TWO KINDS OF m-STIRLING NUMBERS

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ABSTRACT. In this paper, a generalization of the Stirling numbers of the first and second kind which are called by m-Stirling numbers of the first and second kind are derived. Based on colored base-m number system we give a combinatorial interpretation of m-Stirling numbers of the second kind. Some basic properties of two kinds of m-Stirling Numbers including generating functions, explicit expressions and recurrence relations are also obtained.

1. INTRODUCTION

The Stirling numbers of the second kind $S(n,k)$ which satisfy the triangular recurrence relation $S(n,k) = S(n-1,k-1) + kS(n-1,k)$, for $n,k \geq 1$, $S(n,0) = S(0,k) = 0$, except $S(0,0) = 1$ (see [2, P. 208]) and the Stirling numbers of the first kind $s(n,k)$ which satisfy $s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$, for $n,k \geq 1$, $s(n,0) = s(0,k) = 0$, except $s(0,0) = 1$ (see [2, P. 214]) are basic numbers in combinatorics. There are a lot of research and generalizations [1,3,5–7] related to those two kinds of numbers. This paper is motivated by the problem of the m-arithmetic triangle [4]. We construct further generalizations of the m-arithmetic triangle and get two kinds of m-Stirling numbers. By colored base-m number system we obtain a combinatorial interpretation of m-Stirling numbers of the second kind. The generating functions, recursive relations and explicit expressions are also given.

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2. m-STIRLING NUMBERS OF THE SECOND KIND

In this section, we first give a definition and combinatorial explanation of m-Stirling number of the second kind. Several kinds of generating function are also given.

Definition 1. *Let m be a positive integer. We define m-Stirling numbers of the second kind $S_{n,k}^{(m)}$ as*

$$S_{n,k}^{(m)} = S_{n-1,k-m}^{(m)} + S_{n-1,k-m+1}^{(m)} + \cdots + S_{n-1,k-1}^{(m)} + kS_{n-1,k}^{(m)}, \tag{1}$$

with $S_{0,0}^{(m)} = 1$, $S_{0,k}^{(m)} = 0$, if $k \neq 0$.

Obviously, $S_{n,k}^{(1)} = S(n,k)$ is the Stirling number of the second kind. The number $S_{n,k}^{(m)}$ is a generalization of the $S(n,k)$. It is related to the base-$(m+1)$ number system which satisfies (i) There are n positions in the base-$(m+1)$ number system; (ii) $S_{0,0}^{(m)} = 1$, $S_{0,k}^{(m)} = 0$, if $k \neq 0$; (iii) The value of the rightmost position is not equal to 0; (iv) The summation of the values of all positions is equal to k; (v) Every position with 0 is marked by different color. The number of color category of position i whose value is 0 is equal to the summation of all the value on the right side of it. For example, the base-3 number system with $n = 3$, $k = 2$ is 1①1, ①11, ②11, ①①2, ①②2, ②①2, ②②2, therefore $S_{3,2}^{(3)} = 7$. The base-3 number system with $n = 3$, $k = 3$ is 2①1, 111, 1①2, 1②2, ①21, ②21, ③21, ①12, ②12, ③12, therefore $S_{3,3}^{(3)} = 10$. The number ⓘ means 0 that colored by the i-th color.

By the above definition, the base-$(m+1)$ number can be divided into $m+1$ types with whose value of leftmost position is $0, 1, 2, \ldots, m$ respectively. The base-$(m+1)$ number is $S_{n-1,k-1}^{(m)}$, $S_{n-1,k-2}^{(m)}$, $\ldots$, $S_{n-1,k-m}^{(m)}$ respectively if the value of the leftmost position is $1, 2, \ldots, m$. If the value of the leftmost position is 0, by noticing the summation of the rest value is k, the base-$(m+1)$ number of the rest position is $S_{n-1,k}^{(m)}$. The leftmost position with 0 has k types of mark color. Hence the number of this case is $kS_{n-1,k}^{(m)}$ and the number of the base-$(m+1)$ number defined above satisfies the same recursive relation and initial condition.

Theorem 1. *The vertical generating function of $S_{n,k}^{(m)}$ is*

$$S_k^{(m)}(x) := \sum_{n\geq 0} S_{n,k}^{(m)} \frac{x^n}{n!} = \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \frac{(e^x-1)^{i_1}(e^{2x}-1)^{i_2}\cdots(e^{mx}-1)^{i_m}}{1^{i_1}\, i_1! \cdot 2^{i_2}\, i_2! \cdots m^{i_m}\, i_m!}. \tag{2}$$

where $\tau(m,k) = \{(i_1, i_2, \ldots, i_m) \mid i_1 + 2i_2 + \cdots + mi_m = k,\ i_1, i_2, \ldots, i_m \in \mathbb{N}_0\}$, $\mathbb{N}_0$ is a set of nonnegative integer.

Proof. By the recursive relation and the initial relation of $S_{n,k}^{(m)}$, we can see that $S_k^{(m)}(x)$ satisfies the following recursive relation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{S_k^{(m)}(x)\right\} = kS_k^{(m)}(x) + S_{k-1}^{(m)}(x) + \cdots + S_{k-m}^{(m)}(x).$$

We only need to prove that the right hand side of equation of (2) satisfies the above relation. The derivative of the right hand side of (2) by x is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{S_k^{(m)}(x)\right\} &= \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \frac{\mathrm{d}}{\mathrm{d}x}\prod_{r=1}^{m}\frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} \\
&= \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \sum_{\substack{\ell=1\\ i_\ell\geq 1}}^{m} \prod_{\substack{r=1\\ r\neq\ell}}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!}\frac{(e^{\ell x}-1)^{i_\ell-1}}{\ell^{i_\ell-1}(i_\ell-1)!}e^{\ell x} \\
&= \sum_{\substack{(i_1,\ldots,i_m)\\ \in\tau(m,k)}} \sum_{\substack{\ell=1\\ i_\ell\geq 1}}^{m} \prod_{\substack{r=1\\ r\neq\ell}}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!}\left\{\frac{(e^{\ell x}-1)^{i_\ell}}{\ell^{i_\ell}\, i_\ell!}\ell i_\ell + \frac{(e^{\ell x}-1)^{i_\ell-1}}{\ell^{i_\ell-1}(i_\ell-1)!}\right\} \\
&= \sum_{\substack{(i_1,\ldots,i_m)\\ \in\tau(m,k)}} \sum_{\substack{\ell=1\\ i_\ell\geq 1}}^{m} \ell i_\ell \prod_{r=1}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} + \sum_{\ell=1}^{m} \sum_{\substack{(i_1,\ldots,i_m)\\ \in\tau(m,k-\ell)}} \prod_{r=1}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} \\
&= kS_k^{(m)}(x) + \sum_{\ell=1}^{m} S_{k-\ell}^{(m)}(x). \quad \square
\end{aligned}$$

Remark 1. As $m = 1$, the generating function (2) reduces to the generating function of the Stirling numbers of the second kind $S(n,k)$. If $m = 2$, then (2) becomes

$$S_k^{(2)}(x) = \sum_{n\geq 0} S_{n,k}^{(2)}\frac{x^n}{n!} = \sum_{i=0}^{\lfloor k/2\rfloor}\frac{(e^x-1)^{k-2i}}{(k-2i)!}\frac{(e^{2x}-1)^i}{(2i)!!}. \tag{3}$$

By expanding every item in the right side of equation (2), we have

Theorem 2. *The following explicit formula holds:*

$$S_{n,k}^{(m)} = \sum_{\substack{l_1+\cdots+l_m=n\\ l_1,\ldots,l_m\geqslant 0}} \binom{n}{l_1,\cdots,l_m} \sum_{\substack{(i_1,\ldots,i_m)\\ \in\tau(m,k)}} \prod_{r=1}^{m}\sum_{j_r=0}^{i_r}(-1)^{i_r-j_r}\binom{i_r}{j_r}\frac{(i_r-j_r)^{\ell_r}}{r^{i_r-\ell_r}i_r!}, \tag{4}$$

where $(i_r - j_r)^{\ell_r}$ *is changed into 1 if it is* 0^0 *in the above summations.*

Remark 2. If $m = 1$, (4) reduces to the explicit formula of $S(n,k)$ [2, P. 204].

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

Theorem 3. *The double generating function of* $S_{n,k}^{(m)}$ *is*

$$\Psi(x,u) := \sum_{n,k\geq 0} S_{n,k}^{(m)} \frac{x^n}{n!} u^k = \exp\Big\{\sum_{r=1}^{m} \frac{u^r}{r}(e^{rx}-1)\Big\}. \tag{5}$$

Proof. By the result of Theorem 1, we have

$$\begin{aligned}\Psi(x,u) &= \sum_{k\geq 0}\sum_{n\geq 0} S_{n,k}^{(m)} \frac{x^n}{n!} u^k = \sum_{k\geq 0}\sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \prod_{r=1}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} u^k \\ &= \prod_{r=1}^{m}\sum_{i_r\geq 0} \frac{u^{ri_r}(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} = \prod_{r=1}^{m} \exp\Big\{\frac{u^r}{r}(e^{rx}-1)\Big\} \\ &= \exp\Big\{\sum_{r=1}^{m} \frac{u^r}{r}(e^{rx}-1)\Big\}. \quad \Box\end{aligned}$$

Theorem 4. *The rational generating function of* $S_{n,k}^{(m)}$ *is*

$$\Phi_k^{(m)}(u) := \sum_{n\geq 0} S_{n,k}^{(m)} u^n = \sum_{i=0}^{k} \frac{\sigma_{k-i}^{(m)}(1-u, 1-2u, \ldots, 1-(k-1)u)}{(1-u)(1-2u)\ldots(1-ku)} u^i, \tag{6}$$

where $\sigma_k^{(m)}(x_1, x_2, \ldots, x_n)$ *is the summation of the items of elementary symmetric function* $\sigma_k(x_1, x_2, \ldots, x_n)$ *[2, P. 158] except the items with the product of* m *excessive* x_i*.*

Proof. For the sake of convenience, let $x_i = 1 - iu$, for $i = 1, 2, \ldots$. Then (6) becomes

$$\Phi_k^{(m)}(u) = \sum_{i=0}^{k} \frac{\sigma_{k-i}^{(m)}(x_1, x_2, \ldots, x_{k-1})}{x_1 x_2 \cdots x_k} u^i. \tag{7}$$

By the recursive relation and the initial condition of $S_{n,k}^{(m)}$, we have the following recursive relation of $\Phi_k^{(m)}(u)$.

$$\Phi_k^{(m)}(u) = \Big\{\Phi_{k-1}^{(m)}(u) + \Phi_{k-2}^{(m)}(u) + \cdots + \Phi_{k-m}^{(m)}(u)\Big\} \frac{u}{x_k}, \tag{8}$$

with initial condition $\Phi_0^{(m)}(u) = 1$ and $\Phi_k^{(m)}(u) = 0$ for $k < 0$. For $0 \leq k \leq m$, by simple computation we derive

$$\Phi_k^{(m)}(u) = \sum_{i=0}^{k} \frac{\sigma_{k-i}(x_1, x_2, \ldots, x_{k-1})}{x_1 x_2 \cdots x_k} u^i = \sum_{i=0}^{k} \frac{\sigma_{k-i}^{(m)}(x_1, x_2, \ldots, x_{k-1})}{x_1 x_2 \cdots x_k} u^i.$$

For $k > m$, we prove our result by induction. We first assume (7) is valid for the subscript of $\Phi \leq k$. As the subscript of Φ is $k+1$, by noticing (8), we have

$$\Phi_{k+1}^{(m)}(u) = \sum_{i=k-m+1}^{k} \Phi_i^{(m)}(u)\frac{u}{x_{k+1}} = \sum_{i=k-m+1}^{k} \sum_{j=0}^{i} \frac{\sigma_{i-j}^{(m)}(x_1,\ldots,x_{i-1})}{x_1x_2\cdots x_ix_{k+1}}u^{j+1}$$
$$= \sum_{j=1}^{k+1} \frac{u^j}{x_1x_2\cdots x_{k+1}} \sum_{i=\max\{j-1,k-m+1\}}^{k} \sigma_{i-j+1}^{(m)}(x_1,\ldots,x_{i-1}) \prod_{\ell=1}^{k-i} x_{i+\ell}$$
$$= \sum_{j=1}^{k+1} \frac{\sigma_{k-j+1}^{(m)}(x_1,\ldots,x_k)}{x_1x_2\cdots x_{k+1}}u^j.$$

The last equation is valid since $k-i \le m-1$ and $x_{i+1}x_{i+2}\cdots x_k$ is at most the product of $m-1$ successive x_i. □

Remark 3. The recursive relation of $\sigma_k^{(m)}(x_1,\ldots,x_n)$ can be derived from the above proof. For $k \ge m \ge 2$,

$$\sigma_k^{(m)}(x_1,\ldots,x_n) = \sigma_k^{(m)}(x_1,\ldots,x_{n-1}) + \sum_{i=1}^{m-1} \sigma_{k-i}^{(m)}(x_1,\ldots,x_{n-i-1})x_{n-i+1}\cdots x_n, \quad (9)$$

and for $k < m$, $\sigma_k^{(m)}(x_1,\ldots,x_n) = \sigma_k(x_1,\ldots,x_n)$. For example, $\sigma_2^{(2)}(x_1,x_2,x_3,x_4) = \sigma_2^{(2)}(x_1,x_2,x_3) + \sigma_1^{(2)}(x_1,x_2)\,x_4 = x_1x_3 + (x_1+x_2)\,x_4$.

3. m-STIRLING NUMBERS OF THE FIRST KIND

In this section we give the definition of m-Stirling numbers of the first kind and obtain three kinds of generating functions.

Definition 2. *Let m be a positive integer. We define m-Stirling numbers of the first kind $s_{n,k}^{(m)}$ as*

$$s_{n,k}^{(m)} = s_{n-1,k-m}^{(m)} + s_{n-1,k-m+1}^{(m)} + \cdots + s_{n-1,k-1}^{(m)} - (n-1)s_{n-1,k}^{(m)}, \quad (10)$$

with

$$s_{0,0}^{(m)} = 1; \quad s_{0,k}^{(m)} = 0, \text{ if } k \ne 0. \quad (11)$$

The m-Stirling numbers of the first kind $s_{n,k}^{(m)}$ is a natural generalization of Stirling numbers of the first kind and $s_{n,k}^{(1)}$ is the Stirling numbers of the first kind $s(n,k)$. The generating functions of m-Stirling numbers of the first kind are given as follows.

Theorem 5. *The $s_{n,k}^{(m)}$ have the vertical generating function:*

$$s_k^{(m)}(x) := \sum_{n\ge 0} s_{n,k}^{(m)}\frac{x^n}{n!} = \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \frac{\{\ln(1+x)\}^{i_1+\cdots+i_m}}{i_1!\,i_2!\cdots i_m!}. \quad (12)$$

Proof. By the recursive relation (10) and initial condition (11) of $s_{n,k}^{(m)}$, we can see that $s_{n,k}^{(m)}$ satisfies the following differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{s_k^{(m)}(x)\right\} = \left\{s_{k-1}^{m}(x) + s_{k-2}^{(m)}(x) + \cdots + s_{k-m}^{(m)}(x)\right\}\frac{1}{1+x}. \tag{13}$$

It is easy to derive $s_0^{(m)}(x) = 1$ and $s_k^{(m)}(x) = 0$ if $k < 0$. Hence (12) is valid for $k \leq 0$. For $k > 0$, we only need to prove the right hand side of (12) satisfies the equation (13). Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{s_k^{(m)}(x)\right\} &= \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \frac{\mathrm{d}}{\mathrm{d}x}\frac{\{\ln(1+x)\}^{i_1+\cdots+i_m}}{i_1!\,i_2!\cdots i_m!} \\
&= \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \frac{(i_1+\cdots+i_m)\{\ln(1+x)\}^{i_1+\cdots+i_m-1}}{i_1!\,i_2!\cdots i_m!\,(1+x)} \\
&= \sum_{(i_1,\ldots,i_m)\in\tau(m,k)} \sum_{\substack{j=1 \\ i_j\geq 1}}^{m} \prod_{\substack{r=1\\ r\neq j}}^{m} \frac{\ln^{i_r}(1+x)}{i_r!}\,\frac{\ln^{i_j-1}(1+x)}{(i_j-1)!(1+x)}.
\end{aligned}$$

By replacing i_j by i_j+1 in the last equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{s_k^{(m)}(x)\right\} = \sum_{j=1}^{m} \sum_{(i_1,\ldots,i_m)\in\tau(m,k-j)} \prod_{r=1}^{m} \frac{\ln^{i_r}(1+x)}{i_r!} = \sum_{j=1}^{m} \frac{s_{k-j}^{(m)}(x)}{1+x}.$$

This completes the proof. □

Remark 4. As $m = 1$, the generating function (12) reduces to the generating function of Stirling numbers of first kind $s(n,k)$. If $m = 2$, then (12) becomes

$$s_k^{(2)}(x) = \sum_{n\geq 0} s_{n,k}^{(2)}\frac{x^n}{n!} = \sum_{i=0}^{\lfloor k/2\rfloor} \frac{\ln^{k-i}(1+x)}{(k-2i)!\,i!}. \tag{14}$$

Theorem 6. *The $s_{n,k}^{(m)}$ have the following double generating function:*

$$\Phi_m(x,u) := \sum_{n,k\geq 0} s_{n,k}^{(m)}\frac{x^n}{n!}u^k = (1+x)^{u+u^2+\cdots+u^m}. \tag{15}$$

Proof. Multiplying both side of equation (10) by u^k and then taking summation by k from 0 to ∞, we have

$$\sum_{n,k\geq 0} s_{n,k}^{(m)}\frac{x^n}{n!}u^k = \sum_{k\geq 0} s_k^{(m)}(x)u^k = \sum_{k\geq 0}\sum_{\substack{(i_1,\ldots,i_m)\\ \in\tau(m,k)}} \frac{\{\ln(1+x)\}^{i_1+\cdots+i_m}}{i_1!\,i_2!\cdots i_m!}u^k$$

$$= \sum_{i_1,\ldots,i_m\geq 0}\prod_{r=1}^{m}\frac{\{u^r\ln(1+x)\}^{i_r}}{i_r!} = \prod_{r=1}^{m}\sum_{i_r\geq 0}\frac{\{u^r\ln(1+x)\}^{i_r}}{i_r!} = \prod_{r=1}^{m}(1+x)^{u^r}.$$

Therefore $\Phi_m(x,u) = (1+x)^{u+u^2+\cdots+u^m}$. □

Theorem 7. *The rational generating function of $s_{n,k}^{(m)}$ is*

$$\varphi_n(u) := \sum_{k=0}^{nm} s_{n,k}^{(m)} u^{nm-k} = \prod_{k=0}^{n-1}(1+u+u^2+\cdots+u^{m-1}-ku^m). \quad (16)$$

Proof. Multiplying both side of equation (10) by u^{nm-k} and then taking summation by k from 0 to nm, we have

$$\begin{aligned}\varphi_n(u) &= \varphi_{n-1}(u) + u\varphi_{n-1}(u) + \cdots + u^{m-1}\varphi_{n-1}(u) - (n-1)u^m\varphi_{n-1}(u)\\ &= (1+u+\cdots+u^{m-1}-(n-1)u^m)\varphi_{n-1}(u).\end{aligned}$$

By noticing initial condition $\varphi_0(u)=1$, we derive the desired result easily. □

Remark 5. Replacing u by u^{-1} and multiplying both side of (16) by u^{nm}, (16) becomes

$$(u+u^2+\cdots+u^m)_n = \sum_{k=0}^{nm} s_{n,k}^{(m)} u^k, \quad (17)$$

where the symbol $(\cdot)_n$ is called falling factorial of order n. It is defined as $(\alpha)_n := \alpha(\alpha-1)\cdots(\alpha-n+1)$. This is a generalization of horizontal generating function of the $s(n,k)$ [2, P. 213, (5e)].

4. Recursive relation of m-Stirling numbers

In this section, by using generating functions of those two numbers, we obtain some recursive relations related to m-Stirling numbers of the first kind and the second kind.

Theorem 8. *The numbers $S_{n,k}^{(m)}$ have the two vertical recursive relations as follows.*

$$S_{n+1,k}^{(m)} = \sum_{\ell=1}^{m}\sum_{i=0}^{n}\binom{n}{i} S_{n-i,k-\ell}^{(m)} \ell^i. \quad (18)$$

$$S_{n+1,k}^{(m)} = \sum_{\ell=1}^{m}\sum_{i=0}^{n} S_{i,k-\ell}^{(m)} k^{n-i}. \quad (19)$$

Proof. Taking the derivative by x of the vertical generating function $S_k^{(m)}(x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}S_k^{(m)}(x) = \sum_{(i_1,\dots,i_m)\in\tau(m,k)} \sum_{\substack{\ell=1\\ i_\ell\geq 1}}^{m} \prod_{\substack{r=1\\ r\neq\ell}}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} \frac{(e^{\ell x}-1)^{i_\ell-1}}{\ell^{i_\ell-1}(i_\ell-1)!} e^{\ell x}$$

i. e.

$$\begin{aligned}\sum_{n\geq 1} S_{n,k}^{(m)} \frac{x^{n-1}}{(n-1)!} &= \sum_{\ell=1}^{m} \sum_{(i_1,\ldots,i_m)\in\tau(m,k-\ell)} \prod_{r=1}^{m} \frac{(e^{rx}-1)^{i_r}}{r^{i_r}\, i_r!} e^{\ell x} \\ &= \sum_{\ell=1}^{m} S_{k-\ell}^{(m)}(x) e^{\ell x} = \sum_{\ell=1}^{m} \sum_{n,i\geq 0} S_{n,k-\ell}^{(m)} \frac{\ell^i x^{n+i}}{n!\, i!} \\ &= \sum_{n\geq 0} \sum_{\ell=1}^{m} \sum_{i=0}^{n} \binom{n}{i} S_{n-i,k-\ell}^{(m)} \ell^i \frac{x^n}{n!},\end{aligned}$$

where the last equation is obtained by replacing n by $n-i$. Taking the coefficient of $x^n/n!$ we get the first result.
In order to prove the second recursive relation, by noticing $(1-ku)^{-1} = \sum_{j\geq 0} k^j u^j$, the equation (8) can be expanded as

$$\begin{aligned}\Phi_k^{(m)}(u) &= \sum_{\ell=1}^{m} \Phi_{k-\ell}^{(m)}(u) \frac{u}{1-ku} = \sum_{\ell=1}^{m} \sum_{i,j\geq 0} S_{i,k-\ell}^{(m)} k^j u^{i+j+1} \\ &= \sum_{n\geq 0} \sum_{\ell=1}^{m} \sum_{i=0}^{n} S_{i,k-\ell}^{(m)} k^{n-i} u^{n+1}\end{aligned}$$

where the last equation is obtained by replacing $i+j$ by n and j by $n-i$. The second result follows by taking the coefficient of u^{n+1} of both side of the above equation. □

Theorem 9. *The numbers $s_{n,k}^{(m)}$ have the following vertical recursive relations.*

$$k s_{n,k}^{(m)} = \sum_{\ell=1}^{m} \sum_{i=1}^{n} (-1)^{i-1}(i-1)!\ell \binom{n}{i} s_{n-i,k-\ell}^{(m)}, \tag{20}$$

$$s_{n+1,k}^{(m)} = \sum_{\ell=1}^{m} \sum_{i=0}^{n} (-1)^i (n)_i s_{n-i,k-\ell}^{(m)}. \tag{21}$$

Proof. Taking both side of (15) the derivative by u of the double generating function $\Phi_m(x,u)$, we have $\frac{\partial}{\partial u}\Phi_m(x,u) = \sum_{\ell=1}^{m} \ell u^{\ell-1} \Phi_m(x,u) \ln(1+x)$.
i. e.

$$\begin{aligned}\sum_{n\geq 0,k\geq 1} k s_{n,k}^{(m)} \frac{x^n}{n!} u^{k-1} &= \sum_{\ell=1}^{m} \ell u^{\ell-1} \sum_{n,k\geq 0} s_{n,k}^{(m)} \frac{x^n}{n!} u^k \sum_{i\geq 1} (-1)^{i-1} \frac{x^i}{i} \\ &= \sum_{n,k\geq 1} \sum_{\ell=1}^{m} \sum_{i=1}^{n} (-1)^{i-1}(i-1)!\ell \binom{n}{i} s_{n-i,k-\ell}^{(m)} \frac{x^n}{n!} u^{k-1},\end{aligned}$$

where the last equation is derived by replacing n by $n-i$ and k by $k-\ell$. The first recursive relation is obtained by taking the coefficient of $\frac{x^n}{n!}u^k$ of the above equation.

In order to get the second recursive relation, we take the derivative of x of the double generating function and obtain $\frac{\partial}{\partial x}\Phi_m(x,u) = \sum_{\ell=1}^{m} u^\ell \frac{\Phi_m(x,u)}{1+x}$. That is

$$\begin{aligned}
\sum_{n\geq 1, k\geq 0} s_{n,k}^{(m)} \frac{x^{n-1}}{(n-1)!}u^k &= \sum_{\ell=1}^{m} u^\ell \sum_{n,k\geq 0} s_{n,k}^{(m)} \frac{x^n}{n!}u^k \sum_{i\geq 0}(-1)^i x^i \\
&= \sum_{\ell=1}^{m} \sum_{n,k,i\geq 0} (-1)^i s_{n,k}^{(m)} \frac{x^{n+i}}{n!}u^{k+\ell} \\
&= \sum_{n,k\geq 0} \sum_{\ell=1}^{m} \sum_{i=0}^{n} (-1)^i (n)_i s_{n-i,k-\ell}^{(m)} \frac{x^n}{n!}u^k,
\end{aligned}$$

where the last equation is derived by taking the replacements $n \to n-i$ and $k \to k-\ell$. Taking the coefficient of $\frac{x^n}{n!}u^k$ of both side of the above result, the second recursive relation follows. □

Remark 6. If $m=1$, Theorem 8 reduces to the vertical recursive relation of $S(n,k)$ [2, P. 209, Theorem B]. Theorem 9 reduces to the vertical recursive relation of $s(n,k)$ [2, P. 215, Theorem B]. For $m=2$, we can obtain simple vertical recursive relation of $S_{n,k}^{(2)}$ and $s_{n,k}^{(2)}$ related to equations (18)-(21) respectively.

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References

[1] W. C. Chu and C. A. Wei, *Set partitions with restrictions*, Discrete Mathematics, 308 (2008): 3163-3168.

[2] L. Comtet, *Advanced Combinatorics*, Reidel, Boston, Mass., 1974.

[3] B. S. El-Desouky and N. P. Cakić, *Generalized higher order Stirling numbers*, Math. Comput. Modelling, 54 (2011): 2848-2857.

[4] A. R. Moghaddamfar, S. M. H. Pooya, S. Navid Salehy, S. Nima Salehy, *On the matrices related to the m-arithmetic triangle*, Linear Algebra and its Applications, 432 (2010): 53-69.

[5] Y. D. Sun, *Two classes of p-Stirling numbers*, Discrete Mathematics, 306 (2006): 2801-2805.

[6] A. M. Xu, *A Newton interpolation approach to generalized Stirling numbers*, J. Appl. Math.: Art. ID 351935, doi:10.1155/2012/351935.

[7] A. Xu, Z. Cen, *Some identities involving exponential functions and Stirling numbers and applications*, Journal of Computational and Applied Mathematics, 260 (2014), 201-207.