On the vertex-distinguishing proper total colorings of complete p-partite graphs with equipotent parts

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Abstract: A proper k-total coloring of a simple graph G is called k-vertex-distinguishing proper total coloring (k – VDTC) if for any two distinct vertices u and v of G, the set of colors assigned to u and its incident edges differs from the set of colors assigned to v and its incident edges. The minimum number of colors required for a vertex-distinguishing proper total coloring of G, denoted by $\chi_{vt}(G)$, is called the vertex-distinguishing proper total chromatic number. For p is even, $p \geq 4$ and $q \geq 3$, we will obtain vertex-distinguishing proper total chromatic numbers of complete p-partite graphs with each part of cardinality q.

Keywords: complete p-partite graphs with equipotent parts, proper edge coloring, vertex distinguishing proper total coloring, vertex distinguishing proper total chromatic number

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1 Introduction

All graphs mentioned in this article are simple, undirected and finite, we use the standard notation of graph theory (see [1]). Definitions not given here may be found in [1].

A proper edge coloring of G is called vertex-distinguishing proper edge coloring if the set of colors assigned to edges incident to u differs from the set of colors assigned to edges incident to v for any two distinct vertices u and v of G. The minimum number of colors required for a vertex-distinguishing proper edge coloring of G, denoted by $\chi'_s(G)$, is called the vertex-distinguishing proper edge chromatic number of G (or observability of G). The theory about this concept was considered in [2-7].

A proper k-total coloring f of a graph G is an assignment of k colors ,say $1, 2, \cdots, k$, to the vertices and edges of G such that no two adjacent vertices receive the same color, no two adjacent edges receive the same color, and no edge receives the same color as one of its endpoints. Given such a coloring f, for any vertex $u \in V(G)$, let $C(u)$ be the set of colors assigned to vertex u and edges incident to vertex u, i.e., $C(u) = \{f(u)\} \cup \{f(uv)|uv \in E(G),\ v \in V(G)\}$. $C(u)$ is called the color set of vertex u under f. Let $\overline{C}(u) = \{1, 2, \cdots, k\} \setminus C(u)$. $\overline{C}(u)$ is called the complementary color set of vertex u. If $C(u) \neq C(v)$, then we say that u and v are distinguishing or u is distinguished from v. If for any two distinct vertices u and v of G, we have $C(u) \neq C(v)$, then we say that f is a k-vertex-distinguishing proper total coloring of graph G (in brief $k-$VDTC). The minimum number of colors required for a VDTC of G, denoted by $\chi_{vt}(G)$, is called the vertex-distinguishing proper total chromatic number. The theory about this concept was studied in [8-11].

For a graph G, let $n_i(G)$ denote the number of vertices of degree i in G, $\delta(G) \le i \le \Delta(G)$. Set

$$\mu(G) = \min\{\theta | \tbinom{\theta}{i+1} \ge n_i(G),\ \delta(G) \le i \le \Delta(G)\}.$$

In [8], the vertex-distinguishing proper total chromatic numbers of some special graphs, such as paths, cycles, complete graphs, complete bipartite graphs, wheels, fans, the joins of two paths with the same order, the joins of two cycles with the same order, the joins of a path of order n with a cycle of order n are obtained and the following Conjecture 1(VDTC Conjecture) is proposed.

Conjecture 1[8]. For any graph G, $\chi_{vt}(G) = \mu(G)$ or $\mu(G)+1$.

Let G be a simple graph and the vertex set $V(G)$ can be partitioned into p stable sets $V_1,\ V_2,\ \cdots,$
V_p, where $V_i = \{v_i^{(j)} : j = 1, 2, \cdots, q\}$, $|V_i| = q$, $i = 1, 2, \cdots, p$. If each vertex in V_i is adjacent to any one vertex in V_k, where $i \ne k$, $i, k = 1, 2, \cdots, p$, then G is called a complete p-partite graph with each part of cardinality q and denoted by $K(p \times q)$.

In [6], the vertex-distinguishing proper edge chromatic number of $K(p \times q)$ is obtained. Thus, we will discuss the vertex-distinguishing proper total chromatic number of $K(p \times q)$ in this paper.

Proposition 1[6]. If $p \ge 3$ and $q \ge 2$, then $\chi'_s(K(p \times q)) = (p-1)q + 2$.

2 Main results

Lemma 1.
$$\mu(K(p \times q)) = min\{\theta | \binom{\theta}{pq-q+1} \ge pq\} = \begin{cases} 2p, & p \ge 3, q = 2; \\ (p-1)q+3, & p, q \ge 3. \end{cases}$$

Proof. The graph $K(p\times q)$ consists of pq vertices and each vertex has degree $(p-1)q$.

According to the definition of $\mu(G)$, we have

$$\mu(K(p\times q)) = min\{\theta | \binom{\theta}{pq-q+1} \geq pq\}.$$

Suppose $\theta = pq-q+2$, we have

$$\binom{pq-q+2}{pq-q+1} = \binom{pq-q+2}{1} = pq-q+2 \geq pq, \text{ then } q \leq 2.$$

Thus, $\mu(K(p\times 2)) = 2p$.

Suppose $\theta = pq-q+3$, for $p,\ q \geq 3$, we have

$$\binom{pq-q+3}{pq-q+1} = \binom{pq-q+3}{2} = \frac{[(p-1)q+3][(p-1)q+2]}{2} \geq$$

$\frac{(2q+3)[(p-1)q+2]}{2} > \frac{(2q+2)[(p-1)q+2]}{2} = (q+1)[(p-1)q+2] = (q+1)(p-1)q+2(q+1) > (p-1)q+2(q+1) > (p-1)q+q = pq$.

Thus, for $p,\ q \geq 3$, $\mu(K(p\times q)) = (p-1)q+3$.

In [11], we have showed that

$$\chi_{vt}(K(p\times q)) = \begin{cases} 2p, & q=2,\ p\geq 3; \\ (p-1)q+3, & q=3,4,5,\ p\geq 3. \end{cases}$$

Thus, we can propose a conjecture as follows.

Conjecture 2. If p is odd, $p \geq 3$ and $q \geq 3$, then $\chi_{vt}(K(p\times q)) = (p-1)q+3$.

Theorem 1. If p is even, $p \geq 4$ and $q \geq 3$, then $\chi_{vt}(K(p\times q)) = (p-1)q+3$.

In Section 3 we will prove Theorem 1 by giving four algorithm steps of coloring as follows.

Step 1. Constructing a proper edge coloring τ of K_p.

Step 2. Constructing a proper edge coloring ψ of $K(p \times q)$.

Step 3. Constructing a vertex-distinguishing proper edge coloring φ of $K(p \times q)$ based on ψ.

Step 4. Constructing a vertex-distinguishing proper total coloring f of $K(p \times q)$ based on φ.

Obviously, the result of Theorem 1 satisfies VDTC Conjecture.

3 The proof of the Theorem 1

Step 1. Constructing a proper edge coloring τ of K_p.

Consider the complete graph K_p with vertex set $\{v_1, v_2, \cdots, v_p\}$. We will give a proper edge coloring τ of K_p using $p-1$ colors $1, 2, \cdots, p-1$ as follows.

Arrange clockwisely vertices $v_1, v_2, \cdots, v_{p-1}$ on the apices of a regular $(p-1)$-gon with center point v_p. We draw each edge of K_p by a line segment. First, we assign colors $1, 3, 5, \cdots, p-1, 2, 4, 6, \cdots, p-2$ to edges $v_1v_p, v_2v_p, \cdots, v_{p-1}v_p$ respectively. Then let all edges perpendicular to line v_iv_p receive the same color as the color of v_iv_p, $i = 1, 2, \cdots, p-1$. The resulting edge coloring is denoted by τ and τ is proper. For the case $p = 6$, $p = 8$ and $p = 10$, we can see Fig. 1 and Fig. 2.

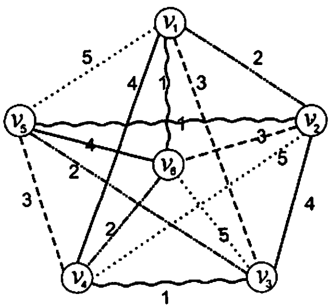

Fig. 1. The proper 5-edge coloring of K_6

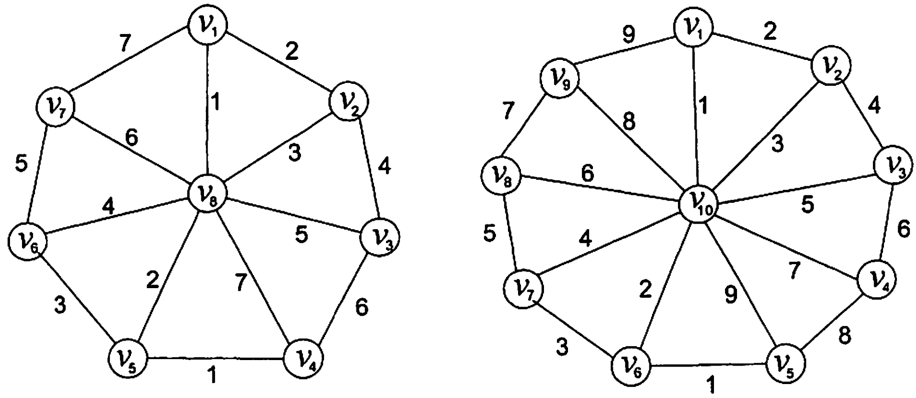

Fig. 2. The partial proper edge colorings of K_8 and K_{10}

Step 2. Constructing a proper edge coloring ψ of $K(p \times q)$.

A Latin square M of order n is an $n \times n$ array in which places are occupied by elements from an n-element set $\{1, 2, \cdots, n\}$, and each element from the set occurs exactly once in each row and column of M.

According to [12], there exists a Latin square M of order n.

Now according to the result in [12], we can construct a special Latin square $A = (a_{jl})_{qq}$ of order $q(\geq 3)$ such that the diagonal element a_{jj} of A is j, where a_{jl} is the element located in the j-th row and l-th column of A, $j, l = 1, 2, \cdots, q$.

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1q} \\ a_{21} & 2 & \cdots & a_{2q} \\ \vdots & \vdots & & \vdots \\ a_{q1} & a_{q2} & \cdots & q \end{pmatrix} .$$

For example, the following two matrixes are the special Latin squares .

$$\begin{pmatrix} 1 & 3 & 2 & 5 & 6 & 4 \\ 3 & 2 & 4 & 6 & 1 & 5 \\ 5 & 6 & 3 & 1 & 4 & 2 \\ 2 & 5 & 6 & 4 & 3 & 1 \\ 6 & 4 & 1 & 2 & 5 & 3 \\ 4 & 1 & 5 & 3 & 2 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 & 2 & 3 & 6 & 5 \\ 3 & 2 & 5 & 6 & 1 & 4 \\ 2 & 6 & 3 & 5 & 4 & 1 \\ 6 & 5 & 1 & 4 & 2 & 3 \\ 4 & 3 & 6 & 1 & 5 & 2 \\ 5 & 1 & 4 & 2 & 3 & 6 \end{pmatrix}$$

We have already given a proper edge coloring τ of K_p in Step 1 and constructed a special Latin square $A = (a_{jl})_{qq}$. According to these, we may obtain a proper edge coloring ψ of $K(p \times q)$ using $(p-1)q$ colors $(1,1),(1,2),\cdots,(1,q),(2,1),(2,2),\cdots,(2,q),\cdots,(p-1,1),(p-1,2),\cdots,(p-1,q)$. We call these $(p-1)q$ colors the complex colors, each complex color has two components.

For each edge v_iv_k of K_p, $i,k = 1,2,\cdots,p$, $i < k$, if the edge v_iv_k receives color s under τ, $s \in \{1,2,\cdots,p-1\}$, then we assign color (s, a_{jl}) to edge $v_i^{(j)}v_k^{(l)}$ of $K(p \times q)$, $j,l = 1,2,\cdots,q$.

Obviously, ψ is a proper edge coloring of $K(p \times q)$.

Step 3. Constructing a vertex-distinguishing proper edge coloring φ of $K(p \times q)$ based on ψ.

On the basis of the proper edge coloring ψ of $K(p \times q)$ given in Step 2, we will give a vertex-distinguishing proper edge coloring φ of $K(p \times q)$ using $(p-1)q+3$ colors by recoloring edges $v_i^{(j)}v_{i+1}^{(j)}$ using new three colors α, β, γ. We call α, β, γ the simple colors.

For $1 \le i \le p-1$, $1 \le j \le q$, then

$$\varphi(v_i^{(j)}v_{i+1}^{(j)})) = \begin{cases} \alpha, & i \equiv 1(mod3); \\ \beta, & i \equiv 2(mod3); \\ \gamma, & i \equiv 0(mod3). \end{cases}$$

Note that, (i) for $1 \le j \le q$, we have

$$\psi(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} (2i, j), & 1 \le i \le \frac{p-2}{2}; \\ (2i-p+1, j), & \frac{p}{2} \le i \le p-2; \\ (p-2, j), & i = p-1. \end{cases}$$

(ii) The $p-2$ sets $\{2,4\}, \{4,6\}, \cdots, \{p-4, p-2\}, \{p-2, 1\}, \{1,3\}, \{3,5\}, \cdots, \{p-5, p-3\}, \{p-3, p-2\}$ are distinct.

Thus, by comparing the complementary color set of each vertex, we can easily verify that the resulting edge coloring φ is a vertex-distinguishing proper edge coloring of $K(p \times q)$.

For the cases $p = 6$, $p = 8$ and $p = 10$, we can see Fig. 3, Fig. 4 and Fig. 5, respectively. The symbol "$a \to b$" indicates that the original color a under ψ is substituted by a color b under φ.

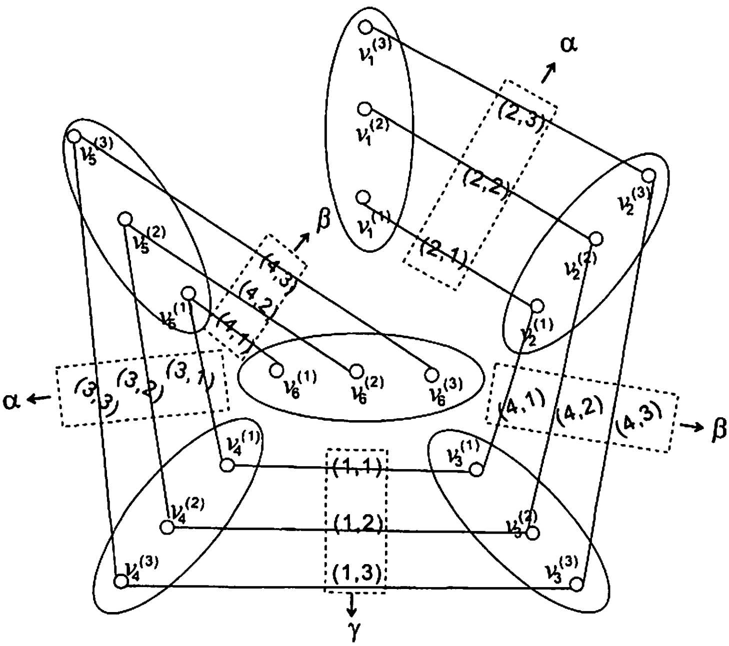

Fig. 3. The partial vertex-distinguishing proper edge coloring of $K(6 \times 3)$

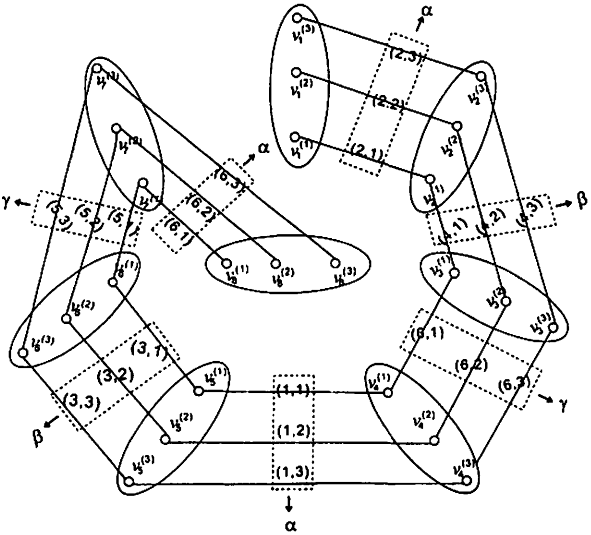

Fig. 4. The partial vertex-distinguishing proper edge coloring of $K(8 \times 3)$

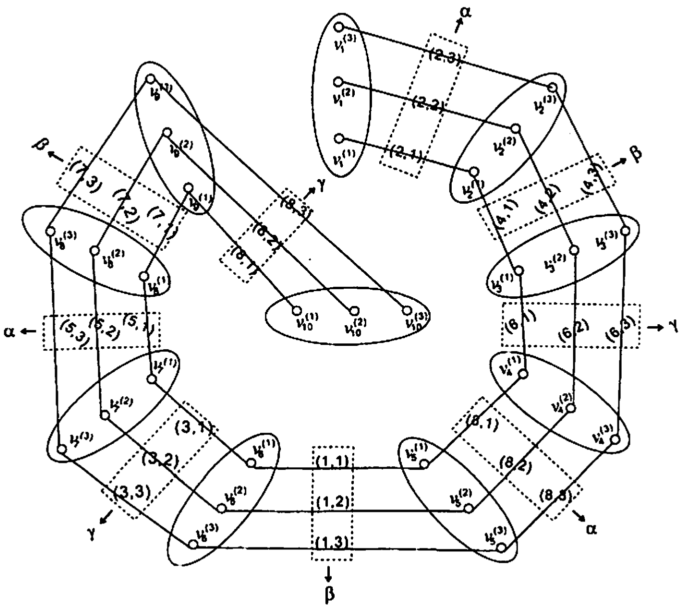

Fig. 5. The partial vertex-distinguishing proper edge coloring of $K(10 \times 3)$

Step 4. Constructing a vertex-distinguishing proper total coloring f of $K(p\times q)$ based on φ.

Based of the vertex-distinguishing proper edge coloring φ of $K(p\times q)$ given in Step 3, we may construct a $((p-1)q+3)$ – VDTC f of $K(p\times q)$ by coloring all vertices $v_i^{(j)}$, $i=1,2,\cdots,p$, $j=1,2,\cdots,q$.

We will distinguish 3 cases in order to color vertices.

Case 1: $p\equiv 0\ (mod\ 3)$.

For $j=1,2,\cdots,q$, we let

$$f(v_i^{(j)})=\begin{cases}\beta, & i=1;\\ (2i-2,j), & 2\le i\le \frac{p-2}{2};\\ \alpha, & i=\frac{p}{2};\\ (2i-p-1,j), & \frac{p+2}{2}\le i\le p-2;\\ (p-2,j), & i=p-1;\\ \gamma, & i=p.\end{cases}$$

Clearly, f is a proper total coloring.

Now we will give the complementary color set of each vertex $v_i^{(j)}$, $i=1,2,\cdots,p$, $j=1,2,\cdots,q$.

$\overline{C}(v_1^{(j)})=\{(2,j),\ \gamma\}$, $j=1,2,\cdots,q$.

$$\overline{C}(v_i^{(j)})=\begin{cases}\{(2i,j),\ \alpha\}, & i\equiv 0\ (mod\ 3);\\ \{(2i,j),\ \beta\}, & i\equiv 1\ (mod\ 3);\\ \{(2i,j),\ \gamma\}, & i\equiv 2\ (mod\ 3);\end{cases}\quad i=2,3,\cdots,\tfrac{p-2}{2},\ j=1,2,\cdots,q.$$

$\overline{C}(v_{\frac{p}{2}}^{(j)})=\{(1,j),\ (p-2,j)\}$, $j=1,2,\cdots,q$.

$$\overline{C}(v_i^{(j)})=\begin{cases}\{(2i-p+1,j),\ \alpha\}, & i\equiv 0\ (mod\ 3);\\ \{(2i-p+1,j),\ \beta\}, & i\equiv 1\ (mod\ 3);\\ \{(2i-p+1,j),\ \gamma\}, & i\equiv 2\ (mod\ 3);\end{cases}\quad i=\tfrac{p+2}{2},\tfrac{p+4}{2},$$

$\cdots, p-2$, $j = 1, 2, \cdots, q$.

$\overline{C}(v_{p-1}^{(j)}) = \{(p-3, j),\ \gamma\}$, $j = 1, 2, \cdots, q$.

$\overline{C}(v_p^{(j)}) = \{(p-2, j),\ \alpha\}$, $j = 1, 2, \cdots, q$.

Next, we will verify that the complementary color sets of any two vertices are different each other.

• Obviously, for each $i \in \{1, 2, \cdots, p\}$, $\overline{C}(v_i^{(1)})$, $\overline{C}(v_i^{(2)})$, $\cdots$, $\overline{C}(v_i^{(q)})$ are distinct, since all of them contain complex colors with the same first components and the different second components.

• As each $\overline{C}(v_{\frac{p}{2}}^{(j)})$ contains no simple color, $\overline{C}(v_k^{(l)})$ has a simple color, therefore $\overline{C}(v_{\frac{p}{2}}^{(j)})$ is distinguished from any vertex $v_k^{(l)}$, $k = 1, 2, \cdots, p$, $k \neq \frac{p}{2}$, $j.l = 1, 2, \cdots, q$.

• From $\frac{p-2}{2} \equiv 2\ (mod\ 3)$, we have that

$\overline{C}(v_p^{(j)}) = \{(p-2, j),\ \alpha\} \neq \{(2i, l),\ \alpha\} = \overline{C}(v_i^{(l)})$ with $i = 2, 3, \cdots, \frac{p-2}{2}$ and $i \equiv 0\ (mod\ 3)$, thus each $v_p^{(j)}$ is distinguished from any vertex $v_k^{(l)}$, where $k = 1, 2, 3, \cdots, \frac{p-2}{2}$, $j, l = 1, 2, \cdots, q$.

Each $v_p^{(j)}$ is also distinguished from any vertex $v_i^{(l)}$ with $i \in \{\frac{p+2}{2}, \frac{p+4}{2}, \cdots, p-1\}$ since in this time $\overline{C}(v_p^{(j)})$ has only one complex color with the first component even and $\overline{C}(v_i^{(l)})$ has only one complex color with the first component odd, $j, l = 1, 2, \cdots, q$.

• For $i = 1, 2, \cdots, \frac{p-2}{2}$, $k = \frac{p+2}{2}, \frac{p+4}{2}, \cdots, p-2$, since $2i$ is even, $2k - p + 1$ and $p - 3$ are odd, we have that

$$\overline{C}(v_i^{(j)}) \neq \overline{C}(v_{p-1}^{(l)}) \text{ and } \overline{C}(v_i^{(j)}) \neq \overline{C}(v_k^{(l)}),\ j, l = 1, 2, \cdots, q.$$

• For $1 \leq i < k \leq \frac{p-2}{2}$ or $\frac{p+2}{2} \leq i < k \leq p-2$, the first component of the unique complex color in $\overline{C}(v_i^{(j)})$ is less than the first component of the unique complex color in $\overline{C}(v_k^{(l)})$. Thus $v_i^{(j)}$ is distinguished from $v_k^{(l)}$ in this time, $j, l = 1, 2, \cdots, q$.

• From $p - 2 \equiv 1\ (mod\ 3)$, we have that $\overline{C}(v_{p-1}^{(j)}) = \{(p - 3, j),\ \gamma\} \neq \{(2i-p+1, j),\ \gamma\} = \overline{C}(v_i^{(l)})$ with $i = \frac{p+2}{2}, \frac{p+4}{2}, \cdots, p-2$

and $i \equiv 2\ (mod\ 3)$, thus each $v_{p-1}^{(j)}$ is distinguished from any vertex $v_i^{(l)}$, $j, l = 1, 2, \cdots, q$.

By comparing the complementary color sets of any two vertices, it is easy to see that f is a $((p-1)q+3)$ − VDTC of $K(p \times q)$.

Case 2: $p \equiv 1\ (mod\ 3)$.

For $j = 1, 2, \cdots, q$, we let

$$f(v_i^{(j)}) = \begin{cases} \gamma, & i = 1; \\ (2i-2, j), & 2 \le i \le \frac{p}{2}; \\ (2i-p-1, j), & \frac{p+2}{2} \le i \le p-2; \\ \alpha, & i = p-1; \\ \beta, & i = p. \end{cases}$$

Clearly, f is a proper total coloring.

Now we will give the complementary color set of each vertex $v_i^{(j)}$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$.

$$\overline{C}(v_i^{(j)}) = \begin{cases} \{(2i, j),\ \alpha\}, & i \equiv 0\ (mod\ 3); \\ \{(2i, j),\ \beta\}, & i \equiv 1\ (mod\ 3); \\ \{(2i, j),\ \gamma\}, & i \equiv 2\ (mod\ 3); \end{cases} \quad i = 1, 2, \cdots, \frac{p-2}{2}, j =$$

$1, 2, \cdots, q$.

$$\overline{C}(v_i^{(j)}) = \begin{cases} \{(2i-p+1, j),\ \alpha\}, & i \equiv 0\ (mod\ 3); \\ \{(2i-p+1, j),\ \beta\}, & i \equiv 1\ (mod\ 3); \\ \{(2i-p+1, j),\ \gamma\}, & i \equiv 2\ (mod\ 3); \end{cases} \quad i = \frac{p}{2}, \frac{p+2}{2}, \cdots,$$

$p-2$, $j = 1, 2, \cdots, q$.

$\overline{C}(v_{p-1}^{(j)}) = \{(p-3, j),\ (p-2, j)\}$, $j = 1, 2, \cdots, q$.

$\overline{C}(v_p^{(j)}) = \{(p-2, j),\ \alpha\}$, $j = 1, 2, \cdots, q$.

Next, we will verify that the complementary color sets of any two vertices are different each other.

• Obviously, for each $i \in \{1, 2, \cdots, p\}$, $\overline{C}(v_i^{(1)})$, $\overline{C}(v_i^{(2)})$, $\cdots$, $\overline{C}(v_i^{(q)})$ are distinct, since all of them contain complex colors with the same first components and the different second components.

• As each $\overline{C}(v_{p-1}^{(j)})$ contains no simple color, $\overline{C}(v_k^{(l)})$ has a simple color, therefore $\overline{C}(v_{p-1}^{(j)})$ is distinguished from any vertex $v_k^{(l)}$, $k = 1, 2, \cdots, p$, $k \neq p-1$, $j.l = 1, 2, \cdots, q$.

• From $\frac{p-2}{2} \equiv 1$ $(mod\ 3)$, we have that

$\overline{C}(v_p^{(j)}) = \{(p-2, j),\ \alpha\} \neq \{(2i, l),\ \alpha\} = \overline{C}(v_i^{(l)})$ with $i = 1, 2, \cdots, \frac{p-2}{2}$ and $i \equiv 0$ $(mod\ 3)$, thus each $v_p^{(j)}$ is distinguished from any vertex $v_i^{(l)}$, $j, l = 1, 2, \cdots, q$.

Each $v_p^{(j)}$ is also distinguished from any vertex $v_i^{(l)}$ with $i \in \{\frac{p}{2}, \frac{p+2}{2}, \cdots, p-2\}$ since in this time $\overline{C}(v_p^{(j)})$ has only one complex color with the first component even and $\overline{C}(v_i^{(l)})$ has only one complex color with the first component odd, $j, l = 1, 2, \cdots, q$.

• For $i = 1, 2, \cdots, \frac{p-2}{2}$, $k = \frac{p}{2}, \frac{p+2}{2}, \cdots, p-2$, since $2i$ is even and $2k - p + 1$ is odd, we have that

$$\overline{C}(v_i^{(j)}) \neq \overline{C}(v_k^{(l)}),\quad j, l = 1, 2, \cdots, q.$$

• For $1 \leq i < k \leq \frac{p-2}{2}$ or $\frac{p}{2} \leq i < k \leq p-2$, the first component of the unique complex color in $\overline{C}(v_i^{(j)})$ is less than the first component of the unique complex color in $\overline{C}(v_k^{(l)})$. Thus $v_i^{(j)}$ is distinguished from $v_k^{(l)}$ in this time, $j, l = 1, 2, \cdots, q$.

By comparing the complementary color sets of any two vertices, we can easily have that f is a $((p-1)q+3)$ – VDTC of $K(p \times q)$.

Case 3: $p \equiv 2$ $(mod\ 3)$.

For $j=1,2,\cdots,q$, we let

$$f(v_i^{(j)})=\begin{cases}\gamma, & i=1;\\ (2i-2,j), & 2\le i\le \frac{p}{2};\\ (2i-p-1,j), & \frac{p+2}{2}\le i\le p-1;\\ \beta, & i=p.\end{cases}$$

Clearly, f is a proper total coloring.

Now we will give the complementary color set of each vertex $v_i^{(j)}$, $i=1,2,\cdots,p$, $j=1,2,\cdots,q$.

$$\overline{C}(v_i^{(j)})=\begin{cases}\{(2i,j),\ \alpha\}, & i\equiv 0\ (mod\ 3);\\ \{(2i,j),\ \beta\}, & i\equiv 1\ (mod\ 3);\\ \{(2i,j),\ \gamma\}, & i\equiv 2\ (mod\ 3);\end{cases}\quad i=1,2,\cdots,\tfrac{p-2}{2},\ j=$$

$1,2,\cdots,q$.

$$\overline{C}(v_i^{(j)})=\begin{cases}\{(2i-p+1,j),\ \alpha\}, & i\equiv 0\ (mod\ 3);\\ \{(2i-p+1,j),\ \beta\}, & i\equiv 1\ (mod\ 3);\\ \{(2i-p+1,j),\ \gamma\}, & i\equiv 2\ (mod\ 3);\end{cases}\quad i=\tfrac{p}{2},\tfrac{p+2}{2},\cdots,$$

$p-2$, $j=1,2,\cdots,q$.

$\overline{C}(v_{p-1}^{(j)})=\{(p-2,j),\ \beta\}$, $j=1,2,\cdots,q$.

$\overline{C}(v_p^{(j)})=\{(p-2,j),\ \gamma\}$, $j=1,2,\cdots,q$.

Next, we will verify that the complementary color sets of any two vertices are different each other.

• Obviously, for each $i\in\{1,2,\cdots,p\}$, $\overline{C}(v_i^{(1)})$, $\overline{C}(v_i^{(2)})$, $\cdots$, $\overline{C}(v_i^{(q)})$ are distinct, since all of them contain complex colors with the same first components and the different second components.

• From $\frac{p-2}{2}\equiv 0\ (mod\ 3)$, we have that

$\overline{C}(v_{p-1}^{(j)})=\{(p-2,j),\ \beta\}\neq\{(2i,l),\ \beta\}=\overline{C}(v_i^{(l)})$ with $i=1,2,\cdots,\frac{p-2}{2}$ and $i\equiv 1\ (mod\ 3)$, thus each $v_{p-1}^{(j)}$ is distinguished from any vertex $v_i^{(l)}$, $j,l=1,2,\cdots,q$.

$\overline{C}(v_p^{(j)}) = \{(p-2,j),\ \gamma\} \neq \{(2i,l),\ \gamma\} = \overline{C}(v_i^{(l)})$ with $i = 1,2,\cdots,\frac{p-2}{2}$ and $i \equiv 2\ (mod\ 3)$, thus each $v_p^{(j)}$ is distinguished from any vertex $v_i^{(l)}$, $j,l = 1,2,\cdots,q$.

Each $v_{p-1}^{(j)}$ and $v_p^{(j)}$ are also distinguished from any vertex $v_i^{(l)}$ with $i \in \{\frac{p}{2}, \frac{p+2}{2}, \cdots, p-2\}$ since in this time $\overline{C}(v_{p-1}^{(j)})$ and $\overline{C}(v_p^{(j)})$ have only one complex color with the first component even and $\overline{C}(v_i^{(l)})$ has only one complex color with the first component odd, $j,l = 1,2,\cdots,q$.

• Obviously, $\overline{C}(v_{p-1}^{(j)}) \neq \overline{C}(v_p^{(l)})$, $j,l = 1,2,\cdots,q$.

• For $i = 1,2,\cdots,\frac{p-2}{2}$, $k = \frac{p}{2}, \frac{p+2}{2}, \cdots, p-2$, since $2i$ is even and $2k-p+1$ is odd, we have that

$$\overline{C}(v_i^{(j)}) \neq \overline{C}(v_k^{(l)}),\quad j,l = 1,2,\cdots,q.$$

• For $1 \leq i < k \leq \frac{p-2}{2}$ or $\frac{p}{2} \leq i < k \leq p-2$, the first component of the unique complex color in $\overline{C}(v_i^{(j)})$ is less than the first component of the unique complex color in $\overline{C}(v_k^{(l)})$. Thus $v_i^{(j)}$ is distinguished from $v_k^{(l)}$ in this time, $j,l = 1,2,\cdots,q$.

By comparing the complementary color sets of any two vertices, we can easily have that f is a $((p-1)q+3)-$ VDTC of $K(p \times q)$.

The proof of Theorem 1 is finished.

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