Degree Associated Reconstruction Number of Graphs with Regular Pruned Graph

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Abstract

A vertex-deleted unlabeled subgraph of a graph G is called a *card* of G. A card of G with which the degree of the deleted vertex is also given is called a *degree associated card* or *dacard* of G. The *degree associated reconstruction number, drn(G)*, of a graph G is the size of the smallest collection of dacards of G that uniquely determines G. The maximal subgraph without end vertices of a graph G that is not a tree is called the *pruned graph* of G. It is shown that *drn* of some connected graphs with regular pruned graph is 2 or 3.

Keywords: Isomorphism, reconstruction number, vertex-transitive graph, caterpillar.

1. Introduction

All graphs considered in this paper are finite and simple. We shall mostly follow the graph theoretic terminology of [2]. A vertex of degree m is called an m*-vertex.* A 1-vertex is called an *end vertex* and the vertex adjacent to a 1-vertex is called its *support vertex.* A path in a graph is *trivial* if it has only one vertex; otherwise it is *nontrivial.* The maximal subgraph without end vertices of a graph G that is not a tree is called the *pruned graph* of G and is denoted by $\widetilde{G}$. If v is a vertex on $\widetilde{G}$, then the maximal subtree T of G such that $V(T) \cap V(\widetilde{G}) = \{v\}$ is called the *rooted tree* at v; the vertex v is called the *root vertex.* A maximal subtree T' of the rooted tree T at v such that v is not a cut vertex of T' is called a *limb* of T. The graph obtained from $P_{l+1} \cup K_{1,r}$ by identifying an end vertex of P_{l+1} (a path on $l+1$ (≥ 2) vertices) and the r-vertex of $K_{1,r}$ is called a *broom graph* and is denoted by $B_{l,r}$; the end vertex other than the identified vertex of P_{l+1} is called the

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special vertex of $B_{l,r}$ (In Figure 1, the dark vertex is the special vertex). The *skeleton* of a tree is the subtree obtained by deleting all the leaves from the tree. *Caterpillar* is a tree whose skeleton is a path.

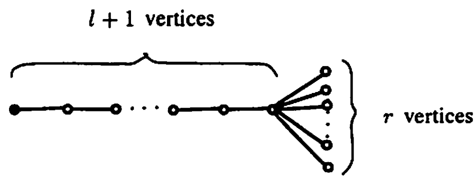


Figure 1. The graph $B_{l,r}$

A *card* $G-v$ of a graph G is obtained from G by deleting a vertex v and all edges incident with v. The *deck* of a graph G, denoted by $\mathscr{D}(G)$ is the multiset of all its cards. The graph H is said to be a *reconstruction* of G if $\mathscr{D}(H) = \mathscr{D}(G)$. A graph G is said to be *reconstructible* if every reconstruction of G is isomorphic to G. The Reconstruction Conjecture (RC) [3] asserts that every graph on at least three vertices is reconstructible. For a reconstructible graph G, Harary and Plantholt [4] defined the *reconstruction number* of G, $rn(G)$, to be the minimum number of cards from the deck of G that suffices to determine G uniquely. An extension of RC to digraphs is the *Digraph Reconstruction Conjecture* (DRC) proposed by Harary [3]. It was disproved by Stockmeyer [12] by exhibiting several infinite families of counter-examples. Ramachandran [9] then proposed a variation in DRC and introduced the degree associated reconstruction.

For a vertex v of a digraph, the ordered triple (r, s, t) is called the *degree triple* of v where r, s and t are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pairs of arcs incident with v. A *degree associated card* or *dacard* of a graph (digraph) G is a pair $(d, G-v)$ consisting of a card $G-v$ from the deck and the degree (degree triple) d of the deleted vertex. The *degree associated deck* or *dadeck* of G is the multiset of all its dacards. A graph (digraph) is said to be degree associated reconstructible if it can be determined uniquely from its dadeck. For a degree associated reconstructible graph (digraph) G, Ramachandran [10] defined the *degree (degree triple) associated reconstruction number*, $drn(G)$, to be the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other graph (digraph) H, $H \not\cong G$. It is clear, from their definitions, that $drn(G) \leq rn(G)$.

An *s-blocking set* of a graph G is a family $\mathscr{F}$ of graphs not isomorphic to G such that every collection of s dacards of G will appear in the dadeck of some graph of $\mathscr{F}$ and every graph in $\mathscr{F}$ will have s dacards in common with G.

In their paper [1], Barrus and West have recently proved the following theorem.

Theorem 1. *For all caterpillars G except stars and one example on 6 vertices (Figure 2), $drn(G) = 2$.*

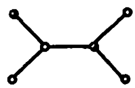


Figure 2.

In this paper, we show that drn of a complete multipartite graph is 1, 2 or 3 and prove that drn of a connected graph G, in which $\tilde{G}$ is regular, is 2 or 3 if one of the following holds.

(i) Every limb is a nontrivial path.

(ii) Every limb is a broom graph and the root vertex is the special vertex of the broom graph.

(iii) Every rooted tree is a caterpillar and the root vertex is a support vertex in a longest path of the caterpillar.

(iv) Every rooted tree is not a path and has exactly one 1-vertex at a distance n from the corresponding root vertex and all other 1-vertices are at distance at most $n-2$ from the corresponding root vertices.

2. Complete Multipartite Graph

In their paper [1], Barrus and West have proved the following theorem and corollary.

Theorem 2. *The dacard $(d, G-v)$ belongs to the dadeck of only one graph (up to isomorphism) if and only if one of the following holds:*

(i) $d = 0$ *or* $d = |V(G-v)|$;

(ii) $d = 1$ *or* $d = |V(G-v)| - 1$, *and* $G - v$ *is vertex-transitive;*

(iii) $G - v$ *is complete or edgeless.*

Corollary 3. *A graph G satisfies $drn(G) = 1$ if and only if G or $\overline{G}$ has an isolated vertex or has a leaf whose deletion leaves a vertex-transitive graph.*

Since $K_{1,n}$ has a vertex adjacent to all other vertices, it follows that $drn(K_{1,n}) = 1$. Ramachandran [9] proved that $drn(K_{m,n}) = 2$ for $2 \leq m < n$ and $drn(K_{m,m}) = 3$ for $m \geq 2$. We shall prove that drn of a complete multipartite graph is 1, 2 or 3.

An *extension* of a dacard $(d(v), G-v)$ of G is a graph obtained from the dacard by adding a new vertex x and joining it to $d(v)$ vertices of the dacard and it is denoted by $H(d(v), G-v)$ (or simply by H). Throughout this paper, H and x are used in the sense of this definition.

Theorem 4. *If G is a complete multipartite graph, then $drn(G)$ is* 1, 2 *or* 3.

Proof. Let G be a complete s-partite graph on n vertices. If at least one partite set has size 1, then G has a dacard with associated degree $n-1$ and so $drn(G) = 1$.

Now, we assume that each partite set in G has size at least 2. If all the partite sets have same size, then $drn(G) = 3$ by Corollary 2.9 of [11] and so we assume that at least two partite sets have different size. Let $r_1, r_2, \dots, r_s$ be the size of each partite set in G such that $r_1 \leq r_i$ for all i. Consider the dacards $(n - r_1, G - u)$ and $(n - r_s, G - v)$ obtained from G by removing the vertices u and v of the partite sets of size r_1 and r_s, respectively. In the dacard $(n - r_1, G - u)$, let $B_1, B_2, \dots, B_s$ denote the partite sets of size $r_1 - 1, r_2, \dots, r_s$, respectively. Let G' be a graph having the two dacards $(n-r_1, G-u)$ and $(n-r_s, G-v)$ in its dadeck. Then G' can be constructed from $(n - r_1, G - u)$ by adding a new vertex x and joining it to some set of $n - r_1$ vertices in $G - u$. In $(n - r_1, G - u)$, if we join x to $n - r_1$ vertices other than the vertices in B_1, then $G' \cong G$. For all other possibilities the resulting graph G' would not have the dacard $(n - r_s, G - v)$ in its dadeck, since the removal of any $(n - r_s)$-vertex in G' results in a dacard with at least one vertex of degree $n - r_1$. Hence $G' \cong G$ and $drn(G) \leq 2$. Also $drn(G) \geq 2$ by Corollary 3. □

2. Unicyclic Graphs

All unicyclic graphs G on at most eight vertices, except seven graphs given in Table I, have $drn(G) \leq 2$. Dark vertex of graphs shown in Table I and Figure 3 denotes the vertex whose removal results in a dacard common with G.

Lemma 5. *Let G be a unicyclic graph in which every rooted tree is K_2. Then $drn(G) = 3$.*

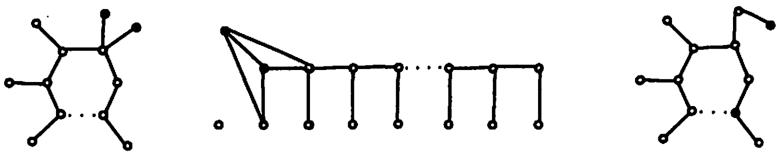
Proof. The graph G has only two types of dacards namely $(1, G - u)$ and $(3, G - v)$ obtained from G by removing the vertices u and v of degree 1 and 3, respectively.
Upper bound: Let G' be a graph having three dacards isomorphic to $(1, G-u)$ in its dadeck. Then G' can be constructed from $(1, G - u)$ by adding a new vertex x and joining it to one vertex in $G - u$. The possibilities are to join x to a 1-vertex, a 2-vertex or a 3-vertex. If we join x to a 2-vertex in $(1, G - u)$, then $G' \cong G$. Otherwise, G' would have at most two dacards isomorphic to $(1, G - u)$, a contradiction. Hence $G' \cong G$ and so $drn(G) \leq 3$.
Lower bound: Consider the following three extensions.

(i) $H_1(1, G - u)$ obtained by joining a new vertex to a 3-vertex.

(ii) $H_2(3, G - v)$ obtained by joining a new vertex to a 2-vertex and its neighbours.

(iii) $H_3(1, G-u)$ obtained by joining a new vertex to a 1-vertex whose support vertex adjacent, in addition, to a 2-vertex and a 3-vertex.

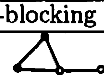
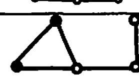
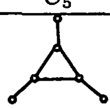
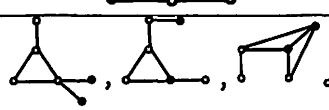
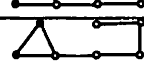
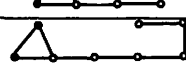
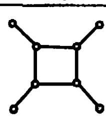
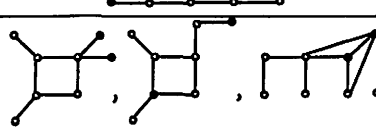
The set $\{ H_1(1, G - u),\ H_2(3, G - v),\ H_3(1, G - u) \}$, where the graphs H_i as indicated in Figure 3, is a 2-blocking set of G and hence $drn(G) \geq 3$.



$H_1(1, G - u)$ $H_2(3, G - v)$ $H_3(1, G - u)$

Figure 3. □

Table I. Unicyclic graphs G on at most eight vertices with $drn(G) = 3$.

$\lvert V(G)\rvert$	G	$drn(G)$	2-blocking set
4	C_4	3	
5	C_5	3	
6		3	
6	C_6	3	
7	C_7	3	
8	C_8	3	
8		3	

3. Regular Pruned Graph

Here we prove our main result that, if G is a connected graph with $\widetilde{G}$ is regular such that every limb is a nontrivial path, then $drn(G) = 2$ or 3. As a prelude, we prove few theorems and lemmas.

If G is a connected graph such that $\widetilde{G}$ is regular and at least two rooted trees are nontrivial, then $drn(G) \geq 2$. For, suppose that drn of G was 1, then the graph G would have a dacard, say $(d, G-v)$ such that it would belong only to the dadeck of G. Since $d \neq 0$ and $d \neq |V(G-v)|$, it follows from Theorem 2 that $G-v$ would be vertex-transitive, contradicting to the fact that none of the dacards of G are vertex-transitive.

Hereafter, all graphs considered are connected with regular pruned graph such that every rooted tree is nontrivial; that is connected graphs G, in which $\widetilde{G}$ is regular, such that $drn(G)$ is at least 2.

By an *m-vertex u at distance k*, we mean an m-vertex u at a distance k from the root vertex of the rooted tree containing u.

Theorem 6. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each rooted tree in G is not a path and has exactly one 1-vertex at distance n and all other 1-vertices are at distance at most $n-2$, then $drn(G) = 2$.*

Proof. Consider the dacards $(1, G-u)$ and $(1, G-v)$ obtained from G by removing, respectively a 1-vertex u at distance n and a 1-vertex v at minimum distance from the corresponding root vertex. In $(1, G-v)$, each rooted tree has exactly one 1-vertex a distance n. Let G' be a graph having the dacards $(1, G-u)$ and $(1, G-v)$ in its dadeck. Then G' can be constructed from $(1, G-u)$ by adding a new vertex x and joining it to one vertex in $G-u$. In $(1, G-u)$, if we join x to the 1-vertex at a distance $n-1$, then the resulting graph $G' \cong G$. Otherwise, the resulting graph G' would not have the dacard $(1, G-v)$ in its dadeck, since the removal of any 1-vertex in G' results in a dacard with a rooted tree having all 1-vertices at distance at most $n-1$. Hence $G' \cong G$ and $drn(G) \leq 2$. □

Theorem 7. *Let G be a connected graph such that $\widetilde{G}$ is regular. If one of the rooted trees in G is K_2 but not all are K_2, then $drn(G) = 2$.*

Proof. Let d be the minimum degree of root vertices in G. Let $(1, G-u)$ be a dacard obtained from G by removing a 1-vertex u of a rooted tree with maximum number of vertices. Let $(1, G-v)$ be a dacard obtained from G by removing the 1-vertex v adjacent to a root vertex of degree d. Then the dacard $(1, G-u)$ guarantees that every extension $H(1, G-u)$ must be connected with $\widetilde{H} \cong \widetilde{G}$ and that every rooted tree must be nontrivial. Consequently, the degree of every root vertex is at least d. Since the dacard $(1, G-v)$ has a root vertex of degree $d-1$, the only possibility to obtain G from $(1, G-v)$ is to add a new vertex x and is to make x adjacent to the root vertex of degree $d-1$, which completes the proof. □

Lemma 8. *Let G be a connected graph such that $\widetilde{G}$ is regular. If every rooted tree in G has exactly m limbs, all are K_2, where m is a constant greater than one, then $drn(G) = 2$. Moreover, if $m = 1$ and $\widetilde{G}$ is not a cycle, then also $drn(G) = 2$.*

Proof. Let d (≥ 4) be the degree of each root vertex in G. Each dacard of G is isomorphic to $(1, G-u)$ or $(d, G-v)$ where u is a 1-vertex and v is a d-vertex in G. We use the two dacards $(1, G-u)$ and $(d, G-v)$ to reconstruct G. The dacard $(1, G-u)$ forces that every extension $H(1, G-u)$ must be connected and $\widetilde{H} \cong \widetilde{G}$ (each vertex is of degree $d-m$ in $\widetilde{G}$) and that the degree of at most one root vertex must be $d-1$. The dacard $(d, G-v)$ has m isolated vertices and the other vertices have degree 1, $d-1$ or d. Since $(1, G-u)$ forces every extension to be connected, it follows that, the newly added vertex to the dacard $(d, G-v)$, must be adjacent to the m isolated vertices. Also, if the new vertex were adjacent to a 1-vertex or a d-vertex, then the resulting extension would have a root vertex of degree 2 or the pruned graph of the resulting extension would have a vertex of degree $d-m+1$, giving a contradiction. Hence in $H(d, G-v)$, the other neighbours of the new vertex must be the vertices of degree $d-1$ (≥ 3). Hence the resulting extension $H(d, G-v)$ must be unique and isomorphic to G, which completes the proof. □

Lemma 9. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each limb in G is K_2, then $drn(G) = 2$ or 3.*

Proof. By virtue of Lemmas 5 and 8, we can assume that at least two rooted trees in G have different number of limbs. Let d_1 and d_2 be, respectively, the minimum and maximum degree of the root vertices in G. Consider the dacards $(1, G-u)$ and $(1, G-v)$ obtained from G by removing, respectively, a 1-vertex u adjacent to a d_1-vertex and a 1-vertex v adjacent to a d_2-vertex. The dacards we use to reconstruct G are $(1, G-u)$ and $(1, G-v)$. The dacard $(1, G-v)$ forces that every extension must be connected in which each root vertex has degree at least d_1 and $\widetilde{H} \cong \widetilde{G}$. The dacard $(1, G-u)$ has a vertex of degree $d_1 - 1$. Hence $H(1, G-u)$ must be obtained from $(1, G-u)$ by adding a new vertex and joining it to the $(d_1 - 1)$-vertex of $G-u$. Therefore the resulting extension $H(1, G-u)$ is unique and isomorphic to G. Hence $drn(G) \leq 2$. □

Lemma 10. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each limb in G is P_k, where k is a constant greater than one, then $drn(G) = 2$ or 3.*

Proof. In view of Lemma 9, we can assume that $k > 2$. Let d be the minimum degree of root vertices in G. We proceed by two cases depending upon whether $k = 3$ or $k > 3$. For $k = 3$, consider the dacards $(1, G-u)$ and $(2, G-v)$, obtained from G by removing a 1-vertex u and a 2-vertex v, respectively of a limb P_k rooted at a vertex of degree d. The dacard $(1, G-u)$ forces that every extension H of it must be connected and $\widetilde{H} \cong \widetilde{G}$ and that the degree of each root vertex must be at least d. Thus $H(2, G-v)$ must be obtained from $(2, G-v)$ by adding a new vertex and joining it to the unique isolated vertex and to the root vertex of degree $d-1$. The resulting extension $H(2, G-v)$ is therefore unique and it is G.

We now assume that $k > 3$. Consider the dacards $(2, G-u)$ and $(2, G-v)$ obtained from G by removing a 2-vertex u at distance $\left\lceil \frac{k}{2} \right\rceil - 1$ and a 2-vertex v at distance $\left\lceil \frac{k}{2} \right\rceil$, respectively of a limb P_k rooted at a vertex of degree d. The dacard $(2, G-v)$ has a component isomorphic to $P_{k-\lceil \frac{k}{2} \rceil - 1}$ and a component whose pruned graph is $\widetilde{G}$ in which exactly one limb is isomorphic to $P_{\lceil \frac{k}{2} \rceil}$ and the remaining limbs are isomorphic to P_k. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G-u)$ by adding a new vertex x and joining it to two vertices in $G-u$. In $(2, G-u)$, if we join x to the root vertex of degree $d-1$ (when $k = 4$) or to the 1-vertex of the limb $P_{\lceil \frac{k}{2} \rceil - 1}$ (when $k > 4$) and to a 1-vertex of the component $P_{k-\lceil \frac{k}{2} \rceil}$, then $G' \cong G$. If we join x to two vertices of the component whose pruned graph is $\widetilde{G}$, then each dacard $(2, D)$ of the resulting graph G' has a component which is either $P_{k-\lceil \frac{k}{2} \rceil}$ or P_t, $1 \leq t \leq k - \left\lceil \frac{k}{2} \right\rceil - 2$ (when $k > 5$). Since the dacard $(2, G-v)$ has none of these structures, it is not a dacard of G', a contradiction. If we join x to two vertices of the component $P_{k-\lceil \frac{k}{2} \rceil}$, then each dacard $(2, D)$ of the resulting graph G' has at least one of the following structures.

(i) A component K_3 for $k = 4$ and $d = 3$ (when $\widetilde{G'}$ is a cycle).
(ii) A root vertex of degree either $d-1$ or $d-2$.
(iii) A limb $P_{\lceil \frac{k}{2} \rceil - 1}$.
(iv) A component P_t, $1 \leq t \leq \left\lceil \frac{k}{2} \right\rceil - 3$ for $k > 6$.

Since the dacard $(2, G-v)$ has none of these structures, it is not a dacard of G', a contradiction. Finally, if we join x to two vertices one in each component, then each dacard $(2, D)$ of the resulting graph G' has at least one of the following structures.

(i) No cycles for $k = 4$ and $d = 3$ (when $\widetilde{G'}$ is a cycle).
(ii) A root vertex of degree either $d-1$ or $d-2$.
(iii) A limb $P_{\lceil \frac{k}{2} \rceil - 1}$.
(iv) A limb with at least two pendant vertices.
(v) A component P_t, $1 \leq t \leq \left\lceil \frac{k}{2} \right\rceil - 3$ for $k > 6$.
(vi) A limb P_t, $2 \leq t \leq \left\lceil \frac{k}{2} \right\rceil - 4$ for $k > 10$.

Since the dacard $(2, G-v)$ has none of these structures, it is not a dacard of G', a contradiction. Hence $G' \cong G$ and $drn(G) \leq 2$. □

Theorem 11. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each limb in G is a nontrivial path, then $drn(G) = 2$ or 3.*

Proof. In view of Lemma 10, we can assume that the length of at least two limbs are distinct. Let d be the minimum degree of root vertices in G; let $\mathscr{L}$ be the family of all limbs rooted at a vertex of degree d; let s be the minimum order of a limb in $\mathscr{L}$. For $s = 2$, consider the dacards $(1, G - u)$ and $(1, G - v)$ obtained from G by removing the 1-vertex u of a limb P_2 in $\mathscr{L}$ and the 1-vertex v of a limb P_t of maximum length among all the limbs in G. We use these two dacards to reconstruct G. The dacard $(1, G - v)$ forces that every extension H of it must be connected with $\widetilde{H} \cong \widetilde{G}$ and that the degree of each root vertex must be at least d. Hence $H(1, G - u)$ must be obtained from $(1, G - u)$ by adding a new vertex and joining it to the $(d-1)$-vertex. Hence the resulting extension $H(1, G-u)$ must be unique and isomorphic to G. For $s = 3$, the proof is just similar to the case when $k = 3$ in Lemma 10.

For $s > 3$, consider the dacards $(2, G-u)$ and $(2, G-v)$ obtained from G by removing a 2-vertex u at distance $\lceil \frac{s}{2} \rceil - 1$ and a 2-vertex v at distance $\lceil \frac{s}{2} \rceil$, respectively of a limb P_s in $\mathscr{L}$. The number, say k of limbs isomorphic to $P_{\lceil \frac{s}{2} \rceil - 1}$ in $(2, G - u)$ is one more than that in $(2, G - v)$. Also let m be the number of root vertices of degree d in $(2, G - u)$ (when $s > 4$) and in $(2, G - v)$. Then clearly $m - 1$ is the number of root vertices of degree d in $(2, G - u)$ (when $s = 4$). Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G - u)$ by adding a new vertex x and joining it to two vertices in $G - u$. In $(2, G - u)$, if we join x to the root vertex of degree $d - 1$ (when $s = 4$) or to the 1-vertex of the limb $P_{\lceil \frac{s}{2} \rceil - 1}$ ($s > 4$) rooted at a vertex of degree d and to a 1-vertex of the component $P_{s - \lceil \frac{s}{2} \rceil}$, then $G' \cong G$. If we join x to two vertices of the component whose pruned graph is $\widetilde{G}$, then each dacard $(2, D)$ of the resulting graph G' has a component isomorphic to either P_t, $1 \le t \le s - \lceil \frac{s}{2} \rceil - 2$ (when $s > 5$) or $P_{s - \lceil \frac{s}{2} \rceil}$. Since the dacard $(2, G - v)$ has none of these structures, it is not a dacard of G', a contradiction. If we join x to two vertices of the component $P_{s - \lceil \frac{s}{2} \rceil}$, then each $(2, D)$ dacard of the resulting graph G' has at least one of the following structures.

(i) A component K_3 for $s = 4$ and $d = 3$ (when $\widetilde{G'}$ is a cycle).
(ii) A limb $P_{\lceil \frac{s}{2} \rceil - 1}$ rooted at a vertex of degree d.
(iii) A root vertex of degree either $d - 1$ or $d - 2$.
(iv) A component P_t, $1 \le t \le \lceil \frac{s}{2} \rceil - 3$ for $k > 6$.

Since the dacard $(2, G - v)$ has none of these structures, it is not a dacard of G', a contradiction. If we join x to two vertices one in each component, then each dacard $(2, D)$ of the resulting graph G' has at least one of the following structures.

(i) A limb $P_{\lceil \frac{s}{2} \rceil - 1}$ rooted at a vertex of degree d.

(ii) At most $m-1$ root vertices of degree d.

(iii) A limb with at least two pendant vertices.

(iv) At least k limbs isomorphic to $P_{\lceil \frac{s}{2} \rceil - 1}$.

(v) No cycles for $s=4$ and $d=3$ (when $\widetilde{G'}$ is a cycle).

(vi) A component P_t, $1 \le t \le \left\lceil \frac{s}{2} \right\rceil - 3$ for $s > 6$.

(vii) A limb P_t rooted at vertex of degree d, $2 \le t \le \left\lceil \frac{s}{2} \right\rceil - 4$ for $s > 10$.

Since the dacard $(2, G-v)$ has none of these structures, it is not a dacard of G', a contradiction. Hence $G' \cong G$ and $drn(G) \le 2$. □

4. Caterpillar

Here it is proved that if G is a connected graph in which $\widetilde{G}$ is regular, each limb is a broom graph and the root vertex is the special vertex of the broom graph, then $drn(G) = 2$. We begin with a lemma.

Lemma 12. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each limb in G is $B_{l,r}$ and the root vertex is the special vertex of $B_{l,r}$, then $drn(G) = 2$.*

Proof. By virtue of Lemma 10, we can assume that $r > 1$. Let d be the minimum degree of root vertices in G. Consider the dacards $(1, G-u)$ and $(r+1, G-v)$ obtained from G by removing a pendant vertex u and a support vertex v, respectively of a limb $B_{l,r}$ rooted at a vertex of degree d. Let G' be a graph having the dacards $(1, G-u)$ and $(r+1, G-v)$ in its dadeck. Then G' can be constructed from $(1, G-u)$ by adding a new vertex x and joining it to one vertex in $G-u$. In $(1, G-u)$, if we join x to a support vertex of degree r, then G' is isomorphic to G. Otherwise, the removal of any $(r+1)$-vertex in G' would result in a dacard $(r+1, D)$ with at least one of the following structures.

(i) $\widetilde{D} \ncong \widetilde{G'}$.

(ii) A limb which is either P_{l+2} or P_{l+3} for $r = 2$.

(iii) A limb with $r-1$ pendant vertices for $r > 2$.

(iv) A limb with r pendant vertices, removal of exactly one of these pendant vertices results in $B_{l,r-1}$ for $l > 1$.

(v) Two nontrivial components for $l > 1$ and $r = 2$.

Since the dacard $(r+1, G-v)$ has none of these structures, it would not be a dacard of G', a contradiction. Hence $G' \cong G$ and $drn(G) \le 2$. □

Theorem 13. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each limb in G is a broom graph and the root vertex is the special vertex of the broom graph, then $drn(G) = 2$.*

Proof. In view of Theorem 11 and Lemma 12, we can assume that at least two limbs are non-isomorphic and not all of them are paths. Let d be the minimum degree of root vertices in G; let $\mathscr{L}$ be the family of all limbs rooted at a vertex of degree d. Let $B_{l,r}$ be a limb of minimum diameter in $\mathscr{L}$ such that r is minimum. For $l = 1$, consider the dacards $(1, G - u)$ and $(r+1, G-v)$ obtained from G by removing a 1-vertex u and an $(r+1)$-vertex v, respectively of a limb $B_{l,r}$ in $\mathscr{L}$ and proceed as in the case when $k = 3$ in Lemma 10, we get any graph G' having these two dacards in its dadeck is isomorphic to G. For $l > 1$ and $r = 1$, consider the dacards $(2, G - u)$ and $(2, G - v)$ obtained from G by removing a 2-vertex u at distance $\left\lceil \frac{l}{2} \right\rceil$ and a 2-vertex v at distance $\left\lceil \frac{l}{2} \right\rceil + 1$, respectively of a limb P_{l+2} rooted at a vertex of degree d and proceed as in the case when $s > 3$ in Theorem 11, we get any graph G' having these two dacards in its dadeck is isomorphic to G. The proof is just similar to the proof of Theorem 11. If $l > 1$ and $r > 1$, consider the dacards $(1, G - u)$ and $(r + 1, G - v)$ obtained from G by removing a 1-vertex u and an $(r+1)$-vertex v, respectively of a limb $B_{l,r}$ in $\mathscr{L}$. In $(1, G-u)$ and $(r+1, G-v)$, let the number of limbs isomorphic to $B_{l,r}$ be k and the number of root vertices of degree d be m; the number, say t of limbs isomorphic to P_l in $(r + 1, G - v)$ is clearly one more than that in $(1, G - u)$. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(1, G - u)$ by adding a new vertex x and joining it to one vertex in $G - u$. In $(1, G - u)$, if we join x to the 1-vertex of a limb $B_{l,r-1}$ rooted at a vertex of degree d, then $G' \cong G$. Otherwise, the removal of any $(r + 1)$-vertex in G' would result in a dacard $(r + 1, D)$ with at least one of the following structures.

(i) $\widetilde{D} \ncong \widetilde{G'}$.
(ii) A limb $B_{l,r-1}$ rooted at a vertex of degree d.
(iii) A limb with r pendant vertices, removal of exactly one of these pendant vertices results in $B_{l,r-1}$ rooted at a vertex of degree d.
(iv) Two nontrivial components.
(v) $k - 1$ limbs isomorphic to $B_{l,r}$ or $t - 1$ limbs isomorphic to P_l.
(vi) $m - 1$ root vertices of degree d.

Since the dacard $(r + 1, G - v)$ has none of these structures, it would not be a dacard of G', a contradiction. Hence $G' \cong G$ and $drn(G) \leq 2$. □

Barrus and West [1] have proved that $drn(G) = 2$ for all caterpillars G except stars and one example on 6 vertices (Figure 2). Here it is shown that $drn(G) = 2$ for some connected graphs G in which $\widetilde{G}$ is regular and each rooted tree is a caterpillar.

Theorem 14. *Let G be a connected graph whose $\widetilde{G}$ is regular. If every rooted tree in G is a caterpillar and the root vertex is a support vertex in a longest path of the caterpillar, then $drn(G) = 2$.*

Proof. By virtue of Lemma 9 and Theorem 11, we can assume that at least one caterpillar is neither a star nor a path.

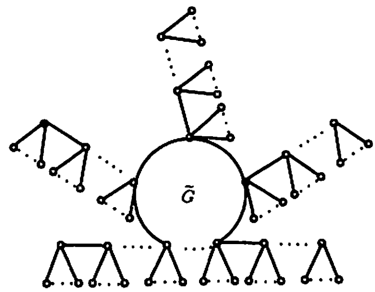


Figure 4. The graph G stated in Theorem 14

Consider the dacards $(1, G-u)$ and $(1, G-v)$ obtained from G by removing, respectively, a 1-vertex whose support vertex is a root vertex u of degree d and a 1-vertex v at a maximum distance from the root vertex of a caterpillar with maximum diameter, where d is the minimum degree of root vertices of G. We use these two dacards to determine G. The dacard $(1, G-v)$ forces that every extension H of it must be connected with $\widetilde{H} \cong \widetilde{G}$ and that the degree of each root vertex must be at least d. But the dacard $(1, G-u)$ has a vertex of degree $d-1$. Hence $H(1, G-u)$ must be obtained from $(1, G-u)$ by joining the newly added vertex x to the root vertex of degree $d-1$ in $G-u$. Hence the resulting extension $H(1, G-u)$ must be unique and isomorphic to G, which completes the proof. □

5. Conclusion

In the above sections, we have proved that drn of a connected graph G in which $\widetilde{G}$ is regular, is 2 or 3 if one of the following holds.

(i) Every limb is a nontrivial path.

(ii) Every limb is a broom graph and the root vertex is the special vertex of the broom graph.

(iii) Every rooted tree is a caterpillar and the root vertex is a support vertex in a longest path of the caterpillar.

(iv) Every rooted tree is not a path and has exactly one 1-vertex at a distance n and all other 1-vertices are at distance at most $n-2$ from the corresponding root vertex.

By this, in view of Lemma 5, we believe that the following problem can be true.

Problem 1. *Let G be a connected graph such that $\widetilde{G}$ is regular. If each rooted tree in G is a nontrivial tree, then $drn(G) = 2$ except when $\widetilde{G}$ is a cycle and all rooted trees are K_2.*

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