The theta-complete graph Ramsey number $R(\theta_n, K_5) = 4n - 3$ for $n = 6$ and $n \geq 10$

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Abstract

For any two graphs F_1 and F_2, the graph Ramsey number $r(F_1, F_2)$ is the smallest positive integer N with the property that every graph of at least N vertices contains F_1 or its complement contains F_2 as a subgraph. In this paper, we consider the Ramsey numbers for theta-complete graphs. In fact, we prove that $r(\theta_n, K_5) = 4n - 3$ for $n = 6$ and $n \geq 10$.

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1 Introduction.

Graphs considered in this paper are finite, undirected and have neither loops nor multiple edges. The symbols C_m, P_m, T_m and K_m stand for a cycle, a path, a tree and a complete graph of order m, respectively. A theta graph of order n is obtained by adding a single edge between non-adjacent vertices in C_n. For a given graph G, we denote the vertex set of a graph G by $V(G)$, and the edge set by $E(G)$.

The graph Ramsey number $r(F_1, F_2)$ is the smallest positive integer N with the property that every graph of at least N vertices contains F_1 or its complement contains F_2 as a subgraph. It is known from the history of known results on Ramsey numbers that the difficulty in computing $r(F_1, F_2)$ increases with the density of edges in F_1 or F_2, but it is more significant when F_1 or F_2 are sparse. Chvátal [7] proved the following well known results of sparse graphs that has a relatively easy proof: $r(K_m, T_n) = (m-1)(n-1)+1$. The above result holds in particular for $T_n = P_n$. The problem becomes much harder when we try to find $r(C_m, K_n)$, in fact Erdös et al.[9] conjectured that $r(C_m, K_n) = (m-1)(n-1)+1$, for all $m \geq n \geq 3$ except $r(C_3, K_3) = 6$. The conjecture was studied by many authors and confirmed only in some cases see [1, 3, 6 ,10, 12, 14, 15, 18, 20-24]. In this work, we consider the problem of finding $r(\theta_m, K_n)$, where θ_n is a theta graph of order n, that is a cycle of length n and a new edge joining any two non adjacent vertices of the cycle.

The Ramsey number of the theta graphs verses complete deleting edge and the theta-complete graph Ramsey numbers were studied by a number of authors. Chvátal and Harary [8], proved that $r(\theta_4, K_4) = 11$. Bolze and Harborth [4] and Faudree et al. [11] showed that $r(\theta_4, K_5) = 16$ and $r(\theta_4, K_5 - e) = 13$, respectively. McNamara [16] determined that $r(\theta_4, K_6) = 21$ and McNamara Radziszowski [17] gave the following two results: $r(\theta_4, K_6 - e) = 17$ and $r(\theta_4, K_7 - e) = 28$. An upper bound for $r(\theta_4, K_7)$ and the exact number for $r(\theta_4, K_8)$ were established by Boza [5], in fact, he proved that $r(\theta_4, K_7) \leq 31$ and $r(\theta_4, K_8) = 42$. Bataineh, Jaradat and Bateeha [2], established that $r(\theta_n, K_m) = (n-1)(m-1)+1$ for $m = 3, 4$ and $n > m$. Bolze and Harborth [4] proved that $r(\theta_4, K_5) = 16$. Finally, Hendry [12] determined that $r(\theta_5, K_5) = 17$. For more results concerning Ramsey numbers of graphs, we direct the reader to the updated bibliography by Radziszowski [19].

In this paper, we continue studying the theta-complete Ramsey number by proving that $r(\theta_n, K_5) = 4n - 3$ for $n \geq 10$. Further we prove that $r(\theta_6, K_5) = 21$. Hence the remaining open values are for $n = 7, 8$ and 9.

For completeness we recall the following definitions: An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident with u. The maximum degree is denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. The neighborhood of the vertex u is the set of all vertices of G that are adjacent to u, denoted by $N(u)$, and we denote $N(u) \cup \{u\}$ by $N[u]$. Suppose that $V_1 \subseteq V(G)$ and V_1 is a non empty, the subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $\langle V_1 \rangle_G$. Finally, the graph nG

is formed by n vertex disjoint copies of the graph G.

2 Main Results.

In this section we determine the Ramsey number of theta graphs of order $n \geq 10$ and 6 versus the complete graph of order 5. By taking $G = (m-1)K_{n-1}$, one can notice that G contains neither θ_n nor an m-element independent set. Thus, we have established that $r(\theta_n, K_m) \geq (n-1)(m-1)+1$.

Theorem 2.1 : $r(\theta_n, K_5) = 4n - 3$ for $n \geq 10$.
Proof : By the inequality, $r(\theta_n, K_m) \geq (n-1)(m-1) + 1$, it suffices to prove that $r(\theta_n, K_5) \leq 4n - 3$ for $n \geq 10$. Let G be a graph of order $4n - 3$ for $n \geq 10$. Suppose that G contains no θ_n as a subgraph and $\alpha(G) \leq 4$. Since $r(C_n, K_5) = 4n - 3$ for $n \geq 10$ (see [3]), and G contains no 5-element independent set, then G must contain C_n as a subgraph. Note that no vertex of C_n can be adjacent to another vertex of C_n as otherwise θ_n is produced for $n \geq 10$ and so a 5-element independent set is produced, a contradiction. The proof of the Theorem is complete. ■

From the above results, we notice that the only cases left over to prove the general case $r(\theta_n, K_5) = 4n - 3$ for $n \geq 5$ are for $n = 6, 7, 8$ and 9. In the rest of this work we only consider the case $n = 6$. The following two graphs F_1 and F_2 in Figure 1 will be needed in our work:

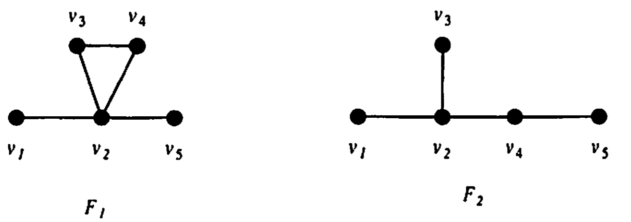

Figure 1: This figure represent two important graphs in our work

Lemma 2.2. Let G be a graph of order greater than or equal to 21 that contains neither θ_6 nor a 5-element independent set. Then $\delta(G) \geq 5$.
Proof: Suppose that G contains a vertex of degree lesser than 5, say u. Then $| v(G - N[u]) | \geq 16$. Since $r(\theta_6, K_4) = 16$ (see [2]), as a result $G - N[u]$ has an independent set consisting of 4 vertices. This set with

the vertex u is a 5-element independent set of vertices of G. This is a contradiction. The proof of the Lemma is complete. ■

Theorem 2.3. $r(\theta_6, K_5) = 21$.

Proof: By the inequality $r(\theta_n, K_m) \geq (n-1)(m-1)+1$, it suffices to prove that $r(\theta_6, K_5) \leq 21$. Let G be a graph of order 21. Suppose that G contains no θ_6 as a subgraph and $\alpha(G) \leq 4$. We consider five cases according to the maximum degree of G.

Case 1: $\Delta(G) \geq 9$. Let u be a vertex of degree equal to $\Delta(G)$. Let $\{v_1, v_2, ..., v_9\} \subseteq N(u)$ and $H = \langle v_1, v_2, ..., v_9 \rangle_G$. Since each vertex of H is adjacent to u, H has neither P_5 nor a 5-element independent set as otherwise θ_6 is produced, a contradiction. Thus, H must be one of the following possibilities: $H = 2K_4 \cup K_1$, $H = K_4 \cup C_4 \cup K_1$, $H = K_4 \cup (K_4 - e) \cup K_1$, $H = K_4 \cup (K_4 - P_3) \cup K_1$, $H = K_4 \cup P_4 \cup K_1$, $H = K_4 \cup K_3 \cup K_2$, $H = K_4 \cup P_3 \cup K_2$, $H = C_4 \cup K_3 \cup K_2$, $H = (K_4 - e) \cup K_3 \cup K_2$, $H = (K_4 - P_3) \cup K_3 \cup K_2, H = P_4 \cup K_3 \cup K_2, H = K_4 \cup K_3 \cup 2K_1$, $H = K_4 \cup 2K_2 \cup K_1$, $H = 3K_3$, $H = 2K_3 \cup P_3$, $H = 2K_3 \cup K_2 \cup K_1$, $H = K_3 \cup 3K_2$, $H = F_1 \cup K_4$ and $H = F_2 \cup K_4$. Let $S = \{v_{10}, v_{11}, v_{12} ..., v_{20}\}$ be the remaining vertices of G. Next we rule out all of these possibilities of H.

Possibility 1: $H = 2K_4 \cup K_1$. Let $\{v_1, v_2, v_3, v_4\}$, $\{v_5, v_6, v_7, v_8\}$ and $\{v_9\}$ be vertex sets of the two complete graphs of order 4 and the complete graph of order one, respectively. Note that no vertex of S can be adjacent to two vertices of $\{v_1, v_2 ..., v_9\}$ as otherwise θ_6 is produced. Since $|S| = 11$ and $r(\theta_6, K_3) = 11$ (see [2]), S has 3 independent vertices, say t, v, and w. Note that there is at least one vertex of $\{v_1, v_2, v_3, v_4\}$, say v_1, adjacent to no vertex of $\{t, v, w\}$. Indeed, if every vertex of $\{v_1, v_2, v_3, v_4\}$ is adjacent to at least one vertex of $\{t, v, w\}$, then there is at least one vertex of $\{t, v, w\}$, say t, adjacent to at least two vertices of $\{v_1, v_2, v_3, v_4\}$, say v_1, v_2, hence $v_1 t v_2 v_3 v_4 u v_1 v_2$ is a θ_6 subgraph, a contradiction. Similarly, there is at least one vertex of $\{v_5, v_6, v_7, v_8\}$, say v_5, adjacent to no vertex of $\{t, v, w\}$. Hence, $\{t, v, w, v_1, v_5\}$ is a 5-element independent set. This is a contradiction.

By applying the same argument as in Possibility 1 on each possibility of $H = K_4 \cup C_4 \cup K_1$, $H = K_4 \cup (K_4 - e) \cup K_1$, $H = K_4 \cup (K_4 - P_3) \cup K_1$ and $H = K_4 \cup P_4 \cup K_1$ we rule out all of these possibilities.

Possibility 2: $H = K_4 \cup K_3 \cup K_2$. Let $\{v_1, v_2, v_3, v_4\}$, $\{v_5, v_6, v_7\}$ and $\{v_8, v_9\}$ be the vertex sets of the complete graph of order 4, the complete graph of order 3 and the complete graph of order two, respectively. Since $\delta(G) \geq 5$, v_1 is adjacent to at least one vertex of S, say v. Similarly, since $\delta(G) \geq 5$ and since no vertex of S can be adjacent to two vertices one of from $\{v_1, v_2, v_3, v_4\}$ and the other from $\{v_5, v_6, v_7\}$ (To see that, assume that t is a vertex of S and adjacent to a vertex of $\{v_1, v_2, v_3, v_4\}$, say v_1, and a vertex of $\{v_5, v_6, v_7\}$, say v_5. Then $v_1tv_5uv_3v_4v_1v_3$ is a θ_6 subgraph, a contradiction), as a result v_5 is adjacent to at least two vertices of $S-\{v\}$, say t, w. Also, one can easily see that v is adjacent to no vertex of $\{v_2, v_3, \ldots, v_9, w, t\}$ (otherwise θ_6 is produced). Similarly, neither t nor w is adjacent to any vertex of $\{v_1, v_2, v_3, v_4, v_8, v_9, v\}$. To this end, if $tw \notin E(G)$, then $\{v, w, t, v_2, v_8\}$ is a 5-element independent set, a contradiction. If $tw \in E(G)$, then neither t nor w is adjacent to any vertex of $\{v_6, v_7\}$ (as otherwise, we have a θ_6, a contradiction) and so $\{v, t, v_2, v_6, v_8\}$ is a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 2 on each possibility of $H = K_4 \cup P_3 \cup K_2$, $H = C_4 \cup K_3 \cup K_2$, $H = (K_4 - e) \cup K_3 \cup K_2$, $H = (K_4 - P_3) \cup K_3 \cup K_2$ and $H = P_4 \cup K_3 \cup K_2$ we rule out all of these possibilities.

Possibility 3: $H = K_4 \cup K_3 \cup 2K_1$. Let $\{v_1, v_2, v_3, v_4\}$, $\{v_5, v_6, v_7\}$, $\{v_8\}$ and $\{v_9\}$ be the vertex sets of complete graphs of order 4, 3, 1, and 1, respectively. Since $\delta(G) \geq 5$, v_1 is adjacent to a vertex of S, say v. As above, v cannot be adjacent to any vertex of $\{v_2, v_3, \ldots, v_9\}$ as otherwise θ_6 is produced. Thus, $\{v, v_2, v_5, v_8, v_9\}$ is a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 3 of Case 1 on the possibility $H = K_4 \cup 2K_2 \cup K_1$ we also get a contradiction.

Possibility 4: $H = 3K_3$. Let $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ be the vertex sets of the three complete graphs of order 3, respectively. Note that no vertex of S can be adjacent to two vertices of two K_3's as otherwise θ_6 is produced. Since $\delta(G) \geq 5$, v_1 is adjacent to at least two vertices of S, say v_{10} and v_{11}, and v_4 is adjacent to at least two vertices of $S - \{v_{10}, v_{11}\}$, say v_{12} and v_{13}. Further, neither v_{10} nor v_{11} can be adjacent to either v_{12} or v_{13} as otherwise a θ_6 is produced. So, according to the existence of $v_{10}v_{11}$ and $v_{12}v_{13}$ in $E(G)$ we consider the following three subpossibilities:

4.1: $v_{10}v_{11} \notin E(G)$ and $v_{12}v_{13} \notin E(G)$. Then $\{v_{10}, v_{11}, v_{12}, v_{13}, v_7\}$ is a 5-element independent set, a contradiction.

4.2: Exactly one of $v_{10}v_{11}$ and $v_{12}v_{13}$ belongs to $E(G)$, say $v_{12}v_{13}$. Note that neither v_5 nor v_6 can be adjacent to any of v_{12} and v_{13} as otherwise θ_6 is produced. Hence, $\{v_5, v_7, v_{10}, v_{11}, v_{12}\}$ is a 5-element independent set, a contradiction.

4.3: $v_{10}v_{11} \in E(G)$ and $v_{12}v_{13} \in E(G)$. Note that neither v_2 nor v_3 can be adjacent to any of v_{10} and v_{11} as otherwise θ_6 is produced. Also, neither v_5 nor v_6 can be adjacent to any of v_{12} and v_{13}. Hence, $\{v_2, v_5, v_7, v_{10}, v_{12}\}$ is a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 4 of Case 1 on the possibility $H = 2K_3 \cup P_3$ we also get a contradiction.

Possibility 5: $H = 2K_3 \cup K_2 \cup K_1$. Let $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, $\{v_7, v_8\}$, and $\{v_9\}$ be the vertex sets of complete graphs of order 3, 3, 2, and 1, respectively. Since $\delta(G) \geq 5$, v_1 is adjacent to at least two vertices of S, say v and w. Note that v and w cannot be adjacent to any vertex of $\{v_2, v_3, \ldots, v_8, v_9\}$ as otherwise a θ_6 is produced. If $vw \notin E(G)$, then $\{v, w, v_4, v_7, v_9\}$ is a 5-element independent set, a contradiction; if $vw \in E(G)$, then neither v nor w can be adjacent to either v_2 or v_3 and so $\{v, v_2, v_4, v_7, v_9\}$ is a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 5 of the Case 1 on the possibility $H = K_3 \cup 3K_2$ we also get a contradiction.

Possibility 6: $H = F_1 \cup K_4$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of the graph F_1 as in Figure 1 and $\{v_6, v_7, v_8, v_9\}$ be the vertex set of K_4. Since $\delta(G) \geq 5$, v_6 is adjacent to at least one vertices in S, say v . Note that v can not be adjacent to any vertex of $\{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9\}$ as otherwise a θ_6 is produced. Hence, $\{v, v_7\}$ with three independent vertices of F_1 form a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 6 of the Case 1 on the possibility $H = F_2 \cup K_4$. we also get a contradiction.

Case 2: $\Delta(G) = 8$. Let u be a vertex of degree equal to 8. Let $N(u) = \{v_1, v_2, ..., v_8\}$ and $H = \langle N(u) \rangle_G$. Since H has neither a P_5 nor a 5-element independent set, H must be one of the following possibilities: $2K_4$, $K_4 \cup (K_4 - e)$, $K_4 \cup (K_4 - P_3)$, $K_4 \cup C_4$, $K_4 \cup P_4$, $2(K_4 - e)$, $(K_4 - e) \cup (K_4 - P_3)$, $(K_4 - e) \cup C_4$, $(K_4 - e) \cup P_4$, $2(K_4 - P_3)$, $(K_4 - P_3) \cup C_4$, $(K_4 - P_3) \cup P_4$, $2C_4$, $C_4 \cup P_4$, $2P_4$, $K_4 \cup K_{1,3}$, $K_4 \cup K_3 \cup K_1$, $(K_4 - e) \cup K_3 \cup K_1$, $(K_4 - P_3) \cup K_3 \cup K_1$, $C_4 \cup K_3 \cup K_1$, $P_4 \cup K_3 \cup K_1, K_4 \cup P_3 \cup K_1$, $K_4 \cup 2K_2$, $(K_4 - e) \cup 2K_2$, $(K_4 - P_3) \cup 2K_2, C_4 \cup 2K_2$, $P_4 \cup 2K_2$, $4K_2$, $2K_3 \cup K_2$, $P_3 \cup K_3 \cup K_2$, $K_3 \cup 2K_2 \cup K_1$, $2K_3 \cup 2K_1$, $F_1 \cup K_3$ and $F_2 \cup K_3$. For the first 15 possibilities the arguments are similar to the argument in possibility 1 of Case 1. In possibilities $H = K_4 \cup K_3 \cup K_1, H = (K_4 - e) \cup K_3 \cup K_1$, $H = (K_4 - P_3) \cup K_3 \cup K_1$, $H = C_4 \cup K_3 \cup K_1$ and $H = P_4 \cup K_3 \cup K_1$ the arguments are similar to the argument in possibility 2 of Case 1. For $H = 2K_3 \cup K_2$, we use similar argument to Possibility 4 of Case 1. In possibilities $H = K_3 \cup 2K_2 \cup K_1$ and $H = 2K_3 \cup 2K_1$ we use similar arguments to Possibility 5 of Case 1. Also, in possibilities $H = K_4 \cup K_{1,3}$ we use similar arguments to Possibility 6 of Case 1. Let $S = \{v_9, v_{10}, ..., v_{20}\}$ be the remaining vertices of G. Now we consider the remaining possibilities:

Possibility 1: $H = F_1 \cup K_3$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of the graph F_1 as in Figure 1 and $\{v_6, v_7, v_8\}$ be the vertex set of K_3. Since $\delta(G) \geq 5$, then v_6 is adjacent to at least two vertices of S, say v and w. Note that v and w cannot be adjacent to any vertex of $\{v_1, v_2, v_3, v_4, v_5, v_7, v_8\}$ as otherwise θ_6 is produced. If $vw \notin E(G)$, then v and w with the 3 independent vertices of F_1 produce a 5-element independent set, if $vw \in E(G)$, then neither v_7 nor v_8 can be adjacent to v or to w as otherwise a θ_6 is produced. So, $\{v\ , v_7\}$ with the 3 independent vertices of F_1 produce a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 5 of Case 1 to $H = F_2 \cup K_3$ we also get a contradiction.
Possibility 2: $H = P_3 \cup K_3 \cup K_2$. Let $v_1v_2v_3$ be a path of order 3 and $\{v_4, v_5, v_6\}$ and $\{v_7, v_8\}$ be the vertex sets of complete graphs of order 3 and 2, respectively. Since $\delta(G) \geq 5$, v_2 is adjacent to at least two vertices of S, say v_9 and v_{10}. If $v_9v_{10} \in E(G)$, then neither v_1 nor v_3 can be adjacent to either v_9 or v_{10} (if v_1 is adjacent to, say v_9, then $uv_1v_9v_{10}v_2v_3uv_2$ is a θ_6 subgraph). Moreover, each of v_9 and v_{10} can not be adjacent to any of $\{v_4, v_5, v_6, v_7, v_8\}$. Thus, $\{v_1, v_3, v_4, v_7, v_9\}$ is a 5-element independent set. If $v_9v_{10} \notin E(G)$, then v_9 and v_{10} are independent with at least one vertex of $\{v_1, v_3\}$, say v_1 (if $v_1v_{10} \in E(G)$ and $v_3v_9 \in E(G)$, then $uv_1v_{10}v_2v_9v_3uv_2$ is a θ_6 subgraph). Also, neither v_9 nor v_{10} can be adjacent to any of $\{v_4, v_5, v_6, v_7, v_8\}$ (if $v_8v_{10} \in E(G)$, then $uv_7v_8v_{10}v_2v_1uv_2$ is a θ_6 subgraph). Therefore, $\{v_1, v_4, v_7, v_9, v_{10}\}$ is a 5-element independent set, a contradiction.
Possibility 3: $H = K_4 \cup P_3 \cup K_1$. Let $\{v_1, v_2, v_3, v_4\}$ and $\{v_8\}$ be the vertex set of complete graphs of order 4 and 1, respectively, and let $v_5v_6v_7$ the path of order 3. Since $r(\theta_6, K_4) = 16$ and $| S \cup \{v_5, v_6, v_7, v_8\} | = 16$, then $S \cup \{v_5, v_6, v_7, v_8\}$ has a 4-element independent set, one vertex of which must be in $\{v_5, v_6, v_7, v_8\}$ as otherwise those 4 independent vertices with u will produce a 5-element independent set. As in Possibility 1 of Case 1, at least one vertex of $\{v_1, v_2, v_3, v_4\}$ is non-adjacent to any of the above 4 independent vertices, say v_1. Therefore, v_1 with the above mentioned 4 independent vertices form a 5-element independent set. This is a contradiction.
By applying the same argument as in Possibility 3 of Case 2 on the possibilities $H = K_4 \cup 2K_2$, $H = (K_4 - e) \cup 2K_2$, $H = (K_4 - P_3) \cup 2K_2$, $H = C_4 \cup 2K_2$ and $H = P_4 \cup 2K_2$ we get the same contradiction.
Possibility 4: $H = 4K_2$. Let $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$ and $\{v_7, v_8\}$ be the vertex sets of 4 complete graphs of order two. One can easily see that no vertex of S can be adjacent to two vertices of two different sets of the above four sets. Since $\delta(G) \geq 5$, v_1 must be adjacent to three vertices of S, say t, v, w. If $\langle\{t, v, w\}\rangle_G \neq K_3$, say $tv \notin E(G)$, then $\{t, v, v_3, v_4, v_5\}$ is a 5-element independent set. If $\langle\{t, v, w\}\rangle_G = K_3$, then non of t, v and w can be adjacent to v_2 (if one of t, v, w is adjacent to v_2, say t, then $twvv_1uv_2tv$ is a θ_6, a contradiction). Thus, $\{t, v_2, v_3, v_5, v_7\}$ is a 5-element independent set. This is a contradiction.
Case 3: Suppose that $\Delta(G) = 7$. Let u be a vertex of degree equal 7. Let $N(u) = \{v_1, v_2, ..., v_7\}$ and $H = \langle N(u)\rangle_G$. Since H has neither a P_5 nor a 5-element independent set, H must be one of the following possibilities: $K_4 \cup K_3$, $(K_4 - e) \cup K_3$, $(K_4 - P_3) \cup K_3, C_4 \cup K_3$, $P_4 \cup K_3$, $K_4 \cup P_3$, $(K_4 - e) \cup P_3$, $(K_4 - P_3) \cup P_3$, $C_4 \cup P_3$, $P_4 \cup P_3$, $K_{1,3} \cup K_3$, $K_4 \cup K_2 \cup K_1$, $(K_4 - e) \cup K_2 \cup K_1$, $(K_4 - P_3) \cup K_2 \cup K_1$, $C_4 \cup K_2 \cup K_1$, $P_4 \cup K_2 \cup K_1$,

$K_4 \cup 3K_1$, $2K_3 \cup K_1$, $K_3 \cup P_3 \cup K_1$, $K_3 \cup 2K_2$, $P_3 \cup 2K_2$, $K_3 \cup K_2 \cup 2K_1$, $3K_2 \cup K_1$, $F_1 \cup K_2$ and $F_2 \cup K_2$. Let $S = \{v_8, v_9, ..., v_{20}\}$ be the remaining vertices of G. Next, we role out all of these possibilities of H.

Possibility 1: $H = K_4 \cup K_3$. Let $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7\}$ be the vertex set of complete graphs of order 4 and 3, respectively. Since $r(\theta_6, K_4) = 16$ (see [2]) and $| S \cup \{v_5, v_6, v_7\} |= 16$, then $|S \cup \{v_5, v_6, v_7\}|$ has a 4-element independent set, say S^*, one vertex of which must be one of $\{v_5, v_6, v_7\}$, say v_5, as otherwise those 4 independent vertices with u will produce a 5-element independent set. As in Possibility 1 of Case 1, at least one vertex of $\{v_1, v_2, v_3, v_4\}$, say v_1 is non adjacent to any vertex of $S^* - \{v_5\}$ and so non adjacent to any vertex of S^*. Therefore, $S^* \cup \{v_1\}$ is a 5-element independent set. This is a contradiction.

For the next 16 possibilities we use a similar argument to Possibility 1 of Case 3. For $2K_3 \cup K_1$ we use the same argument as in Possibility 4 of Case 1. For $K_3 \cup P_3 \cup K_1$ and $P_3 \cup 2K_2$ we use the same argument as in Possibility 2 of Case 2. For $K_3 \cup K_2 \cup 2K_1$ we use the same argument as in Possibility 5 of Case 1. Also, for $K_{3,1} \cup K_3$ we use the same argument as in Possibility 1 of Case 2. Now, we consider the remaining possibilities of H.

Possibility 2: $H = F_1 \cup K_2$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of the graph F_1 as in Figure 1 and $\{v_6, v_7\}$ be the vertex set of K_2. Note that no vertex of S can be adjacent to a vertex of F_1 and to a vertex of K_2 as otherwise θ_6 is produced. Since $\delta(G) \geq 5$, v_6 is adjacent to at least three vertices of S, say v, t, and w. If there exists two independent vertices of $\{v, t, w\}$, without loss of generality say v and t, then $\{v\ , t\}$ with the 3 independent vertices of F_1 produce a 5-element independent set, if $\langle\{v, t, w\}\rangle_G = K_3$, then v_7 cannot be adjacent to a vertex of $\{v, t, w\}$ as otherwise θ_6 is produced. So, $\{v\ , v_7\}$ with the 3 independent vertices of F_1 produce a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 2 of Case 3 to $H = F_2 \cup K_2$ we also get a contradiction.

Possibility 3: $H = K_3 \cup 2K_2$. Let $\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_6, v_7\}$ be the vertex sets of complete graphs of order $3, 2, 2$, respectively. Note that no vertex of S can be adjacent to a vertex of K_3 and to a vertex of K_2 or to a vertex of K_2 and to a vertex of the other copy of K_2 as otherwise θ_6 is produced. Since $\delta(G) \geq 5$, then v_1 is adjacent to at least two vertices of S, say v_8 and v_9, and v_4 is adjacent to at least 3 vertices of S, say v_{10}, v_{11} and v_{12}. Note that neither v_8 nor v_9 can be adjacent to a vertex of $\{v_{10}, v_{11}, v_{12}\}$ as otherwise θ_6 is produced. According to the existence of v_8v_9 and the completeness of $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G$, we consider the following four subpossibilities:

2.1: $v_8v_9 \notin E(G)$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G \neq K_3$, say $v_{10}v_{11} \notin E(G)$. Then $\{v_6, v_8, v_9, v_{10}, v_{11}\}$ is a 5-element independent set, a contradiction.

2.2: $v_8v_9 \notin E(G)$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G = K_3$. Then no vertex of

$\{v_{10}, v_{11}, v_{12}\}$ can be adjacent to v_5. Thus, $\{v_8, v_9, v_{10}, v_5, v_6\}$ is a 5-element independent set, a contradiction.

2.3: $v_8v_9 \in E(G)$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G \neq K_3$, say $v_{10}v_{11} \notin E(G)$. Note that neither v_8 nor v_9 can be adjacent to any vertex of $\{v_2, v_3, v_{10}, v_{11}, v_{12}\}$ as otherwise θ_6 is produced, so $\{v_2, v_6, v_8, v_{10}, v_{11}\}$ is a 5-element independent set, a contradiction.

2.4: $v_8v_9 \in E(G)$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G = K_3$. Then as above, no vertex of $\{v_{10}, v_{11}, v_{12}\}$ can be adjacent to v_5 and neither v_8 nor v_9 can be adjacent to any vertex of $\{v_2, v_3, v_{10}, v_{11}, v_{12}\}$. Hence $\{v_2, v_5, v_6, v_8, v_{10}\}$ is a 5-element independent set, a contradiction.

Possibility 4: $H = 3K_2 \cup K_1$. Let $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$ and $\{v_7\}$ be the vertex sets of the three complete graphs of order 2 and a complete graph of order 1, respectively. Note that no vertex of S can be adjacent to two vertices of two sets of $\{v_1, v_2\}$, $\{v_3, v_4\}$ and $\{v_5, v_6\}$ as otherwise θ_6 is produced. Since $\delta(G) \geq 5$, then v_1 is adjacent to at least 3 vertices of S, say $\{v_8, v_9, v_{10}\}$, and v_3 is adjacent to at least 3 vertices of $S - \{v_8, v_9, v_{10}\}$, say $\{v_{11}, v_{12}, v_{13}\}$. Note that no vertex of $\{v_8, v_9, v_{10}\}$ can be adjacent to a vertex of $\{v_{11}, v_{12}, v_{13}\}$. Observe that there exist two independent vertices of $\{v_1, v_2, v_8, v_9, v_{10}\}$ and two independent vertices of $\{v_3, v_4, v_{11}, v_{12}, v_{13}\}$ as otherwise θ_6 is produced. So, this produces 4 independent vertices and those 4 vertices together with v_5 produce a 5-element independent set, a contradiction.

Case 4: $\Delta(G) = 6$. Let u be a vertex of degree 6. Let $N(u) = \{v_1, v_2, ..., v_6\}$ and $H = \langle N(u)\rangle$. Since H has neither P_5 nor a 5-element independent set, as a result H must be one of the following possibilities: $K_4 \cup K_2$, $(K_4 - e) \cup K_2$, $(K_4 - P_3) \cup K_2, C_4 \cup K_2$, $P_4 \cup K_2, K_{1,3} \cup K_2, K_4 \cup 2K_1$, $(K_4 - e) \cup 2K_1$, $(K_4 - P_3) \cup 2K_1$, $C_4 \cup 2K_1$, $P_4 \cup 2K_1$, $2K_3$, $K_3 \cup P_3$, $2P_3$, $K_3 \cup K_2 \cup K_1$, $P_3 \cup K_2 \cup K_1$, $K_3 \cup 3K_1$, $3K_2$, $2K_2 \cup 2K_1$, $F_1 \cup K_1$ and $F_2 \cup K_1$. For the first 11 possibilities, the argument is similar to the argument in Possibility 1 of Case 3. For $K_3 \cup 3K_1$ we use the same arguments as in possibility 5 of Case 1. Also, for $K_{1,3} \cup K_2$ we use the same arguments as in Possibility 2 of Case 3. Let $S = \{v_7, ..., v_{20}\}$ be the remaining vertices of G. Now, we consider the remaining possibilities of H.

Possibility 1: $H = P_3 \cup K_2 \cup K_1$. Let $v_1v_2v_3$ be a path of order 3 and $\{v_4, v_5\}$ and $\{v_6\}$ be the vertex sets of complete graphs of order 2 and 1, respectively. Since $\delta(G) \geq 5$, v_1 is adjacent to at least three vertices of S, say v_7, v_8 and v_9. Similarly v_3 is adjacent to three vertices of S, say v_{10}, v_{11}, and v_{12}. For simplicity we consider $S_1 = \{v_7, v_8, v_9\}$ and $S_2 = \{v_{10}, v_{11}, v_{12}\}$. Note that no vertex of S_1 can be adjacent to any vertex of S_2 as otherwise a θ_6 is produced. Also, no vertex of $S_1 \cup S_2$ can be adjacent to any vertex of $\{v_4, v_5, v_6\}$ as otherwise a θ_6 is produced. Now, according to $|S_1 \cap S_2|$, we consider the following four subpossibilities:

1.1. $|S_1 \cap S_2| = 0$. If $\alpha(\langle S_1\rangle) = 3$, then $S_1 \cup \{v_4, v_6\}$ is a 5-element independent set, a contradiction. If $\alpha(\langle S_1\rangle) < 3$, then there is at least one edge joining two vertices of S_1, say v_7v_8. One can easily note that v_2 can not be adjacent to any of v_7 and v_8, as otherwise a θ_6 is produced (for instance if $v_2v_8 \in E(G)$, then $v_2v_3u\ v_1v_7v_8v_2u$ is a θ_6, a contradiction). Now we consider the following two subsubpossibilities:

1.1.1. v_2 is not adjacent to at least one vertex of S_2, say v_{10}. Then $\{v_2, v_4, v_6, v_7, v_{10}\}$ is a 5-element independent set, a contradiction.

1.1.2. v_2 is adjacent to every vertex of S_2. Then $\alpha(\langle S_2\rangle) = 3$, otherwise a θ_6 is produced (for instance if $v_{10}v_{11} \in E(G)$, then $v_{10}v_2v_1u\ v_3v_{11}v_{10}v_3$ is a θ_6, a contradiction). And so, $S_2 \cup \{v_4, v_6\}$ is a 5-element independent set, a contradiction.

1.2. $|S_1 \cap S_2| = 1$. Consider $v_{12} = v_7$. Then $\alpha(\langle v_7, v_8, v_{10}\rangle) = 3$, as otherwise a θ_6 is produced (if $v_7v_8 \in E(G)$, then $v_7v_8v_1uv_2v_3v_7v_1$ is a θ_6 subgraph of G, a contradiction. If $v_7v_{10} \in E(G)$, then $v_7v_{10}v_3uv_2v_1v_7v_3$ is a θ_6 subgraph of G, a contradiction. Similarly, if $v_8v_{10} \in E(G)$, we get the same contradiction). Thus, $\{v_4, v_6, v_7, v_8, v_{10}\}$ is a 5-element independent set, a contradiction.

1.3. $|S_1 \cap S_2| = 2$. Consider $v_{12} = v_7$ and $v_{11} = v_8$. Then as in 1.2 $\alpha(\langle v_7, v_8, v_{10}\rangle) = 3$, as otherwise a θ_6 is produced. Thus, $\{v_4, v_6, v_7, v_8, v_{10}\}$ is a 5-element independent set, a contradiction.

1.4. $|S_1 \cap S_2| = 3$. Consider $v_{12} = v_7, v_{11} = v_8$ and $v_9 = v_{10}$. Then as in 1.2 $\alpha(\langle v_7, v_8, v_9\rangle) = 3$, as otherwise a θ_6 is produced. Thus, $\{v_4, v_6, v_7, v_8, v_{10}\}$ is a 5-element independent set, a contradiction.

Possibility 2: $H = F_1 \cup K_1$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of the graph F_1 as in Figure 1 and $\{v_6\}$ be the vertex set of K_1. As in Possibility 1 of Case 3, we note that no vertex of S can be adjacent to a vertex of F_1 and to the vertex of K_1 as otherwise θ_6 is produced. Also, no vertex of S can be adjacent to both v_3 and v_4 as otherwise if z is adjacent to both v_3 and v_4, then $v_3v_2v_1uv_4zv_3v_4$ is a θ_6 subgraph, a contradiction. Since $\delta(G) \geq 5$, v_3 is adjacent to at least one vertex of S, say v and v_4 is a adjacent to a vertex of S say w. Note that $vw \notin E(G)$ as otherwise $v_3v_2uv_4wvv_3v_4$ is a θ_6 subgraph, a contradiction. Also, non of v and w can not be adjacent to any vertex of $\{v_1, v_5\}$ as otherwise a θ_6 is produced. Thus, $\{v, w, v_1, v_5, v_6\}$ is a 5-element independent set, a contradiction.

By applying the same argument as in Possibility 2 of Case 4 to $H = F_2 \cup K_1$ we get also a contradiction.

Possibility 3: $H = 2K_3$. Let $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ be the vertex sets of the two complete graphs of order 3. Since $\delta(G) \geq 5$, v_1 is adjacent to at least two vertices of S, say v_7 and v_8, and v_4 is adjacent to at least two vertices of $S - \{v_7, v_8\}$, say v_9 and v_{10}. Note that, neither v_7 nor v_8 can be adjacent to any of v_9 and v_{10} as otherwise θ_6 is produced. According to the existence of v_7v_8 and v_9v_{10} in $E(G)$ we consider the following three subpossibilities:

3.1: $v_7v_8 \notin E(G)$ and $v_9v_{10} \notin E(G)$. Then $\{v_7, v_8, v_9, v_{10}, u\}$ is a 5-element independent set, a contradiction.

3.2: Only one of v_7v_8 and v_9v_{10} belongs to $E(G)$, say $v_7v_8 \in E(G)$. Since $\delta(G) \geq 5$, v_2 is adjacent to two vertices, say v_{11} and v_{12}. Note that, excluding the edge v_7v_8 and the possibility that $v_{11}v_{12} \in E(G)$, any two vertices of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ are not adjacent. Thus, $\{v_7, v_9, v_{10}, v_{11}, u\}$ is a 5-element independent set, a contradiction.

3.3: $v_7v_8 \in E(G)$ and $v_9v_{10} \in E(G)$. As above, Since $\delta(G) \geq 5$, each of v_2 and v_5 are adjacent to two vertices, say v_2 adjacent to v_{11} and v_{12} and v_5 adjacent to v_{13} and v_{14}. Note that, excluding the edges v_7v_8 and v_9v_{10} and the possibilities that $v_{11}v_{12}, v_{13}v_{14} \in E(G)$, any two vertices of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ are not adjacent. Thus, $\{v_7, v_9, v_{11}, v_{13}, u\}$ is a 5-element independent set. This is a contradiction.

By applying the same argument as in Possibility 3 of Case 4 to $K_3 \cup P_3$ and $2P_3$ we rule out these possibilities.

Possibility 4: $H = K_3 \cup K_2 \cup K_1$. Let $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$ and $\{v_6\}$ be the vertex set of complete graphs of order 3, 2, 1, respectively. Since $\delta(G) \geq 5$, v_1 is adjacent to at least two vertices of S, say v_7 and v_8, and v_2 is adjacent to at least two vertices of S, say v_9 and v_{10}. Let $t = |\{v_7, v_8\} \cap \{v_9, v_{10}\}|$. Observe that, $t = 0$ or 1 or 2. Now, according to t, we consider the following three subpossibilities:

4.1: $t = 0$. Then we have the following three subsubpossibilities:

4.1.1: $v_7v_8 \notin E(G)$ and $v_9v_{10} \notin E(G)$. Then $\{v_7, v_8, v_9, v_{10}, u\}$ is a 5-element independent set, a contradiction.

4.1.2: Exactly one of v_7v_8 and v_9v_{10} belongs to $E(G)$, say $v_7v_8 \in E(G)$. Then $\{v_4, v_6, v_7, v_9, v_{10}\}$ is a 5−element independent set, a contradiction.

4.1.3: $v_7v_8 \in E(G)$ and $v_9v_{10} \in E(G)$. Note that v_3 can not be adjacent to any of v_7, v_8, v_9 and v_{10} as otherwise θ_6 is produced. Thus $\{v_3, v_4, v_6, v_7, v_9\}$ is a 5-element independent set, a contradiction.

4.2: $t = 1$. Suppose that $\{v_7, v_8\} \cap \{v_9, v_{10}\} = \{v_7\}$. Then, $\{v_7, v_8, v_9\}$ is a 3-element independent set (If two vertices in this set are adjacent, then θ_6 is produced). So, $\{v_4, v_6, v_7, v_8, v_9\}$ is a 5-element independent set, a contradiction.

4.3: $t = 2$. That is v_1 and v_2 are adjacent to both v_7 and v_8. If $v_7v_8 \in E(G)$, then θ_6 is produced, a contradiction. If $v_7v_8 \notin E(G)$, then v_3 can not be adjacent to any of v_7 and v_8 (if $v_7v_3 \in E(G)$, then $uv_3v_7v_1v_8v_2uv_1$ is a θ_6 subgraph, a contradiction). Thus, $\{v_3, v_4, v_6, v_7, v_8\}$ is a 5-element independent set, a contradiction.

Possibility 5: $H = 3K_2$. Note that no vertex of S can be adjacent to a vertex of K_2 and to a vertex of another copy of K_2 as otherwise θ_6 is produced. Since $r(\theta_6, K_4) = 16$ and $|S\cup\{v_5, v_6\}| = 16$, then $S\cup\{v_5, v_6\}$ has 4 independent vertices. Let $Y = \{t_1, t_2, t_3, t_4\}$ be the independent vertices of $S\cup\{v_5, v_6\}$, if $v_5 \notin Y$ and $v_6 \notin Y$, then $Y\cup\{u\}$ is a 5-element independent set. So, we need to consider the case when $v_5 \in Y$ or $v_6 \in Y$, without loss of generality we may assume that $v_5 = t_1$. For simplicity, we assume that $v_7 = t_2$, $v_8 = t_3$ and $v_9 = t_4$. Consider the set $\{v_5, v_7, v_8, v_9, v\}$. Observe that for each $v \in \{v_1, v_2, v_3, v_4\}$, at least one of vv_7, vv_8 and vv_9 must belong to $E(G)$ as otherwise a 5-element independent set is produced. Further, each of v_7, v_8 and v_9 can not be adjacent to a vertex of $\{v_1, v_2\}$ and to a vertex of $\{v_3, v_4\}$. Thus the induced subgraph $\langle\{v_1, v_2, ..., v_9\}\rangle_G$ contains H_1 as a subgraph where H_1 is as shown in Figure 2 (up to isomorphism).

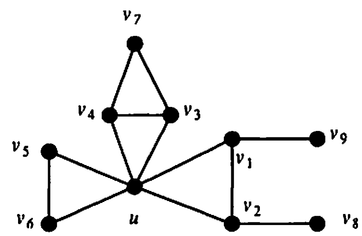

Figure 2: The figure depicts the situation in Possibility 4 of Case 5

Since $\delta(G) \geq 5$, then v_6 is adjacent to at least 3 vertices of $S - \{v_7, v_8, v_9\}$, say v_{10} is one of them. Note that v_{10} can not be adjacent to any of v_7, v_8 and v_9. Thus, $\{u, v_7, v_8, v_9, v_{10}\}$ is a 5-element independent set, a contradiction.

Possibility 6: $H = 2K_2 \cup 2K_1$. Let $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5\}$ and $\{v_6\}$ be the vertex sets of complete graphs of order $2, 2, 1, 1$, respectively. As above, since $\delta(G) \geq 5$, v_1 is adjacent to at least 3 vertices of S, say v_7, v_8 and v_9, and v_3 is adjacent to at least 3 vertices of $S - \{v_7, v_8, v_9\}$, say v_{10} ,v_{11} and v_{12}. Note that no vertex of $\{v_{10}, v_{11}, v_{12}\}$ can be adjacent to any vertex of v_7, v_8 and v_9 as otherwise θ_6 is produced. So, we consider the following three subpossibilities:

6.1: At least 4 vertices of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ are independent. Thus, those 4 independent vertices with u produce a 5-element independent set, a contradiction.

6.2: Exactly 3 vertices of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ are independent. Thus two of them must be in one of the two sets $\{v_7, v_8, v_9\}$ and $\{v_{10}, v_{11}, v_{12}\}$, say of $\{v_7, v_8, v_9\}$. Without loss of generality we may assume that $v_7v_8 \notin E(G)$ and $v_8v_9 \in E(G)$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G = K_3$. Then non of v_4, v_5 and v_6 can be adjacent to any vertex of $\{v_8, v_9, v_{10}, v_{11}, v_{12}\}$ as otherwise θ_6 is produced. Hence $\{v_4, v_5, v_6, v_8, v_{10}\}$ is a 5−element independent set, a contradiction.

6.3: Exactly 2 vertices of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ are independent. That is $\langle\{v_7, v_8, v_9\}\rangle_G = K_3$ and $\langle\{v_{10}, v_{11}, v_{12}\}\rangle_G = K_3$. Note that v_2 can not

be adjacent to any vertex of $\{v_7, v_8, v_9\}$, also v_4 can not be adjacent to any vertex of $\{v_{10}, v_{11}, v_{12}\}$ and v_5 can not be adjacent to a vertex of $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ as otherwise θ_6 is produced. Thus, $\{v_2, v_4, v_5, v_7, v_{10}\}$ is a 5-element independent set, a contradiction.
Case 5: $\Delta(G) = 5$. Then every vertex in G is of degree 5. Hence $\sum_{v \in G} d_G(v) = 105$, which contradicts the fact that the sum of degrees in a graph is even. The proof of the theorem is complete. ■

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