A Note on Co-Maximal Ideal Graph of Commutative Rings *†

S. Akbari[a,c], B. Miraftab[a], R. Nikandish[b]

[a] *Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*

[b] *Department of Basic Sciences, Jundi-Shapur University of Technology, Dezful, Iran*

P.O. Box 64615-334

[c] *School of Mathematics, Institute for Research in Fundamental Sciences,* (IPM)

P.O. Box 19395-5746

s_akbari@sharif.edu babak.math@gmail.com r.nikandish@ipm.ir

Abstract

Let R be a commutative ring with unity. The co-maximal ideal graph of R, denoted by $\Gamma(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. We classify all commutative rings whose co-maximal ideal graphs are planar. In 2012 the following question was posed: If $\Gamma(R)$ is an infinite star graph, can R be isomorphic to the direct product of a field and a local ring? In this paper, we give an affirmative answer to this question.

1. Introduction

When one assigns a graph to an algebraic structure numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, independence number and so on. There are a lot of papers which apply combinatorial methods

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to obtain algebraic results, for instance see [1], [2], [3], [8] and [9].

Let G be a graph with the vertex set $V(G)$. A bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If the size of one of the parts is 1, then the graph is said to be a *star graph.* A *clique* of G is a complete subgraph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. A graph is said to be *planar*, if it can be drawn in the plane so that its edges intersect only at their ends.
Throughout this paper R is a commutative ring with unity. The set of maximal ideals of R and the Jacobson radical of R are denoted by $\mathrm{Max}(R)$ and $\mathrm{J}(R)$, respectively. The ring R is called *local* if $|\mathrm{Max}(R)| = 1$. The ring R is said to be *uniserial* if ideals of R are totally ordered by inclusion. The *co-maximal ideal graph* of R, denoted by $\Gamma(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. This graph was first introduced and studied in [9]. In 2012, Ye and Wu in [9, Question 4.12] asked the following question: If $\Gamma(R)$ is an infinite star graph, can R be isomorphic to the direct product of a field and a local ring? In this paper, we give an affirmative answer to this question. Indeed, we show that there exists a vertex of $\Gamma(R)$ which is adjacent to all other vertices if and only if R is isomorphic to the direct product of a local ring and a field. Also we characterize all commutative rings whose co-maximal ideal graphs are planar.

2. Results

In this section, we classify all rings whose co-maximal ideal graphs have a vertex which is adjacent to all other vertices. We start with the following theorem.

Theorem 1. *Let R be a ring. Then there exists a vertex of $\Gamma(R)$ which is adjacent to all other vertices if and only if R is isomorphic to the direct product of a local ring and a field.*

Proof. One side is clear. For the other side, let I be a vertex adjacent to all other vertices and $a \in I \setminus \mathrm{J}(R)$. Since I is adjacent to all other vertices, we deduce that $I = Ra$ and I is a maximal ideal of R. Also, Ra^2 is a vertex of $\Gamma(R)$ and so $Ra = Ra^2$. Thus $a = ta^2$, for some $t \in R$. Clearly, $1 \neq 1-ta$ is a non-zero idempotent. By [4, Proposition 5.10], $R \cong R_1 \times R_2$, for some rings R_1 and R_2. We show that at least one of the rings R_1 and R_2 is a field. With no loss of generality, we may assume that $I = R_1 \times \mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R_2. Obviously, if $\mathfrak{m} \neq 0$, then I is not adjacent to $R_1 \times 0$, a contradiction. Thus R_2 is a field. Now, we prove that R_1 is a local ring. By contrary, assume that R_1 is not a local ring. Thus there exists an ideal of R, say $J = \mathfrak{m}_1 \times 0$, where $\mathfrak{m}_1$ is a maximal ideal of R_1, and J is a vertex of $\Gamma(R)$. But I and J are not adjacent, a contradiction and the proof is complete. □

In the sequel of this paper, we provide some conditions under which $\Gamma(R)$ is a finite graph.

Theorem 2. *If $\alpha(\Gamma(R)) < \infty$, then $\Gamma(R)$ is a finite graph.*

Proof. Since $\alpha(\Gamma(R)) < \infty$, we deduce that $\alpha(\Gamma(\frac{R}{\mathrm{J}(R)})) < \infty$. Thus $\frac{R}{\mathrm{J}(R)}$ is an Artinian ring and so by [5, Theorem 8.7], $\frac{R}{\mathrm{J}(R)}$ has finitely many maximal ideals. Therefore, $|\mathrm{Max}(R)| < \infty$ and hence by [9, Theorem 3.1], $\omega(\Gamma(R)) < \infty$. Now, the result follows from Ramsey's Theorem, see [6, Theorem 12.5]. □

Theorem 3. *If each vertex of $\Gamma(R)$ has a finite degree, then R has finitely many ideals. Moreover, R is a direct product of finitely many uniserial rings and a finite ring.*

Proof. Let I be a vertex of $\Gamma(R)$. Then there exists an ideal L of R such that $I + L = R$. So there exist two elements $a \in I$ and $b \in L$ such that $a + b = 1$. Since $1 = (a+b)^n = \Sigma_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$, we conclude that $I + Rb^n = R$, for $n = 1, 2, \ldots$. Furthermore since I has finite degree and $Rb^n \nsubseteq \mathrm{J}(R)$, for $n = 1, 2, \ldots$, we find that $(b^t) = (b^{2t})$, for some $t \geq 1$. Therefore $b^t = lb^{2t}$, for some $l \in R$, and so $1 - lb^t$ is a non-trivial idempotent element. Hence $R \cong R_1 \times R_2$, for some rings R_1 and R_2, see [4, Proposition 5.10]. We show that R_i contains finitely many ideals for $i = 1, 2$. If $\{I_i\}_{i=1}^{\infty}$ is an infinite family of ideals of R_1, then the vertex $J = R_1 \times 0$ is adjacent to $I_i \times R_2$, for $i \geq 1$. So the degree of J is not finite, a contradiction. By a similar argument R_2 has finitely many ideals. Thus R contains finitely many ideals. It follows from [7, Theorem 2.4], R is a direct product of finitely many uniserial rings and a finite ring. □

To prove the next result, we need a celebrated theorem due to Kuratowski.

Theorem 4. [6, Theorem 10.30] *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

We close this paper with the following theorem.

Theorem 5. *Let $\Gamma(R)$ be a finite graph. If $\Gamma(R)$ is planar, then one of the following holds:*

(i) *$R \cong R_1 \times R_2$, where R_1 and R_2 are local rings and one of R_i has at most three ideals.*
(ii) *$R \cong R_1 \times R_2 \times R_3$ and each R_i has at most one non-trivial ideal.*

Proof. Assume that $\Gamma(R)$ is planar. Since $\Gamma(R)$ is finite, it follows from Theorem 3 that R is an Artinian ring. By [5, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for $i = 1, \ldots, n$. Now, Kuratowski's Theorem implies that $|\text{Max}(R)| \leq 4$. Assume that $|\text{Max}(R)| = 4$ and $\text{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4\}$. It is not hard to see that $V_1 = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2\}$ and $V_2 = \{\mathfrak{m}_3, \mathfrak{m}_4, \mathfrak{m}_3\mathfrak{m}_4\}$ induce $K_{3,3}$. Thus $|\text{Max}(R)| \leq 3$. If $|\text{Max}(R)| = 2$, then (i) is directly follows from Kuratowski's Theorem and [9, Theorem 4.5]. Hence suppose that $|\text{Max}(R)| = 3$ and so $R \cong R_1 \times R_2 \times R_3$. With no loss of generality, assume that R_1 has at least two non-trivial ideals I and J. Thus two sets $V_1 = \{I \times R_2 \times R_3, J \times R_2 \times R_3, 0 \times R_2 \times R_3\}$ and $V_2 = \{R_1 \times 0 \times R_3, R_1 \times R_2 \times 0, R_1 \times 0 \times 0\}$ imply that $\Gamma(R)$ contains $K_{3,3}$, a contradiction. Therefore, each R_i has at most one non-trivial ideal. This completes the proof. □

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