Distance spectral radius of quasi-trees

Hongying Lin, Bo Zhou*
Department of Mathematics, South China Normal University, Guangzhou 510631, P. R. China

Abstract

A quasi-tree is a graph for which the deletion of some vertex results in a tree. We determine the unique graph with minimum distance spectral radius among quasi-trees with fixed order and the unique graph with maximum distance spectral radius among cycle-containing quasi-trees with fixed order.

1 Introduction

We consider simple graphs. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between vertices $u, v \in V(G)$, denoted by d_{uv}, is the length of a shortest path between them in G. The distance matrix of G, denoted by $D(G)$, is the matrix $D(G) = (d_{uv})_{u,v \in V(G)}$. The eigenvalues of $D(G)$ are known as the distance eigenvalues of G. Since $D(G)$ is real and symmetric, the distance eigenvalues of G are real. The distance spectral radius of G, denoted by $\rho(G)$, is the largest distance eigenvalue of G. If $|V(G)| \geq 2$, then since $D(G)$ is irreducible, we have by the Perron-Frobenius theorem that $\rho(G)$ is simple, positive, and there is a unique positive unit eigenvector corresponding to $\rho(G)$, which is called the distance Perron vector of G.

The study of distance eigenvalues of a connected graph dates back to the classical work of Graham and Pollack [5], Graham and Lovász [4] and Edelberg et al. [3]. Ruzieh and Powers [8] showed that among the connected graphs of order n, the complete graph K_n is the unique graph with minimum distance spectral radius, while the path P_n is the unique graph with maximum distance spectral radius. In recent years, the distance spectral radius has received much attention, see, e.g., [1, 2, 6, 7, 9, 10, 11, 12, 13].

A graph G is called a quasi-tree if there is a vertex $w \in V(G)$ such that the deletion of w from G results in a tree. The trivial graph is a

*Corresponding author. E-mail: zhoubo@scnu.edu.cn

quasi-tree. The class of quasi-trees includes trees, unicyclic graphs, bicyclic graphs without vertex disjoint cycles, and infinite other types of graphs. Recall that the star S_n (the path P_n, respectively) uniquely achieves the minimum (maximum, respectively) distance spectral radius in the class of trees of order n [10, 8]. The unicyclic graphs with minimum (maximum, respectively) distance spectral radius have been determined in [13]. The bicyclic graphs with minimum (maximum, respectively) distance spectral radius have been studied in [2, 7, 12].

By the result of Ruzieh and Powers [8] mentioned above, K_n is the unique quasi-tree of order n with minimum distance spectral radius for $n = 1, 2, 3$, and P_n is the unique quasi-tree of order n with maximum distance spectral radius.

In this paper, we determine the unique graph with minimum distance spectral radius among the quasi-trees of order n for $n \geq 4$ and the unique graph with maximum distance spectral radius among cycle-containing quasi-trees of order n for $n \geq 3$.

2 Distance spectral radius of quasi-trees

Let G be a connected graph. For $v \in V(G)$, let $\delta_G(v)$ be the degree of v in G. A path $u_0u_1 \ldots u_r$ with $r \geq 1$ in G is a pendent path at u_0 if $\delta_G(u_0) \geq 3$, $\delta_G(u_i) = 2$ for $i = 1, 2, \ldots, r-1$, and $\delta_G(u_r) = 1$. For $v \in V(G)$, $G - v$ denotes the graph obtained by deleting v and its incident edges. Similarly, for $uv \in E(G)$, $G - uv$ denotes the graph obtained from G by deleting uv. For an edge uv of the complement of G, $G + uv$ denotes the graph that arises from G by adding uv.

Let G be a connected graph. By the Rayleigh's principle, for a $|V(G)|$-dimensional positive unit column vector X, $\rho(G) \geq X^\top D(G)X$ with equality if and only if X is the distance Perron vector of G. The following lemma is well known.

Lemma 1. [8] *Let G be a connected graph with $u, v \in V(G)$. If $uv \notin E(G)$, then $\rho(G) > \rho(G + uv)$.*

For a graph G, $P_1 \vee G$ denotes the graph obtained from G by adding a new vertex and edges between the added vertex and all vertices of G.

For $V_0 \subseteq V(G)$, let $\sigma(V_0)$ be the sum of the components of the distance Perron vector of G corresponding to the vertices in V_0.

Lemma 2. *Let T be a tree with $u \in V(T)$ such that $\delta_T(u) \geq 3$. Let $w \in N_T(u)$ and let $P_{u,v}$ be a pendent path at u in T not passing w, where v is the pendent vertex in $P_{u,v}$. Let $G = P_1 \vee T$ and $G' = G - uw + vw$. Then G' is a quasi-tree and $\rho(G) > \rho(G')$.*

Proof. Obviously, G is a quasi-tree, and thus G' is also a quasi-tree.

Let X be the distance Perron vector of G', where x_a denotes the component of X corresponding to the vertex a in G'. Note that the distance between any pair of distinct vertices of G is 1 or 2.

Case 1. $x_v \geq x_u$. When we pass from G to G', the distance between u and w is increased by 1, the distance between v and w is decreased by 1, and the distance between any other vertex pair remains unchanged. Thus

$$\frac{1}{2}[\rho(G) - \rho(G')] \geq \frac{1}{2}X^{\top}[D(G) - D(G')]X = (x_v - x_u)x_w \geq 0,$$

implying that $\rho(G) \geq \rho(G')$. If $\rho(G) = \rho(G')$, then $\rho(G) = X^{\top}D(G)X$, and thus X is also the distance Perron vector of G, which implies that $\rho(G)x_v = (D(G))_v X > (D(G'))_v X = \rho(G')x_v$, a contradiction. Thus $\rho(G) > \rho(G')$.

Case 2. $x_v < x_u$. Let T_1 be the subtree of T containing u obtained by deleting uw and the edge incident with u in $P_{u,v}$. Let $H = G' - \sum_{x \in N_{T_1}(u)} ux + \sum_{x \in N_{T_1}(u)} vx$. Obviously, $H \cong G$. When we pass from G' to H, the distance between u and any vertex in $N_{T_1}(u)$ is increased by 1, the distance between v and any vertex in $N_{T_1}(u)$ is decreased by 1, and the distance between any other vertex pair remains unchanged. Thus

$$\begin{aligned} \frac{1}{2}[\rho(G) - \rho(G')] &= \frac{1}{2}[\rho(H) - \rho(G')] \\ &\geq \frac{1}{2}X^{\top}[D(H) - D(G')]X \\ &= (x_u - x_v)\sigma(N_{T_1}(u)) \\ &> 0, \end{aligned}$$

implying that $\rho(G) > \rho(G')$. □

Let $QT(n)$ be the set of quasi-trees of order n, where $n \geq 3$.

Theorem 1. *Let $G \in QT(n)$, where $n \geq 3$. Then $\rho(G) \geq \rho(P_1 \vee P_{n-1})$ with equality if and only if $G \cong P_1 \vee P_{n-1}$.*

Proof. The cases $n = 3, 4$ are easily checked.

Suppose that $n \geq 5$ and that G is a graph in $QT(n)$ with minimum distance spectral radius. Note that $G - w$ is a tree for some $w \in V(G)$. If $wv \notin E(G)$ for some $v \in V(G)$, then $G + wv \in QT(n)$, and we have by Lemma 1 that $\rho(G + wv) < \rho(G)$, a contradiction. Thus $\delta_G(w) = n - 1$, implying that $G = P_1 \vee T$, where $T = G - w$. If T is not a path, then we may choose a vertex, say u with $\delta_T(u) \geq 3$, such that there is a pendent path at u in T, and thus by Lemma 2, there is a graph $G' \in QT(n)$ such that $\rho(G) > \rho(G')$, a contradiction. Thus T is a path. □

It is interesting to note that it is $P_1 \vee P_{n-1}$ not $P_1 \vee S_{n-1}$ that achieves minimum distance spectral radius among quasi-trees of order n for $n \geq 5$.

Let P_n' be the graph obtained by attaching P_{n-3} at one terminal vertex to a vertex of a triangle for $n > 3$ and P_3' be a triangle.

Lemma 3. [13] *Let G be a unicyclic graph on $n \geq 3$ vertices. Then $\rho(G) \leq \rho(P_n')$ with equality if and only if $G \cong P_n'$.*

Let $QTC(n)$ be the set of cycle-containing quasi-trees of order n, where $n \geq 3$.

Theorem 2. *Let $G \in QTC(n)$, where $n \geq 3$. Then $\rho(G) \leq \rho(P_n')$ with equality if and only if $G \cong P_n'$.*

Proof. Suppose G is a graph in $QTC(n)$ with maximum distance spectral radius. There is some $w \in V(G)$ such that $G - w$ is a tree. Obviously, $\delta_G(w) \geq 2$. Suppose that $\delta_G(w) \geq 3$. Since $G - w$ is connected, every edge incident with w in G lies on some cycle. Thus, for $u \in N_G(w)$, $G - wu \in QTC(n)$, and by Lemma 1, we have $\rho(G) < \rho(G - wu)$, a contradiction. It follows that $\delta_G(w) = 2$, and thus G is a unicyclic graph. By Lemma 3, $G \cong P_n'$. □

We mention again that if the condition of 'containing cycles' in Theorem 2 is dropped, then $\rho(G) \leq \rho(P_n)$ with equality if and only if $G \cong P_n$, which is known [8].

Acknowledgement. This work was supported by the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20124407110002).

References

[1] S.S. Bose, M. Nath, S. Paul, Distance spectral radius of graphs with r pendent vertices, Linear Algebra Appl. 435 (2011) 2828–2836.

[2] S.S. Bose, M. Nath, S. Paul, On the maximal distance spectral radius of graphs without a pendent vertex, Linear Algebra Appl. 438 (2013) 4260–4278.

[3] M. Edelberg, M.R. Garey, R.L. Graham, On the distance matrix of a tree, Discrete Math. 14 (1976) 23–39.

[4] R.L. Graham, L. Lovász, Distance matrix polynomials of trees, Adv. Math. 29 (1978) 60–88.

[5] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.

[6] M. Nath, S. Paul, On the distance spectral radius of bipartite graphs, Linear Algebra Appl. 436 (2012) 1285–1296.

[7] S. Paul, On the maximal distance spectral radius in a class of bicyclic graphs, Discrete Math. Algorithms Appl. 4 (2012) 1250061.

[8] S.N. Ruzieh, D.L. Powers, The distance spectrum of the path P_n and the first distance eigenvector of connected graphs, Linear and Multilinear Algebra 28 (1990) 75–81.

[9] D. Stevanović, A. Ilić, Spectral properties of distance matrix of graphs, in: I. Gutman, B. Furtula (Eds.), Distance in Molecular Graphs - Theory, Univ. Kragujevac, Kragujevac, 2011, pp. 139–176.

[10] D. Stevanović, A. Ilić, Distance spectral radius of trees with fixed maximum degree, Electron. J. Linear Algebra 20 (2010) 168–179.

[11] Y. Wang, B. Zhou, On distance spectral radius of graphs, Linear Algebra Appl. 438 (2013) 3490–3503.

[12] R. Xing, B. Zhou, On the distance and distance signless Laplacian spectral radii of bicyclic graphs, Linear Algebra Appl. 439 (2013) 3955–3963.

[13] G. Yu, Y. Wu, Y. Zhang, J. Shu, Some graft transformations and its application on a distance spectrum, Discrete Math. 311 (2011) 2117–2123.