each $i \in \{0, 1, 2\}$.

In [4] was considered the problem of classification of k -orbit medial maps. Here we address the same problem for truncation and leapfrog maps, extending the results in [17] on truncation of k -orbit maps.

4.1. Truncation. Geometrically, when we apply truncation to a map \mathcal{M} , the vertices of \mathcal{M} are replaced by faces, and the faces of \mathcal{M} become faces with twice the number of vertices than the original ones. An example is the truncated octahedron, shown in Figure 2, it has six square faces and eight hexagonal faces that correspond to the vertices and faces of the octahedron, respectively. Hence, there is a correspondence between the set of faces of the truncated map $\text{Tr}(\mathcal{M})$ of \mathcal{M} with the set of vertices and faces of \mathcal{M} . That is, $F(\text{Tr}(\mathcal{M})) = V(\mathcal{M}) \cup F(\mathcal{M})$.

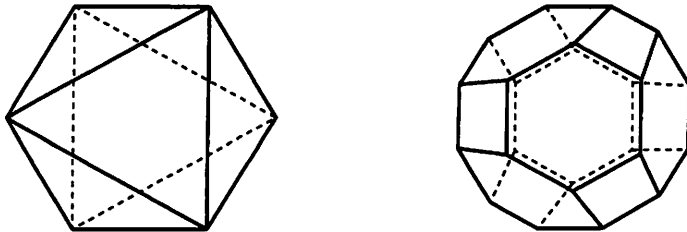


FIGURE 2. Octahedron and truncated octahedron.

Furthermore, for each edge of \mathcal{M} there are exactly two vertices of $\text{Tr}(\mathcal{M})$, two of these vertices have an edge joining them if either both belong to a common edge of \mathcal{M} or if the corresponding edges in \mathcal{M} share a vertex and belong to the same face. Then, the sets of edges and vertices of $\text{Tr}(\mathcal{M})$ are

$$E(\text{Tr}(\mathcal{M})) = E(\mathcal{M}) \cup \{\{\Phi_0, \Phi_2\} | \Phi \in \mathcal{F}(\mathcal{M})\} \text{ and}$$

$$V(\text{Tr}(\mathcal{M})) = \{\{\Phi_0, \Phi_1\} | \Phi \in \mathcal{F}(\mathcal{M})\},$$

respectively. Each vertex of $\text{Tr}(\mathcal{M})$ has valency 3 and therefore, the truncated map $\text{Tr}(\mathcal{M})$ contains $2|E(\mathcal{M})|$ vertices and $3|E(\mathcal{M})|$ edges.

In the Figure 3 is depicted how on the surface every fundamental triangle for \mathcal{M} is divided into three smaller fundamental triangles for the truncated map $\text{Tr}(\mathcal{M})$. This is, for each flag $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ in $\mathcal{F}(\mathcal{M})$, there are

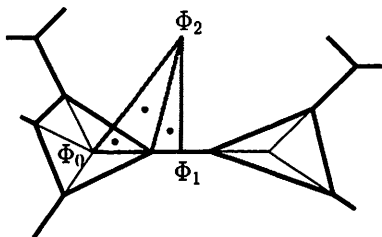


FIGURE 3. The three flags of $\mathcal{F}(\text{Tr}(\mathcal{M}))$ corresponding to the flag $\Phi = (\Phi_0, \Phi_1, \Phi_2) \in \mathcal{F}(\mathcal{M})$.

three flags $(\Phi, 0) := (\{\Phi_0, \Phi_1\}, \{\Phi_0, \Phi_2\}, \Phi_0)$, $(\Phi, 1) := (\{\Phi_0, \Phi_1\}, \Phi_1, \Phi_2)$ and $(\Phi, 2) := (\{\Phi_0, \Phi_1\}, \{\Phi_0, \Phi_2\}, \Phi_2)$ corresponding to Φ in $\mathcal{F}(\text{Tr}(\mathcal{M}))$. The adjacencies between the flags in $\mathcal{F}(\text{Tr}(\mathcal{M}))$ are given as follows.

$$\begin{aligned} (\Phi, 0) \cdot r_0 &= (\Phi^{s_1}, 0), & (\Phi, 0) \cdot r_1 &= (\Phi^{s_2}, 0), & (\Phi, 0) \cdot r_2 &= (\Phi, 2); \\ (\Phi, 1) \cdot r_0 &= (\Phi^{s_0}, 1), & (\Phi, 1) \cdot r_1 &= (\Phi, 2), & (\Phi, 1) \cdot r_2 &= (\Phi^{s_2}, 1); \\ (\Phi, 2) \cdot r_0 &= (\Phi^{s_1}, 2), & (\Phi, 2) \cdot r_1 &= (\Phi, 1), & (\Phi, 2) \cdot r_2 &= (\Phi, 0). \end{aligned}$$

Notice that r_0 , r_1 and r_2 depend of the adjacency between the flags in $\mathcal{F}(\mathcal{M})$. Moreover, r_0 , r_1 and r_2 are fixed-point free involutions that generate the monodromy group $\text{Mon}(\text{Tr}(\mathcal{M}))$ of $\text{Tr}(\mathcal{M})$, where $r_0 r_2 = r_2 r_0$ and, $(r_1 r_2)^3 = id$.

Consequently, in Figure 4 is presented an algorithm to construct the flag graph of $\text{Tr}(\mathcal{M})$ from $\mathcal{G}_{\mathcal{M}}$. Such algorithm induces a partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$

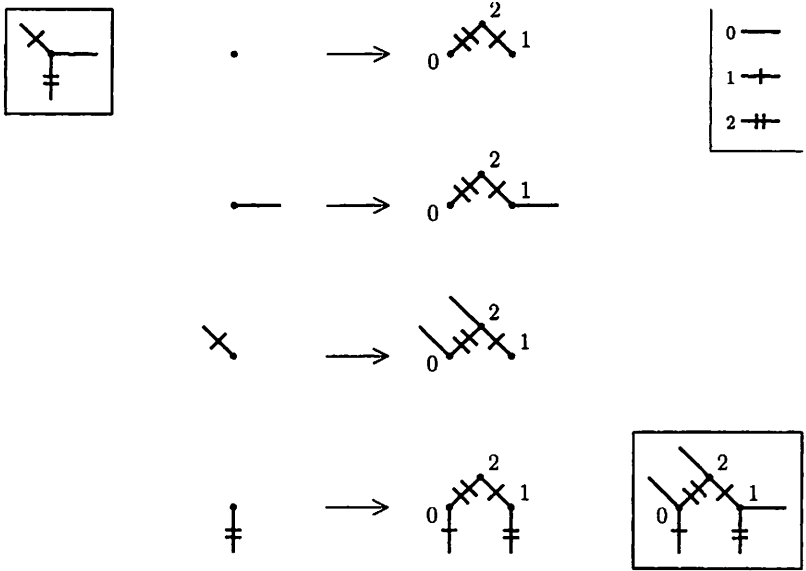


FIGURE 4. Local representation of a flag in $\mathcal{G}_{\mathcal{M}}$, in the left. The image under the truncation operation, locally obtained, in the right.

on the vertices of $\mathcal{G}_{\text{Tr}(\mathcal{M})}$, where $\mathcal{A}_i := \{(\Phi, i) | \Phi \in \mathcal{F}(\mathcal{M})\}$ with $i = 0, 1, 2$, and consequently we have Proposition 1.

Proposition 1. *The flag graph $\mathcal{G}_{\text{Tr}(\mathcal{M})}$, of the truncation map $\text{Tr}(\mathcal{M})$ of any map \mathcal{M} , can be quotient into a graph as the symmetry type graph 3^0 .*

Proof. Let $\mathcal{A}_i = \{(\Phi, i) | \Phi \in \mathcal{F}(\mathcal{M})\}$ be the subset of $\mathcal{F}(\text{Tr}(\mathcal{M}))$ containing all flags of $\text{Tr}(\mathcal{M})$ of the form (Φ, i) with $i = 0, 1, 2$. Then, $\mathcal{F}(\text{Tr}(\mathcal{M})) = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_0 \cap \mathcal{A}_1 \cup \mathcal{A}_2 = \emptyset$. Hence, $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ is a partition of the set of flags $\mathcal{F}(\text{Tr}(\mathcal{M}))$. Based on Figure 4, it is straightforward to see that the quotient of $\mathcal{G}_{\text{Tr}(\mathcal{M})}$ over such partition, is isomorphic to the symmetry type graph of a map with symmetry type 3^0 (see Figure 5). \square

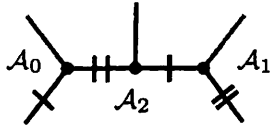


FIGURE 5. Symmetry type graph 3^0 .

As the vertices of the map $\text{Tr}(\mathcal{M})$ have valency three, there are exactly six flags of $\mathcal{F}(\text{Tr}(\mathcal{M}))$ around each vertex in $\text{Tr}(\mathcal{M})$. Recall that the vertices of the map $\text{Tr}(\mathcal{M})$ are identified with the orbits of the subgroup $\langle \tau_1, \tau_2 \rangle$ on $\mathcal{F}(\text{Tr}(\mathcal{M}))$. Since the automorphism group of a map partitions its set of flags into orbits of the same size, then Theorem 1 implies that the 2-factors coloured by 1 and 2 in the symmetry type graph $T(\text{Tr}(\mathcal{M}))$ must be as those in Figure 6.

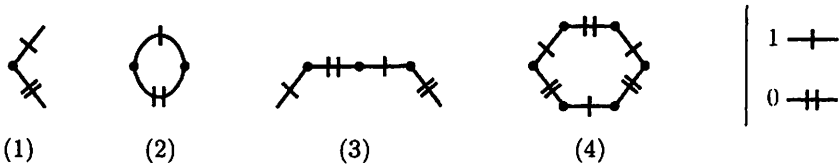


FIGURE 6. Possible quotients of 1-2 coloured 6-cycles of $\mathcal{G}_{\text{Tr}(\mathcal{M})}$.

Orbanić, Pellicer and Weiss proved the following Theorem and present examples of when any of this cases is possible, [17].

Theorem 2. [17] *If \mathcal{M} is a k -orbit map, then the map $\text{Tr}(\mathcal{M})$ is either a k -orbit, a $\frac{3k}{2}$ -orbit (with k even) or a $3k$ -orbit map.*

With regards to this result, using coset enumeration, the authors in [17] show that in particular if a k -orbit map \mathcal{M} which truncation map $\text{Tr}(\mathcal{M})$ is either a $\frac{3k}{2}$ -orbit or a k -orbit map, then we must necessarily find a bipartition on the vertices of $\mathcal{G}_{\mathcal{M}}$ in such way that $\mathcal{G}_{\mathcal{M}}$ can be quotient into a graph isomorphic to the symmetry type graph 20_1 ; i.e. all vertices of \mathcal{M} must have even valence, and the flags of each two adjacent faces in \mathcal{M} can be put in different parts.

By Proposition 1, we have that there exists a partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the vertices of the flag graph $\mathcal{G}_{\text{Tr}(\mathcal{M})}$ of the truncation map $\text{Tr}(\mathcal{M})$ in such

way that for each vertex $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$ in the partition \mathcal{A}_0 , the vertices Ψ^2 and $\Psi^{2,1}$ correspond to the partitions \mathcal{A}_2 and \mathcal{A}_1 , respectively. This latter implies a local arrangement in the flags of $\text{Tr}(\mathcal{M})$: assembling the flags $\Psi, \Psi^2, \Psi^{2,1} \in \mathcal{F}(\text{Tr}(\mathcal{M}))$ into a new flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ where the face Ψ_2 will be considered as an element of $V(\mathcal{M})$, [17].

Finally the authors, in [17], conclude that the flag graph $\mathcal{G}_{\text{Tr}(\mathcal{M})}$ can be quotient in to a graph isomorphic to the symmetry type graph 3^0 under the action of a subgroup $H \leq \Gamma(\text{Tr}(\mathcal{M}))$, of the automorphism group of $\text{Tr}(\mathcal{M})$, if and only if \mathcal{M} is a regular map.

In what follows, supported by Proposition 1 and the local arrangement on the flags of $\text{Tr}(\mathcal{M})$ described above, we discuss the results of Orbanić, Pellicer and Weiss on truncation of regular, 2-orbit and 3-orbits maps, [17]. Further on, we show results on the truncation of k -orbit maps, with $k = 4, \dots, 7$ and $k = 9$, using the same local arrangement on the flags of $\text{Tr}(\mathcal{M})$ that Orbanić, Pellicer and Weiss used for smaller k . Due to space, we leave on aside the truncation of 8-orbit maps.

4.1.1. *Truncation of regular, 2-orbit and 3-orbit maps.* As it was said previously, if \mathcal{M} is a regular map, then $\text{Tr}(\mathcal{M})$ is either regular or a 3-orbit map, which has symmetry type graph 3^0 (see Figure 5). On another hand, if \mathcal{M} is a 2-orbit or a 3-orbit map, Orbanić, Pellicer and Weiss conclude with the following results on the truncation map of \mathcal{M} , [17].

Proposition 2. [17] *If the truncation $\text{Tr}(\mathcal{M})$ of a 2-orbit map \mathcal{M} is a 2-orbit map, then one of the following holds*

- (i) \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type 2,
- (ii) \mathcal{M} is of type 2_{01} and $\text{Tr}(\mathcal{M})$ is of type 2_0 , or
- (iii) \mathcal{M} is of type 2_2 and $\text{Tr}(\mathcal{M})$ is of type 2_{12} .

The symmetry type graphs 2, 2_{01} , 2_0 , 2_2 and 2_{12} are depicted in Figure 7.

Proposition 3. [17] *If the truncation $\text{Tr}(\mathcal{M})$ of a 2-orbit map is a 3-orbit map, then \mathcal{M} is of type 2_{01} and $\text{Tr}(\mathcal{M})$ is of type 3^0 (see Figure 7).*

Proposition 4. [17] *If the truncation $\text{Tr}(\mathcal{M})$ of a 3-orbit map is a 3-orbit map, then \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type 3^{02} (see Figure 7).*

In other words, if \mathcal{M} is a regular, a 2-orbit or a 3-orbit map, and $\text{Tr}(\mathcal{M})$ has either 1, 2 or 3 flag-orbits, then the symmetry type graph of $\text{Tr}(\mathcal{M})$ is one of the six graphs in Figure 7. Which in fact, are the only possible

symmetry type graphs with 1, 2 and 3 vertices, consistent with the (1,2) 2-factors in Figure 6.

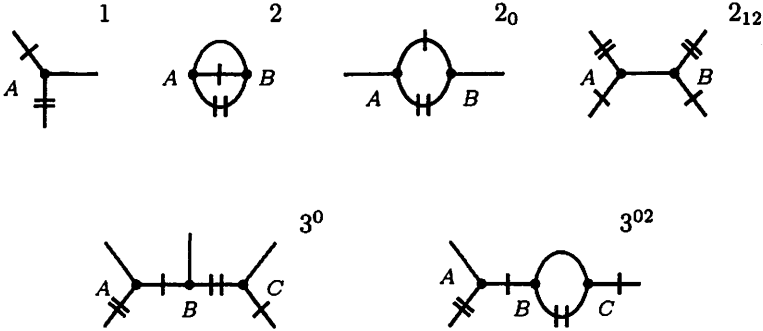


FIGURE 7. Symmetry type graphs of $\text{Tr}(\mathcal{M})$, for a regular map or \mathcal{M} as in Propositions 2, 3, 4.

Orbanić, Pellicer and Weiss prove the Propositions 2, 3 and 4 based on the following facts.

- (i) All the vertices of the map $\text{Tr}(\mathcal{M})$ have valency 3.
- (ii) The vertex set of \mathcal{M} is a proper subset of the set of faces of $\text{Tr}(\mathcal{M})$ (recall that $F(\text{Tr}(\mathcal{M})) = V(\mathcal{M}) \cup F(\mathcal{M})$).
- (iii) Each flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ of \mathcal{M} is divided in exactly three flags $(\Phi_\Psi, 0) =: \Psi$, $(\Phi_\Psi, 2) =: \Psi^2$, $(\Phi_\Psi, 1) =: \Psi^{2,1} \in \mathcal{F}(\text{Tr}(\mathcal{M}))$ of $\text{Tr}(\mathcal{M})$ (Figures 3 and 4), where

$$\Phi_\Psi := \{\Psi, \Psi^2, \Psi^{2,1}\}.$$

This is, given that we can define a proper partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the set $\mathcal{F}(\text{Tr}(\mathcal{M}))$ in such way that for each flag $\Psi \in \mathcal{A}_0$, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$, consequently. Assume that the flag $\Psi \in \mathcal{A}_0$ represents a particular orbit of the flags of $\text{Tr}(\mathcal{M})$, follow its adjacencies determined by the corresponding symmetry type graph $T(\text{Tr}(\mathcal{M}))$ in Figure 7 to define to which flag-orbit of $\mathcal{F}(\text{Tr}(\mathcal{M}))$ belong the other flags in the in the partitions \mathcal{A}_2 and \mathcal{A}_1 . Finally, suppose that the face Ψ_2 of $\text{Tr}(\mathcal{M})$ corresponds to a vertex in $V(\mathcal{M})$, and obtain the representative flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$. Then, determine the symmetry type of \mathcal{M} when $\text{Tr}(\mathcal{M})$ is a 2-orbit or a 3-orbit map, from the types 2, 2_{01} , 2_2 and 3^{02} of maps that are vertex-transitive and also have vertices of valency 3.

On the other hand, if $\text{Tr}(\mathcal{M})$ is a $3k$ -orbit map, applying the algorithm presented in the Figure 4 to the symmetry type graph $T(\mathcal{M})$ with k vertices, we obtain straightforward the symmetry type graph of $\text{Tr}(\mathcal{M})$ with $3k$ vertices, and we will refer to this as the *truncated symmetry type graph* of $T(\mathcal{M})$. For instance, the truncated symmetry type graphs with 6 vertices, that correspond to the seven symmetry type graphs of maps with 2 vertices are depicted in Figure 8. (Recall that the notation of the symmetry types of k -orbit maps, with $k \leq 5$, follows from [4].)

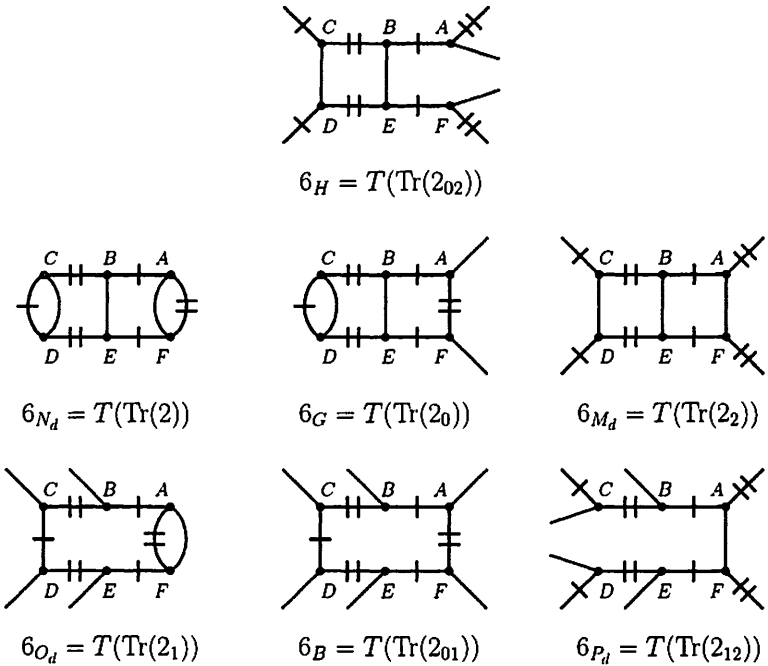


FIGURE 8. Truncation symmetry type graphs with 6 vertices, of the seven 2-orbit symmetry type graphs.

Now we proceed to our first goal: to find the possible symmetry type graphs with k and $\frac{3k}{2}$ vertices, of maps that correspond to truncation of a k -orbit map \mathcal{M} , with $k = 4, 5, 6, 7, 9$. Extending the results of [17], for the truncation of these k -orbit maps. To find the corresponding symmetry type of the map \mathcal{M} , we use the same local arrangement on the flags of $\text{Tr}(\mathcal{M})$, assembling the flags $\Psi, \Psi^2, \Psi^{2,1} \in \mathcal{F}(\text{Tr}(\mathcal{M}))$ into a new flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$, given in [17], under the appropriate partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the flags in $\mathcal{F}(\text{Tr}(\mathcal{M}))$ according to Proposition 1.

4.1.2. *Truncation of 4-orbit maps.* By Theorem 2, if \mathcal{M} is a 4-orbit map, then its truncation $\text{Tr}(\mathcal{M})$ is a map with either 4, 6 or 12 orbits on its flags.

Let \mathcal{M} be a 4-orbit map and suppose that the map $\text{Tr}(\mathcal{M})$ also has 4 orbits on its flags. Then, by a proper combination of the (1,2) 2-factors shown in Figure 6, into a symmetry type graph with 4 vertices (and looking at the twenty two symmetry type graphs with 4 vertices, [4]), it can be seen that $\text{Tr}(\mathcal{M})$ has either symmetry type 4_{D_p} , 4_D or 4_{G_d} (see Figure 9).

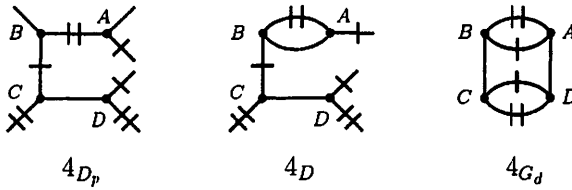


FIGURE 9. Possible symmetry type graphs of the 4-orbit maps that correspond to the truncation of another map.

Notice that the maps with the symmetry type 4_{D_p} have two orbits on their faces. In fact, one of its flag-orbits is completely contained in one of the face-orbits. However, this latter does not limit the choice of the face $\Psi_2 \in V(\mathcal{M})$, corresponding to a flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$. Let $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ be the partition of the set of flags $\mathcal{F}(\text{Tr}(\mathcal{M}))$, and suppose that the flag $\Psi \in \mathcal{A}_0$ belongs to the flag-orbit B . Then, by assembling the flags $\Psi \in \mathcal{A}_0$, $\Psi^2 \in \mathcal{A}_2$, $\Psi^{2,1} \in \mathcal{A}_1$ of a map $\text{Tr}(\mathcal{M})$ with symmetry type 4_{D_p} , we obtain four different flags $\Phi_\Psi, \Phi_{\Psi^1}, \Phi_{\Psi^{1,0}}$ and $\Phi_{\Psi^{1,0,2,1,0,1,2}}$ in $\mathcal{F}(\mathcal{M})$, depicted in Figure 10. Implying that \mathcal{M} is a 4-orbit map, with symmetry type 4_{D_p} , as well.

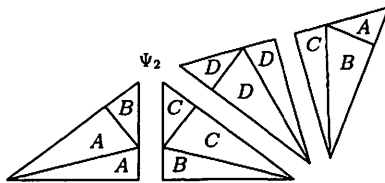


FIGURE 10. Assembled flags in a map $\text{Tr}(\mathcal{M})$, with symmetry type 4_{D_p} as truncation of another 4-orbit map. The element $\Psi_2 \in V(\mathcal{M})$ corresponds to the face of the base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

On the other hand, the maps with symmetry type 4_D and 4_{G_d} are face-transitive maps. Hence the arrangement of their flags is more natural than for 4_{D_p} . In case that $\text{Tr}(\mathcal{M})$ is of type 4_D , we suppose that the flag $\Psi \in \mathcal{A}_0$ belongs to the flag-orbit A , consequently the flag $\Psi^2 \in \mathcal{A}_2$ belongs to the flag-orbit B and $\Psi^{2,1} \in \mathcal{A}_1$ belongs to the flag-orbit C . Inducing the local arrangement on the flags of $\text{Tr}(\mathcal{M})$ as those in Figure 11 (a). Where the other assembled flags

- Φ_{Ψ^0} , with the flag $\Psi^0 \in \mathcal{A}_0$ in the orbit B and the flags $\Psi^{0,2} \in \mathcal{A}_2$ and $\Psi^{0,2,1} \in \mathcal{A}_1$ in the orbit A ;
- $\Phi_{\Psi^{0,1}}$, with the flags $\Psi^{0,1} \in \mathcal{A}_0$ and $\Psi^{0,1,2} \in \mathcal{A}_2$ in the orbit C and the flag $\Psi^{0,1,2,1} \in \mathcal{A}_1$ in the orbit B ; and
- $\Phi_{\Psi^{0,1,0}}$, with the flags $\Psi^{0,1,0} \in \mathcal{A}_0$, $\Psi^{0,1,0,2} \in \mathcal{A}_2$ and the flag $\Psi^{0,1,0,2,1} \in \mathcal{A}_1$ in the orbit D .

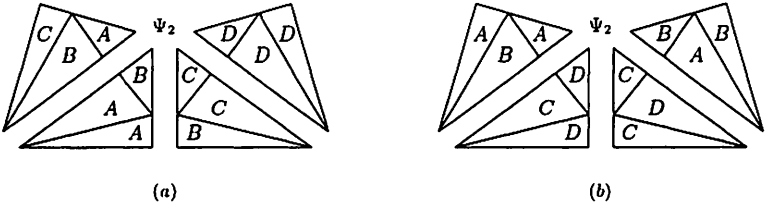


FIGURE 11. Assembled flags in a map $\text{Tr}(\mathcal{M})$, with (a) symmetry type 4_D , and (b) with symmetry type 4_{G_d} as truncation of another 4-orbit map. The element $\Psi_2 \in V(\mathcal{M})$ corresponds to the face of the base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

Proceeding in similar way with the flags of a map $\text{Tr}(\mathcal{M})$ of type 4_{G_d} . Assume that the flags $\Psi \in \mathcal{A}_0$ and $\Psi^{2,1} \in \mathcal{A}_1$ belong to the flag-orbit A , and the flag $\Psi^2 \in \mathcal{A}_2$ belongs to the flag-orbit B . Then the four flags produced by this local arrangement on the flags of $\text{Tr}(\mathcal{M})$ as those in Figure 11 (b). Where the assembled flags are Φ_{Ψ} ;

- Φ_{Ψ^0} , with the flags $\Psi^0 \in \mathcal{A}_0$ and $\Psi^{0,2,1} \in \mathcal{A}_1$ are in the orbit D and the flag $\Psi^{0,2} \in \mathcal{A}_2$ in the orbit C ;
- $\Phi_{\Psi^{0,1}}$, with the flags $\Psi^{0,1} \in \mathcal{A}_0$ and $\Psi^{0,1,2,1} \in \mathcal{A}_1$ in the orbit C and the flag $\Psi^{0,1,2} \in \mathcal{A}_2$ in the orbit D ; and
- $\Phi_{\Psi^{0,1,0}}$, with the flags $\Psi^{0,1,0} \in \mathcal{A}_0$ and $\Psi^{0,1,0,2,1} \in \mathcal{A}_1$ in the orbit B and the flag $\Psi^{0,1,0,2} \in \mathcal{A}_2$ in the orbit A .

Therefore, if we assume that $\text{Tr}(\mathcal{M})$ and \mathcal{M} are 4-orbit maps, then we obtain the following proposition.

Proposition 5. *If the truncation $\text{Tr}(\mathcal{M})$ of a 4-orbit map \mathcal{M} is a 4-orbit map, then exactly one of the following holds.*

- (i) \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type $4D_p$,
- (ii) \mathcal{M} is of type $4E$ and $\text{Tr}(\mathcal{M})$ is of type $4D$, or
- (iii) \mathcal{M} is of type $4G$ and $\text{Tr}(\mathcal{M})$ is of type $4G_d$.

Recall that given a 4-orbit map \mathcal{M} , its truncated map $\text{Tr}(\mathcal{M})$ might be a 6-orbit map. To find the possible symmetry types of maps with six orbits on its flags that would correspond to the symmetry type of a map $\text{Tr}(\mathcal{M})$, when \mathcal{M} is a 4-orbit map, it is necessary to observe the following.

The symmetry type graph of a 6-orbit map $\text{Tr}(\mathcal{M})$, constructed by a proper combination of the (1,2) 2-factors shown in Figure 6, contains either exactly one copy of the 2-factor (4), two copies of the 2-factor (3), or exactly one copy of the 2-factors (1), (2) and (3); joined consistently with the (0, 2) 2-factors in Figure 1. In fact, there are sixteen symmetry type graphs with six vertices that satisfy any of the three later cases. Recall that in the local arrangement on the flags $\Psi, \Psi^2, \Psi^{2,1} \in \mathcal{F}(\text{Tr}(\mathcal{M}))$ representing elements of the tripartition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$, we consider that the face Ψ_2 is an element of $V(\mathcal{M})$.

Thus, by looking at the symmetry type graphs with 6 vertices in Figures 8, 15 and 17, we deduce the following observations on the sixteen symmetry types, that follow the latter paragraph.

- i) 6_B , has three face-orbits, each containing 2 flag-orbits (this corresponds to the truncation symmetry type graph of 2_{01});
- ii) 6_{P_d} , has three face-orbits, one with 4 flag-orbits and the others with one flag-orbit each;
- iii) $6_G, 6_H, 6_{M_d}, 6_{N_d}, 6_{O_d}$, have two face-orbits, one with 4 flag-orbits and the other with 2 flag-orbits;
- iv) 6_{H_p} , has two face-orbits, both with 3 flag-orbits;
- v) 6_{J_d} , has two face-orbits, one with 5 flag-orbits and the other with one flag-orbit; and
- vi) $6_{B_p}, 6_{F_d}, 6_{G_p}, 6_{M_{dp}}, 6_{N_{dp}}, 6_{O_{dp}}, 6_{P_{dp}}$, that are transitive on their faces.

Therefore, the possible cases to be consider in manner to find whenever the map \mathcal{M} is a 4-orbit map such that the truncation map $\text{Tr}(\mathcal{M})$ is a 6-orbit map are the cases *ii)* and *iii)* above. This is, suppose that the map $\text{Tr}(\mathcal{M})$ has symmetry type either $6_G, 6_H, 6_{M_d}, 6_{N_d}, 6_{O_d}$, or 6_{P_d} (Figure 8). First consider the face-orbit in $\text{Tr}(\mathcal{M})$ containing four flag-orbits, then

- (a) If $\text{Tr}(\mathcal{M})$ is a 6-orbit map with symmetry type graph 6_G , 6_{N_d} or 6_{O_d} , and a partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of its set of flags $\mathcal{F}(\text{Tr}(\mathcal{M}))$. Let $\Psi \in \mathcal{A}_0$ be a flag in the orbit A of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, then $\Psi^2 \in \mathcal{A}_2$, and $\Psi^{2,1} \in \mathcal{A}_1$ are flags in the orbits F and E , respectively in $\text{Tr}(\mathcal{M})$. Hence, the induced flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ corresponds to the first flag shown in Figure 12 (a).
- (b) Similarly, if $\text{Tr}(\mathcal{M})$ is a 6-orbit map with symmetry type graph 6_H , 6_{M_d} or 6_{P_d} , and a partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of its set of flags $\mathcal{F}(\text{Tr}(\mathcal{M}))$. Let $\Psi \in \mathcal{A}_0$ be a flag in the orbit A of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, then $\Psi^2 \in \mathcal{A}_2$, and $\Psi^{2,1} \in \mathcal{A}_1$ are flags in the orbits A and B , respectively in $\text{Tr}(\mathcal{M})$. Hence, the induced flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ corresponds to the first flag shown in Figure 12 (b).

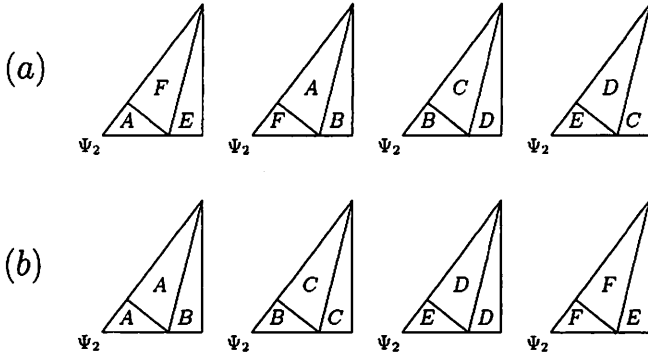


FIGURE 12. Assembled flags in a 6-orbit map $\text{Tr}(\mathcal{M})$, (a) with symmetry type graph 6_G , 6_{N_d} or 6_{O_d} , and (b) with symmetry type graph 6_H , 6_{M_d} or 6_{P_d} , as truncation of a 4-orbit map. The element $\Psi_2 \in V(\mathcal{M})$ corresponds to face of the base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

The remaining three different assembled flags in both cases: (a) and (b), that represent other orbits of the elements in $\mathcal{F}(\mathcal{M})$, different from Φ_Ψ , are obtained straight-forward from the adjacencies given by the symmetry type graph of $\text{Tr}(\mathcal{M})$ and the corresponding partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$.

Consequently, we obtain the Proposition 6, and the remaining symmetry type graphs will be analysed in Section 4.1.4.

Proposition 6. *If the truncation $\text{Tr}(\mathcal{M})$ of a 4-orbit map \mathcal{M} is a 6-orbit map, then exactly one of the following holds.*

- (i) \mathcal{M} is of type 4_B and $\text{Tr}(\mathcal{M})$ are of type 6_{P_d} ,

- (ii) \mathcal{M} is of type 4_C and $\text{Tr}(\mathcal{M})$ is of type 6_{O_d} ,
- (iii) \mathcal{M} is of type 4_G and $\text{Tr}(\mathcal{M})$ is either of type 6_{N_d} or of type 6_{M_d} , or
- (iv) \mathcal{M} is of type 4_H and $\text{Tr}(\mathcal{M})$ is either of type 6_G or of type 6_H .

To conclude with the truncation of 4-orbit maps. There are twenty two truncated symmetry type graphs with 12 vertices that correspond to the twenty two symmetry type graphs with four vertices, determined by applying the algorithm in Figure 4 to each.

4.1.3. *Truncation of 5-orbit maps.* Similarly to the previous section, if \mathcal{M} is a 5-orbit map, its truncation map $\text{Tr}(\mathcal{M})$ is either a 5-orbit or a 15-orbit map, Theorem 2.

Notice that out of the thirteen different symmetry type graphs of 5-orbit maps, [4], the only possible combination of the (1,2) 2-factors with 2 and 3 vertices in Figure 6 is the symmetry type 5_{B_d} (Figure 13). This is, a 5-orbit map $\text{Tr}(\mathcal{M})$ is the truncation of a 5-orbit map \mathcal{M} , if $\text{Tr}(\mathcal{M})$ is of type 5_{B_d} .

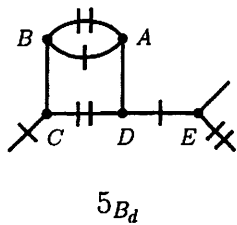


FIGURE 13. Only possible symmetry type graph of a 5-orbit maps $\text{Tr}(\mathcal{M})$, derived from another 5-orbit map.

Let $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ be the partition on the flags of a map $\text{Tr}(\mathcal{M})$ with symmetry type graph 5_{B_d} , such that the flags $\Psi \in \mathcal{A}_0$, $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ induce the flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$. Clearly, the maps of type 5_{B_d} are face-transitive. Then, without loss of generality we can assume that the flag $\Psi \in \mathcal{A}_0$ is in the flag-orbit A of $\text{Tr}(\mathcal{M})$, where the face Ψ_2 is an element in $V(\mathcal{M})$. Consequently, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are flags in the orbits B and A , respectively. Therefore, $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ represents an orbit of the flags of \mathcal{M} , and the flags $\Phi_{\Psi^1}, \Phi_{\Psi^{1,0}}, \Phi_{\Psi^0}, \Phi_{\Psi^{0,1}} \in \mathcal{F}(\mathcal{M})$, with Ψ^1 a flag in the orbit B , $\Psi^{1,0}$ a flag in the orbit C , Ψ^0 a flag in the orbit D and $\Psi^{0,1}$ a flag in the orbit E of the elements in $\mathcal{F}(\text{Tr}(\mathcal{M}))$, represent four other different flag-orbits of the set $\mathcal{F}(\mathcal{M})$ as those in Figure 14. Obtaining the following Proposition.

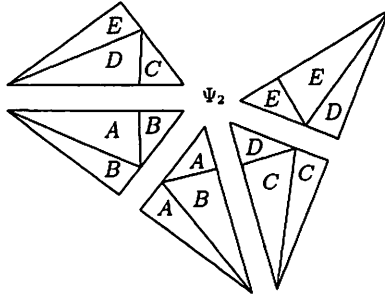


FIGURE 14. Assembled flags of a 5-orbit map $\text{Tr}(\mathcal{M})$ as truncation of another 5-orbit map, where $\Psi_2 \in V(\mathcal{M})$ is the corresponding face in $\text{Tr}(\mathcal{M})$ in a base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

Proposition 7. *If the truncation $\text{Tr}(\mathcal{M})$ of a 5-orbit map is a 5-orbit map, then \mathcal{M} is of type 5_{B_p} and $\text{Tr}(\mathcal{M})$ is of type 5_{B_d} .*

On the other hand, there are thirteen truncated symmetry type graphs with 15 vertices that correspond to the all symmetry type graphs with five vertices shown in [4], determined by applying the algorithm in Figure 4 to each.

4.1.4. *Truncation of 6-orbit maps.* Given a 6-orbit map \mathcal{M} , by Theorem 2 we know that its truncation map $\text{Tr}(\mathcal{M})$ has either 6, 9 or 18 orbits on its flags. In this section we go through the remaining nine possible symmetry type of 6-orbit maps: 6_{H_p} , 6_{J_d} , 6_{B_p} , 6_{F_d} , 6_{G_p} , $6_{M_{dp}}$, $6_{N_{dp}}$, $6_{O_{dp}}$ and $6_{P_{dp}}$ (mentioned in Section 4.1.2); that might correspond to the truncation map $\text{Tr}(\mathcal{M})$ of a map \mathcal{M} . Further on, we find the possible symmetry type graphs with 9 vertices that correspond to the truncation $\text{Tr}(\mathcal{M})$ of a map \mathcal{M} and determine which of them are related to the truncation of a 6-orbit or 9-orbit map.

Let $\text{Tr}(\mathcal{M})$ be a map with symmetry type graph either 6_{B_p} , 6_{G_p} , $6_{N_{dp}}$, $6_{O_{dp}}$ (Figure 15), 6_{F_d} , 6_{H_p} , 6_{J_d} , $6_{M_{dp}}$, or $6_{P_{dp}}$ (Figure 17). Consider, the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ on the flags of a map $\text{Tr}(\mathcal{M})$, such that the flags $\Psi \in \mathcal{A}_0$, $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ induce the flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$.

Observe that a map $\text{Tr}(\mathcal{M})$ with symmetry type graph 6_{B_p} , 6_{G_p} , $6_{N_{dp}}$, or $6_{O_{dp}}$ is transitive on its sets of faces and vertices, under the action of its automorphism group, Figure 15. Then, we can choose any flag Ψ in the flag-orbit A of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, such that the face Ψ_2 is an element in $V(\mathcal{M})$.

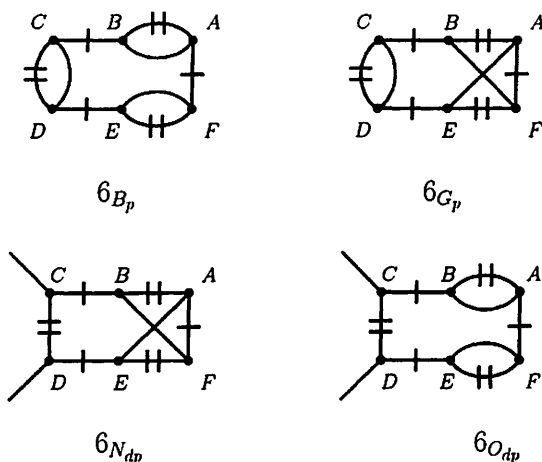


FIGURE 15. Possible symmetry type graphs of 6-orbit vertex-transitive maps $\text{Tr}(\mathcal{M})$, where \mathcal{M} is a 6-orbit map.

Thus, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are in the flag-orbits B and C of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, respectively. Hence, the flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ is assembled as the very left flag in Figure 16. The remaining five orbits are determined by the

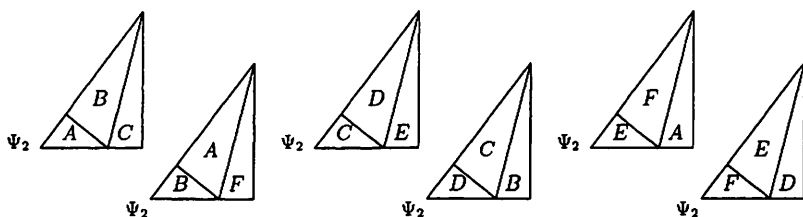


FIGURE 16. Assembled flags of a vertex- and face-transitive 6-orbit map $\text{Tr}(\mathcal{M})$ as truncation of a 6-orbit map, where $\Psi_2 \in V(\mathcal{M})$ is the corresponding face in $\text{Tr}(\mathcal{M})$ in a base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the flags in $\mathcal{F}(\text{Tr}(\mathcal{M}))$ and the adjacency of the flag orbits given by each symmetry type graph in Figure 15.

Furthermore, note that the maps with symmetry graph 6_{F_d} have three orbits on its vertices, while those with symmetry types 6_{H_p} , 6_{J_d} , $6_{M_{dp}}$ and $6_{P_{dp}}$, have two orbits on their vertices, see Figure 17. Then, given the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the flags of $\text{Tr}(\mathcal{M})$. Suppose that the flag $\Psi \in \mathcal{A}_0$

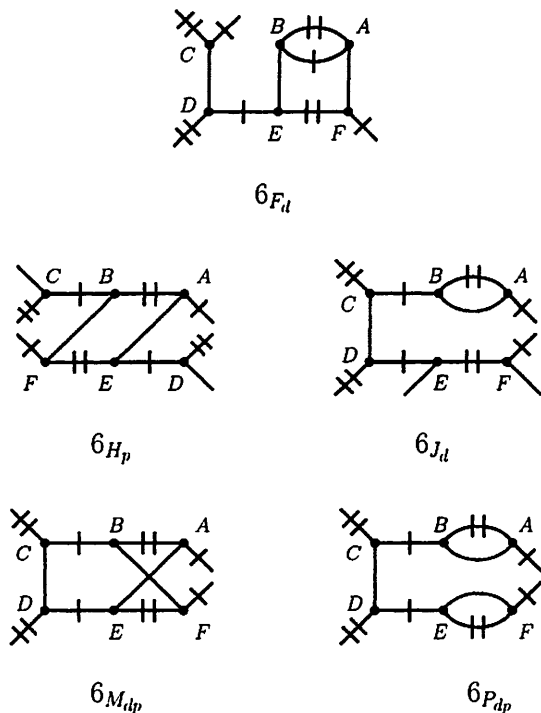


FIGURE 17. Possible symmetry type graphs of 6-orbit maps $\text{Tr}(\mathcal{M})$ which are not vertex-transitive, derived from 6-orbit maps \mathcal{M} .

is a flag in the flag-orbit F in $\text{Tr}(\mathcal{M})$. Assume that the face Ψ_2 is an element in $V(\mathcal{M})$ and $\text{Tr}(\mathcal{M})$ has symmetry type graph either

- (a) 6_{F_d} . Then, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are in the flag-orbits E and B of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, respectively. Inducing a flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ (as the very right one of the Figure 18 (a)), that represents a flag-orbit of \mathcal{M} . Furthermore, is not hard to see that the assembled flags $\Phi_{\Psi^0}, \Phi_{\Psi^{0,1}}, \Phi_{\Psi^{0,1,0}}, \Phi_{\Psi^{0,1,0,1}}, \Phi_{\Psi^{0,1,0,1,0}} \in \mathcal{F}(\mathcal{M})$, with $\Psi^0, \Psi^{0,1}, \Psi^{0,1,0}, \Psi^{0,1,0,1}, \Psi^{0,1,0,1,0} \in \mathcal{A}_0$, represent the other five flag-orbits of \mathcal{M} following the adjacency on the flags of $\text{Tr}(\mathcal{M})$, given by its symmetry type graph; or
- (b) $6_{H_p}, 6_{J_d}, 6_{M_{dp}}$ or $6_{P_{dp}}$. Then, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are in the flag-orbits E and D of $\mathcal{F}(\text{Tr}(\mathcal{M}))$, respectively. Inducing a flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ (as the very right one of the Figure 18 (b)), that represents a flag-orbit of \mathcal{M} . Respecting the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$,

and following the adjacency on the flags in $\mathcal{F}(\text{Tr}(\mathcal{M}))$ given by the respective symmetry type graph of $\text{Tr}(\mathcal{M})$, it is not hard to find that the remaining five different assembled flags in the Figure 18 (b) represent other flag-orbits of $\mathcal{F}(\mathcal{M})$, different of Φ_Ψ .

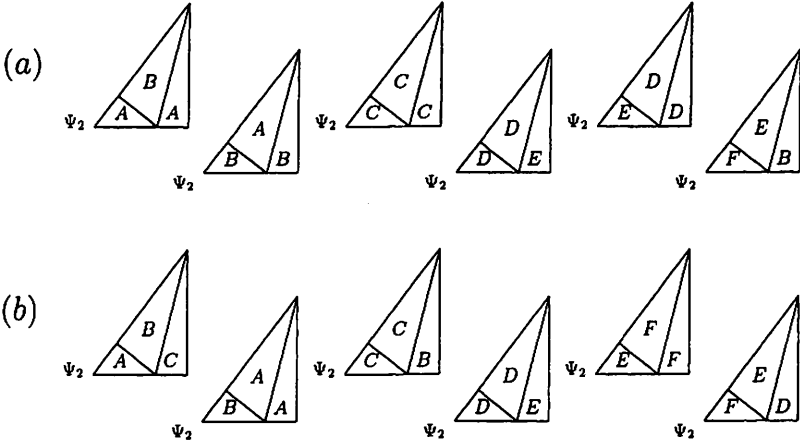


FIGURE 18. Assembled flags of a no vertex-transitive 6-orbit map $\text{Tr}(\mathcal{M})$ with symmetry type 6_{F_d} in (a), 6_{H_p} , 6_{J_d} , $6_{M_{dp}}$ or $6_{P_{dp}}$ in (b), as truncation of another 6-orbit map, where $\Psi_2 \in V(\mathcal{M})$ is the corresponding face in $\text{Tr}(\mathcal{M})$ in a base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

Concluding with Proposition 8.

Proposition 8. *If the truncation $\text{Tr}(\mathcal{M})$ of a 6-orbit map is a 6-orbit map, then one of the following holds.*

- (i) \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type 6_{B_p} ,
- (ii) \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type 6_{G_p} ,
- (iii) \mathcal{M} and $\text{Tr}(\mathcal{M})$ are of type 6_{H_p} ,
- (iv) \mathcal{M} is of type 6_{J_p} and $\text{Tr}(\mathcal{M})$ is of type 6_{J_d} ,
- (v) \mathcal{M} is of type $6_{M_{opp}}$ and $\text{Tr}(\mathcal{M})$ is of type $6_{N_{dp}}$,
- (vi) \mathcal{M} is of type $6_{N_{opp}}$ and $\text{Tr}(\mathcal{M})$ is of type 6_{F_d} or of type $6_{M_{dp}}$,
- (vii) \mathcal{M} is of type $6_{O_{opp}}$ and $\text{Tr}(\mathcal{M})$ is of type $6_{P_{dp}}$, or
- (viii) \mathcal{M} is of type $6_{P_{opp}}$ and $\text{Tr}(\mathcal{M})$ is of type $6_{O_{dp}}$.

The truncation of a 6-orbit map \mathcal{M} can be a 9-orbit or a 18-orbit map. We shall say that there are seventy possible symmetry type graphs with 18 vertices, obtained by applying once more the so called algorithm in Figure

4 to each symmetry type graph $T(\mathcal{M})$ with 6 vertices. (Forty-two of the symmetry type graphs with 6 vertices are depicted together in this paper and in [4], the other twenty-two can easily be founded by applying operations described in [4], as well.)

Finally, to determine which of the many symmetry types of 9-orbit maps correspond to be the symmetry type of the truncation $\text{Tr}(\mathcal{M})$ of a 6-orbit map \mathcal{M} , we recall the necessary 2-factors in Figures 1 and 6, that the symmetry type graph of the truncation of a map must contain. There are exactly ten possible symmetry type graphs with 9 vertices that satisfy the conditions to correspond to the symmetry type of a 9-orbit map $\text{Tr}(\mathcal{M})$, as the truncation on either a 3-orbit, a 6-orbit or a 9-orbit map \mathcal{M} , see Figure 19.

In fact, applying the algorithm in Figure 4 to the three symmetry type graphs of 3-orbit maps, it can be seen that the three graphs labelled by 9_A , 9_B y 9_{C_d} in Figure 19 are isomorphic to the truncated symmetry type graphs of the symmetry type graphs 3^0 , 3^2 and 3^{02} , respectively.

Moreover, if a map $\text{Tr}(\mathcal{M})$ of type 9_A , 9_B or 9_{C_d} has the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the set $\mathcal{F}(\text{Tr}(\mathcal{M}))$ such that a flag $\Psi \in \mathcal{A}_0$, is in the flag-orbit A and the face Ψ_2 is an element of $V(\mathcal{M})$. Then, the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are in the flag-orbits A and B . Consequently, respecting the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$, and following the adjacency on the flags in $\mathcal{F}(\text{Tr}(\mathcal{M}))$ given by the symmetry type graphs 9_A or 9_{C_d} , or 9_B , we obtain six the different flags that represent the flag-orbits of \mathcal{M} , as those shown in Figure 20 (a) or (b), respectively. Inducing the results in Proposition 9.

Proposition 9. *If the truncation $\text{Tr}(\mathcal{M})$ of a 6-orbit map is a 9-orbit map, then one of the following holds.*

- (i) \mathcal{M} is of type 6_D and $\text{Tr}(\mathcal{M})$ is of type 9_A ,
- (ii) \mathcal{M} is of type 6_F and $\text{Tr}(\mathcal{M})$ is of type 9_B , or
- (iii) \mathcal{M} is of type $6_{M_{opp}}$ and $\text{Tr}(\mathcal{M})$ is of type 9_{C_d} .

It can be seen that the remaining symmetry type graphs in Figure 19 of 9-orbit maps, different from 9_A , 9_B and 9_{C_d} , correspond to the truncation of other 9-orbit maps, as it is shown in the following section.

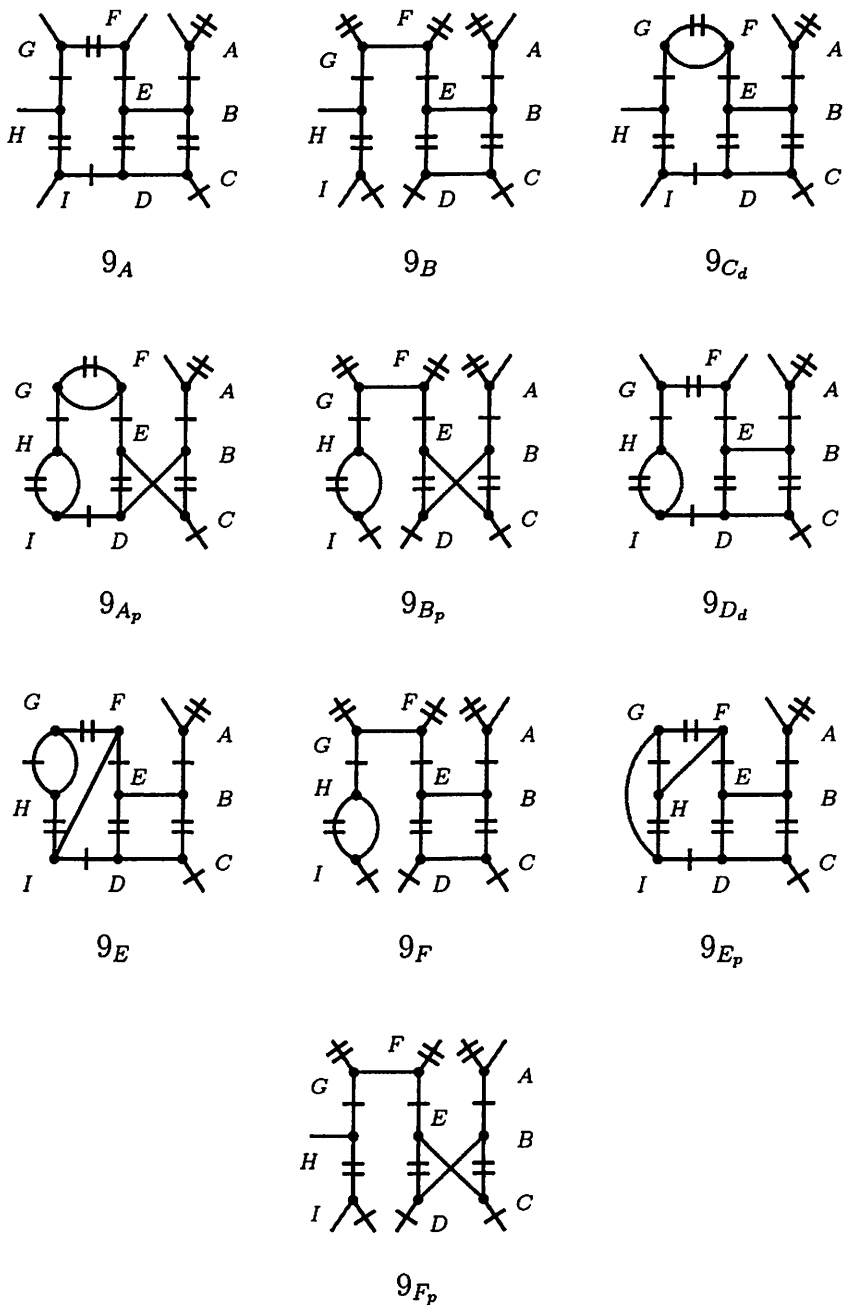


FIGURE 19. Symmetry type graphs with 9 vertices of truncated 3-orbit, 6-orbit or 9-orbit maps.

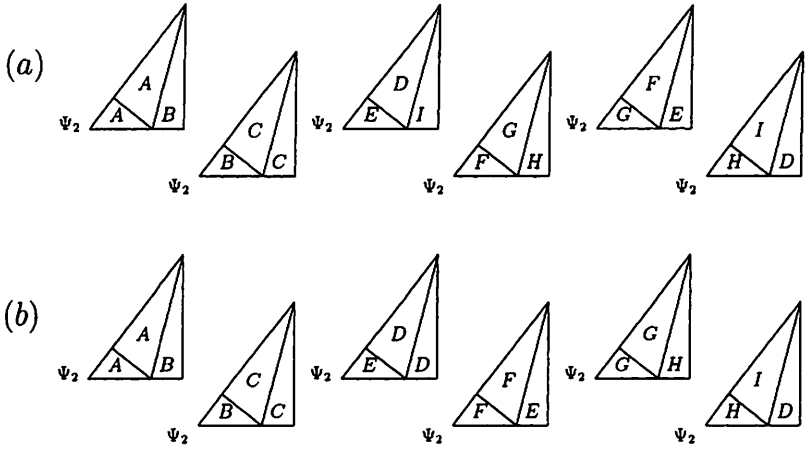


FIGURE 20. Assembled flags of maps $\text{Tr}(\mathcal{M})$ with (a) symmetry type graph 9_A or 9_{C_d} , and (b) with symmetry type graph 9_B , as truncation of a 6-orbit map, where $\Psi_2 \in V(\mathcal{M})$ is the corresponding face in $\text{Tr}(\mathcal{M})$ in a base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

4.1.5. *Truncation of 7-orbit and 9-orbit maps.* In this section we complete the study on truncation of k -orbit maps with $k \leq 7$ and $k = 9$. In what follows we will study the symmetry types of 7-orbit and 9-orbit truncation maps $\text{Tr}(\mathcal{M})$, when \mathcal{M} is a 7-orbit or a 9-orbit map, respectively.

Once again, one can find all truncated symmetry type graphs with 21 and with 27 vertices, associated to each symmetry type graphs with 7 and 9 vertices, respectively, by applying the algorithm in Figure 4.

By a proper combination of the (1,2) 2-factors in Figure 6, it can be seen that there are exactly two different symmetry type graphs with 7 vertices: 7_J and 7_{J_p} , that correspond to the symmetry type of the truncation of a 7-orbit map, depicted in the Figure 21. Let $\text{Tr}(\mathcal{M})$ be a map with symmetry type either 7_J or 7_{J_p} , and $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ the partition on the flags of $\text{Tr}(\mathcal{M})$, such that the the face Ψ_2 of the flag $\Psi \in \mathcal{A}_0$ is an element of $V(\mathcal{M})$. Then, if Ψ is a flag in the orbit A , the flags $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$ are in the orbits A and B , respectively. Thus, the assembled flag $\Phi_\Psi \in \mathcal{F}(\mathcal{M})$ represents the very left flag of the 7 different flags in Figure 22. Moreover, if we respect the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the set $\mathcal{F}(\text{Tr}(\mathcal{M}))$, the adjacency of the flags given by the symmetry type graph, and the previous assumption on the flag $\Psi \in \mathcal{A}_0$, we can easily find the remaining six flags in Figure 22 that will

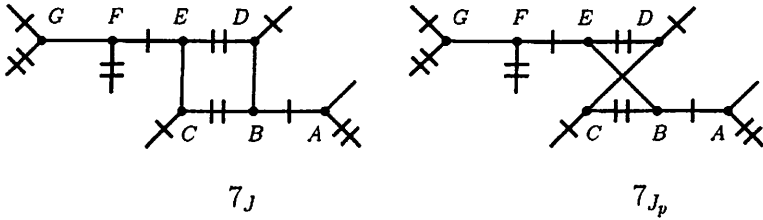


FIGURE 21. Symmetry type graphs with 7 vertices of truncated 7-orbit maps.

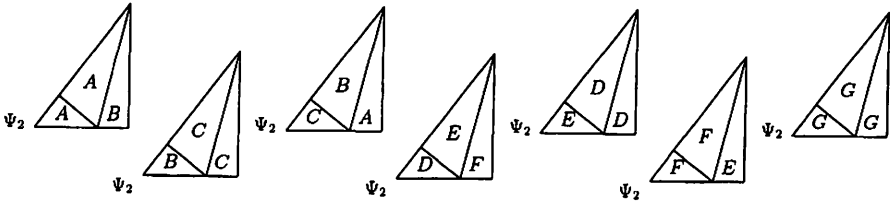


FIGURE 22. Assembled flags of 7-orbit maps $\text{Tr}(\mathcal{M})$ as truncation of another 7-orbit map.

correspond to representing flags in $\mathcal{F}(\mathcal{M})$, according to the symmetry type graph of $\text{Tr}(\mathcal{M})$. In this way it follows Proposition 10.

Proposition 10. *If the truncation $\text{Tr}(\mathcal{M})$ of a 7-orbit map is again a 7-orbit map, then one of the following holds.*

- (i) \mathcal{M} is of type 7_K and $\text{Tr}(\mathcal{M})$ is of type 7_J , or
- (ii) \mathcal{M} is of type 7_L and $\text{Tr}(\mathcal{M})$ is of type 7_{J_p} .

On the other hand, recall the symmetry type graphs of 9-orbit maps, different of 9_A , 9_B and 9_{C_d} , in Figure 19. Let $\text{Tr}(\mathcal{M})$ be a map with symmetry type 9_{A_p} , 9_{B_p} , 9_{D_d} , 9_E , 9_{E_p} , 9_F , or 9_{F_p} , and consider the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ of the set $\mathcal{F}(\text{Tr}(\mathcal{M}))$ such that a flag $\Psi \in \mathcal{A}_0$ implies that $\Psi^2 \in \mathcal{A}_2$ and $\Psi^{2,1} \in \mathcal{A}_1$. Suppose that the face Ψ_2 is an element of $V(\mathcal{M})$, and that the flags Ψ and Ψ^2 are flags in the flag-orbit A of $\text{Tr}(\mathcal{M})$, so as $\Psi^{2,1}$ is a flag in the flag-orbit B . In particular, respecting the partition $(\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_1)$ and following the adjacencies of the flag with respect to Ψ , Ψ^2 and $\Psi^{2,1}$ as before. If $\text{Tr}(\mathcal{M})$ has symmetry type graph

- (a) 9_{A_p} , 9_{D_d} , 9_E , or 9_{E_p} ; or
- (b) 9_{B_p} , 9_F , or 9_{F_p} .

Then we can obtain two sets of 9 different flags of the truncation map $\text{Tr}(\mathcal{M})$ as those in Figure 23. Inducing the following Proposition.

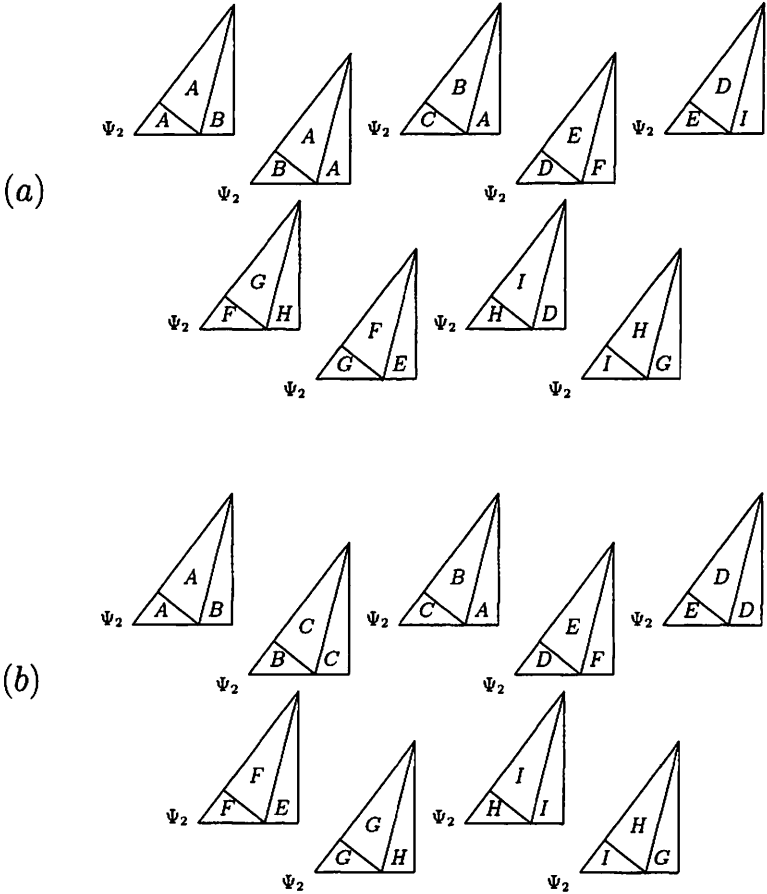


FIGURE 23. Assembled flags of maps $\text{Tr}(\mathcal{M})$ with (a) symmetry type graph 9_{A_p} , 9_{D_4} , 9_E , or 9_{E_p} , and (b) with symmetry type graph 9_{B_p} , 9_F , or 9_{F_p} , as truncation of a 9-orbit map, where $\Psi_2 \in V(\mathcal{M})$ is the corresponding face in $\text{Tr}(\mathcal{M})$ in a base flag $\Psi \in \mathcal{F}(\text{Tr}(\mathcal{M}))$.

Proposition 11. *If the truncation $\text{Tr}(\mathcal{M})$ of a 9-orbit map is again a 9-orbit map, then one of the following holds.*

- (i) \mathcal{M} is of type 9_C and $\text{Tr}(\mathcal{M})$ is of type 9_{B_p} ,

- (ii) \mathcal{M} is of type 9_{G_p} and $\text{Tr}(\mathcal{M})$ is of type 9_{E_p} ,
- (iii) \mathcal{M} is of type 9_{H_p} and $\text{Tr}(\mathcal{M})$ is of type 9_{A_p} .
- (iv) \mathcal{M} is of type 9_I and $\text{Tr}(\mathcal{M})$ is of type 9_{D_d} ,
- (v) \mathcal{M} is of type 9_J and $\text{Tr}(\mathcal{M})$ is of type 9_E ,
- (vi) \mathcal{M} is of type 9_K and $\text{Tr}(\mathcal{M})$ is of type 9_F , or
- (vii) \mathcal{M} is of type 9_L and $\text{Tr}(\mathcal{M})$ is of type 9_{F_p} .

In this way, we conclude with the results obtained so far on truncation of k -orbit maps, with $k = 1, \dots, 6, 7, 9$ are listed in the Tables 1 and 2.

Sym type of \mathcal{M}	Sym type of $\text{Tr}(\mathcal{M})$ with		
	k orbits	$\frac{3k}{2}$ orbits	$3k$ -orbits
1	1	—	3^0
2	2	—	6_{N_d}
2_0	—	—	6_G
2_2	2_{12}	—	6_{M_d}
2_1	—	—	6_{O_d}
2_{01}	2_0	3^0	6_B
2_{12}	—	—	6_{P_d}
2_{02}	—	—	6_H
3^0	—	—	9_A
3^2	—	—	9_B
3^{02}	3^{02}	—	9_{C_d}
4_B	—	6_{P_d}	12_B
4_C	—	6_{O_d}	12_C
4_{D_p}	4_{D_p}	—	12_D
4_E	4_D	—	12_E
4_G	—	$6_{N_d}, 6_{M_d}$	12_G
4_{G_d}	4_G	—	12_H
4_H	—	$6_G, 6_H$	12_C
5_{B_p}	5_{B_d}	—	15_A

TABLE 1. Truncation symmetry types of k -orbit maps, with $1 \leq k \leq 5$.

Sym type of \mathcal{M}	Sym type of $\text{Le}(\mathcal{M})$ with		
	k orbits	$\frac{3k}{2}$ orbits	$3k$ -orbits
6_{B_p}	6_{B_p}	—	18_B
6_D	—	9_A	18_D
6_F	—	9_B	18_F
6_{G_p}	6_{G_p}	—	18_G
6_{H_p}	6_{H_p}	—	18_H
6_{J_p}	6_{J_d}	—	18_J
$6_{M_{opp}}$	$6_{N_{dp}}$	9_{C_d}	18_M
$6_{N_{opp}}$	$6_{F_d}, 6_{M_{dp}}$	—	18_N
$6_{O_{opp}}$	$6_{P_{dp}}$	—	18_O
$6_{P_{opp}}$	$6_{O_{dp}}$	—	18_P
7_K	7_J	—	21_K
7_L	7_{J_p}	—	21_L
9_C	9_{B_p}	—	27_C
9_{G_p}	9_{E_p}	—	27_G
9_{H_p}	9_{A_p}	—	27_H
9_I	9_{D_d}	—	27_I
9_J	9_E	—	27_J
9_K	9_F	—	27_K
9_L	9_{F_p}	—	27_L

TABLE 2. Truncation symmetry types of k -orbit maps, with $6 \leq k \leq 7$ and $k = 9$.

4.2. Composition of a dual and truncation. For a given map \mathcal{M} , the vertices of its truncation map $\text{Tr}(\mathcal{M})$ have valency 3. Then, the dual map of $\text{Tr}(\mathcal{M})$ is a map with triangular faces. Hence, there is a correspondence between the sets of vertices and faces of \mathcal{M} with the vertex set of the dual map $(\text{Tr}(\mathcal{M}))^*$, also known as the *two-dimensional subdivision* of \mathcal{M} , [19]. Its symmetry type graph can easily be founded by exchanging the colours i and $2-i$, with $i \in \{0, 1, 2\}$, on the edges of the graphs $\mathcal{G}_{\text{Tr}(\mathcal{M})}$ and $T(\text{Tr}(\mathcal{M}))$ mentioned through the past section.

However, if we proceed in the other order and consider the truncation of the dual of a map \mathcal{M}^* , then we produce a map isomorphic to the *leapfrog map* $\text{Le}(\mathcal{M})$ of a map \mathcal{M} , [19, 9, 5]. In other words, $\text{Le}(\mathcal{M}) \cong \text{Tr}(\mathcal{M}^*)$. And this gives a completely different result than the map $(\text{Tr}(\mathcal{M}))^*$ as one can see below.

4.2.1. *Leapfrog map.* As we said before, the leapfrog map $\text{Le}(\mathcal{M})$ of \mathcal{M} is isomorphic to the truncation of the dual map \mathcal{M}^* of \mathcal{M} . One way to construct the leapfrog map $\text{Le}(\mathcal{M})$ of the map \mathcal{M} is by drawing, on the surface, a perpendicular edge to each edge of \mathcal{M} and joining by an edge the two end points of two edges if the corresponding edges in \mathcal{M} share a vertex and belong to the same face. In this way, we obtain a one-to-one correspondence between the faces of $\text{Le}(\mathcal{M})$ and the set of faces and vertices of \mathcal{M} . In Figure 24 is shown the image of the octahedron after apply the leapfrog operation.

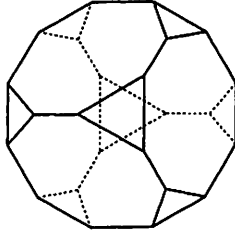


FIGURE 24. Leapfrog image of the octahedron, isomorphic to the truncated cube.

Notice that all the vertices of the map $\text{Le}(\mathcal{M})$ have valence three. The faces of $\text{Le}(\mathcal{M})$ that are in correspondence with the faces of \mathcal{M} remain of the same length, while the faces of $\text{Le}(\mathcal{M})$ that correspond to the vertices of \mathcal{M} are of length two times the valence of its corresponding vertex. It is not hard to see that the map $\text{Le}(\mathcal{M})$ contains $3|E(\mathcal{M})|$ edges.

As is depicted in the Figure 25, every flag in $\mathcal{F}(\mathcal{M})$ is divided into three different flags of the leapfrog map $\text{Le}(\mathcal{M})$. Let $\Phi = (\Phi_0, \Phi_1, \Phi_2) \in \mathcal{F}(\mathcal{M})$ be a flag in $\mathcal{F}(\mathcal{M})$, then $(\Phi, 0) := (\{\Phi_1, \Phi_2\}, \Phi_1, \Phi_0)$, $(\Phi, 1) := \{\{\Phi_1, \Phi_2\}, \{\Phi_0, \Phi_2\}, \Phi_0\}$ and $(\Phi, 2) := \{\{\Phi_1, \Phi_2\}, \{\Phi_0, \Phi_2\}, \Phi_2\}$ denote the three corresponding flags of Φ in $\mathcal{F}(\text{Le}(\mathcal{M}))$. The adjacency between these is given as follows.

$$\begin{aligned}
 (\Phi, 0) \cdot r'_0 &= (\Phi^{s_2}, 0), & (\Phi, 0) \cdot r'_1 &= (\Phi, 1), & (\Phi, 0) \cdot r'_2 &= (\Phi^{s_0}, 0); \\
 (\Phi, 1) \cdot r'_0 &= (\Phi^{s_1}, 1), & (\Phi, 1) \cdot r'_1 &= (\Phi, 0), & (\Phi, 1) \cdot r'_2 &= (\Phi, 2); \\
 (\Phi, 2) \cdot r'_0 &= (\Phi^{s_1}, 2), & (\Phi, 2) \cdot r'_1 &= (\Phi^{s_0}, 2), & (\Phi, 2) \cdot r'_2 &= (\Phi, 1).
 \end{aligned}$$

Note that once again r'_0 , r'_1 and r'_2 depend of the adjacency between the flags in $\mathcal{F}(\mathcal{M})$. Thus, we also present the algorithm shown in Figure 26 to construct, from $\mathcal{G}_{\mathcal{M}}$, the flag graph of $\text{Le}(\mathcal{M})$.

Recall that the flag graph $\mathcal{G}_{\mathcal{M}}$ of a k -orbit map \mathcal{M} which truncation map $\text{Tr}(\mathcal{M})$ is a k -orbit or a $\frac{3k}{2}$ -orbit map, can be quotient into a graph

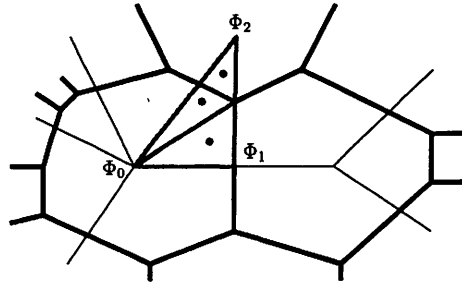


FIGURE 25. The three respective flags of $\mathcal{F}(\text{Le}(\mathcal{M}))$ to the flag $\Phi = (\Phi_0, \Phi_1, \Phi_2) \in \mathcal{F}(\mathcal{M})$.

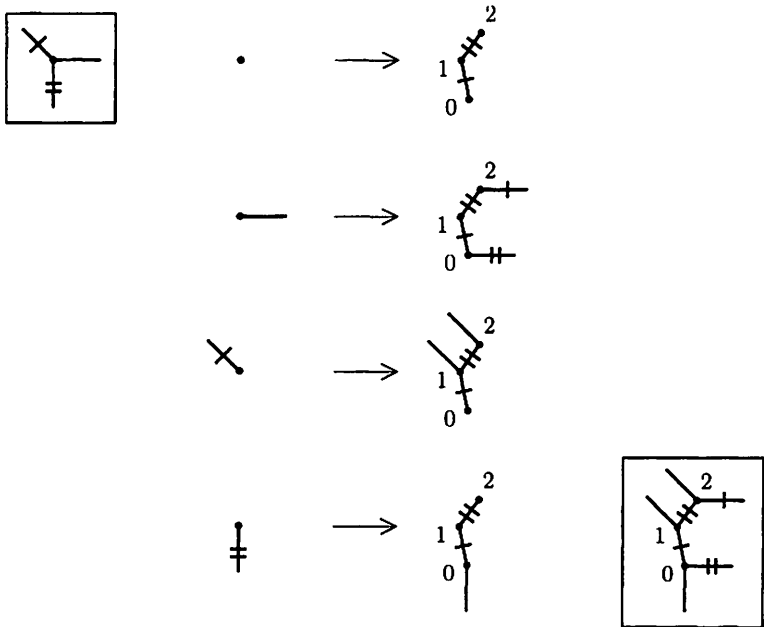


FIGURE 26. Local representation of a flag in \mathcal{G}_M , in the left. The image under the leapfrog operation, locally obtained, in the right.

isomorphic to 2_{01} , [17]. Meaning that there exist a bipartition (A, B) on the vertices of \mathcal{G}_M such that each vertex in the partition A is adjacent to a vertex in the partition B by an edge of colour 2. In addition, recall that there is a bijection $\delta : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M}^*)$ such that for each $\Phi \in \mathcal{F}(\mathcal{M})$ and each $i \in \{0, 1, 2\}$, $\Phi^i \delta = (\Phi \delta)^{2-i}$. Since $\text{Le}(\mathcal{M}) \cong \text{Tr}(\mathcal{M}^*)$, we shall say

that the flag graph $\mathcal{G}_{\mathcal{M}}$ of a k -orbit map \mathcal{M} such that its leapfrog map is a k -orbit or a $\frac{3k}{2}$ -orbit map, can be quotient in to a graph isomorphic to 2_{12} , i.e. there exist a bipartition (A', B') on the vertices of $\mathcal{G}_{\mathcal{M}}$ such that each vertex in the partition A' is adjacent to a vertex in the partition B' by and edge of colour 0.

However, the flag graph $\mathcal{G}_{\text{Le}(\mathcal{M})}$ of the leapfrog map $\text{Le}(\mathcal{M})$, of any map \mathcal{M} , can be quotient into a graph isomorphic to 3^0 , with the corresponding partition $(\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0)$ of the vertices of the flag graph $\mathcal{G}_{\text{Le}(\mathcal{M})}$, in such way that for each vertex $\Psi \in \mathcal{F}(\text{Le}(\mathcal{M}))$ in the partition \mathcal{A}_2 , the vertices Ψ^2 and $\Psi^{2,1}$ correspond to the partitions \mathcal{A}_1 and \mathcal{A}_2 , respectively. Thus, the assembling these flags we obtain a new flag $\Phi_{\Psi} := \{\Psi, \Psi^2, \Psi^{2,1}\} \in \mathcal{F}(\mathcal{M})$, where the face Ψ_2 will be considered as an element of $F(\mathcal{M})$.

Therefore, by the results on truncation, shown in the Tables 1 and 2, we can obtain the classification given in the Tables 3 and 4 where are listed the symmetry types that the map $\text{Le}(\mathcal{M})$ can have.

Sym type of \mathcal{M}	Sym type of $\text{Le}(\mathcal{M})$ with		
	k orbits	$\frac{3k}{2}$ orbits	$3k$ -orbits
1	1	—	3^0
2	2	—	6_{N_d}
2_0	2_{12}	—	6_{M_d}
2_2	—	—	6_G
2_1	—	—	6_{O_d}
2_{01}	—	—	6_{P_d}
2_{12}	2_0	3^0	6_B
2_{02}	—	—	6_H
3^0	—	—	9_B
3^2	—	—	9_A
3^{02}	3^{02}	—	9_{C_d}
4_{B_d}	—	6_{P_d}	12_B
4_{C_d}	—	6_{O_d}	12_C
4_{D_p}	4_{D_p}	—	12_D
4_{E_d}	4_D	—	12_E
4_G	4_G	—	12_H
4_{G_d}	—	$6_{N_d}, 6_{M_d}$	12_G
4_{H_d}	—	$6_G, 6_H$	12_C
5_{B_p}	5_{B_d}	—	15_A

TABLE 3. Leapfrog symmetry types k -orbit maps, with $1 \leq k \leq 5$.

Sym type of \mathcal{M}	Sym type of $\text{Le}(\mathcal{M})$ with		
	k orbits	$\frac{3k}{2}$ orbits	$3k$ -orbits
$6B_p$	$6B_p$	—	$18B$
$6D_d$	—	$9A$	$18D$
$6F_d$	—	$9B$	$18F$
$6G_p$	$6G_p$	—	$18G$
$6H_p$	$6H_p$	—	$18H$
$6J_p$	$6J_d$	—	$18J$
$6M_{dp}$	$6N_{dp}$	$9C_d$	$18M$
$6N_{dp}$	$6F_d, 6M_{dp}$	—	$18N$
$6O_{dp}$	$6P_{dp}$	—	$18O$
$6P_{dp}$	$6O_{dp}$	—	$18P$
$7K_d$	$7J$	—	$21K$
$7L_d$	$7J_p$	—	$21L$
$9C_d$	$9B_p$	—	$27C$
$9G_{pd}$	$9E_p$	—	$27G$
$9H_{pd}$	$9A_p$	—	$27H$
$9I_d$	$9D_d$	—	$27I$
$9J_d$	$9E$	—	$27J$
$9K_d$	$9F$	—	$27K$
$9L_d$	$9F_p$	—	$27L$

TABLE 4. Leapfrog symmetry types k -orbit maps, with $6 \leq k \leq 7$ and $k = 9$.

5. CONCLUSION AND ACKNOWLEDGEMENTS

As in [4], the symmetry type graph is used to find the possible symmetry types of the medial k -orbit map, with $k \leq 5$. Here, we complete a classification of possible symmetry types for the truncation and leapfrog of k -orbit maps for $k \leq 7$ and $k = 9$, extending the results in [17], using the symmetry type graph of a map as a tool. Hence, with the help of the symmetry type graph, we can continue classifying k -orbit maps with their image under other operations on maps.

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