On the incidence energy of graphs under three graph decorations

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Abstract In this paper, we obtain that the characteristic polynomials of the signless Laplacian matrix of $Q(G)$, $R(G)$, $T(G)$ can be expressed in terms of the characteristic polynomial of G when G is a regular or semiregular graph, from which upper bounds for the incidence energy of $Q(G)$, $R(G)$, $T(G)$ are deduced.

Keywords Incidence energy; Middle graph; Total graph
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1 Introduction

Let G be a simple graph with order n and $A(G)$ be the adjacency matrix of G with eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $D(G) = diag(d_1, d_2, \cdots, d_n)$ be the diagonal matrix of vertex degree. The Laplacian matrix $L(G)$ of G is $L(G) = D(G) - A(G)$ with eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$, and the signless Laplacian matrix $L^+(G)$ of G is $L^+(G) = D(G) + A(G)$ with eigenvalues ${\mu_1}^+(G) \geq {\mu_2}^+(G) \geq \cdots \geq {\mu_n}^+(G)$. All eigenvalues of both $L(G)$ and $L^+(G)$ are real and non-negative. If G is a connected bipartite graph, then $\mu_i(G) > 0$ for $i = 1, 2, ..., n-1$

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and $\mu_n(G) = 0$ [16]. If G is a connected nonbipartite graph, then $\mu_i(G) > 0$ for $i = 1, 2, ..., n$ [1]. Denote by $P_G(x)$ ($P_{L^+(G)}(x)$) the characteristic polynomial of the adjacency matrix (the signless Laplacian matrix) of the graph G. It is known that $P_{L^+(G)}(x) = det(xI - D(G) - A(G))$, $P_G(x) = det(xI - A(G))$.

The energy of G was defined by I. Gutman in [3] as

$$E(G) = \sum_{j=1}^{n} |\lambda_j|.$$

This quantity has a clear connection to chemical problems and for more details on graph energy see the reviews [4,5,9,10,12-14,18,21]. Nikiforov [17] recently extended the concept of energy to all (not necessarily square) matrices, defining the energy of a matrix M as the sum of the singular values of M. Recall that the singular values of a matrix M are equal to the square roots of the eigenvalues of the (square) matrix MM^t.

In line with Nikiforov's idea, Jooyandeh et al. [11] introduced the incidence energy $IE(G)$ of a graph G, $IE(G)$ was defined as the energy of its incidence matrix. Gutman et al. [6] showed that

$$IE(G) = \sum_{j=1}^{n} \sqrt{\mu_j^+(G)}.$$

More details on incidence energy can be found in [6,7,11].

Suppose G is a connected graph. We can define three related graph subdivision graph $S(G)$, the middle graph $Q(G)$, the total graph $T(G)$ as follows [2]:

Subdivision graph: $S(G)$ is the graph obtained from G by inserting an additional vertex into each edge of G.
The middle graph: $Q(G)$ is the graph obtained from G by inserting a new vertex (of degree 2) on each edge of G and then joining by edges those pairs of the new added vertices which lie on adjacent edges of G.
The total graph: $T(G)$ is the graph obtained from $Q(G)$ by joining pairs of vertices which are adjacent in G by edges.

One extra subdivision-related graphs named $R(G)$ which is the graph obtained from G by adding, for each edge uv, a new vertex whose neighbours are u and v.

Various (upper and lower) bounds on the energy and incidence energy have been established. Z. Liu [15] obtained upper bounds for the energy and Laplacian energy of line graph, middle graph and total graph. B. Zhou [20] obtained the upper bounds for the incidence energy using the first Zagreb index. I. Gutman *et al.* [7] gave several lower and upper bounds for incidence energy. In particular, W. Wang *et al.* [19] gave upper bounds for the incidence energy of the line graph of a semiregular graph and the paraline graph of a regular graph.

In this paper, we obtain that the characteristic polynomials of the signless Laplacian matrix of $Q(G)$, $R(G)$, $T(G)$ can be expressed in terms of the characteristic polynomial of G when G is a regular or semiregular graph, from which upper bounds for the incidence energy of $Q(G)$, $R(G)$, $T(G)$ are deduced.

2 Main result

Let G be a simple graph with n vertices and m edges. Let $B(G)$ be the incident matrix of G. It is known that $B(G)B(G)^t = D(G) + A(G)$ and $B(G)^t B(G) = 2I_m + A(L(G))$ [2]. The following result is well known [8].

Lemma 1 [8] *Let A be a matrix. Then, the matrices AA^t and A^tA have the same nonzero eigenvalues.*

Lemma 2 [8] *If M is a non-singular square matrix, then we have*
$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|$.

Theorem 3 *Let G be an r-regular graph with n vertices and m edges. If the eigenvalues of $A(G)$ are $\lambda_1 = r$, λ_2, $\cdots$, λ_n, then*

$$P_{L^+(R(G))}(x) = (x-2)^{m-n}\Pi_{i=1}^n((x^2 - 2(r+1)x + 3r) - (x-1)\lambda_i)$$

$$= (x-2)^m P_G(\tfrac{x^2-2(r+1)x+3r}{x-1}).$$

Proof. Let A be the adjacency matrix of G and B be the incident matrix of G. By the definition of $R(G)$, we have $A(R(G)) = \begin{pmatrix} 0 & B^t \\ B & A \end{pmatrix}$, $D(R(G)) = \begin{pmatrix} 2I_m & 0 \\ 0 & 2rI_n \end{pmatrix}$, so

$$P_{L^+(R(G))}(x) = \begin{vmatrix} (x-2)I_m & -B^t \\ -B & (x-2r)I_n - A \end{vmatrix}$$

$$= (x-2)^m |(x-2r)I_n - A - \tfrac{1}{x-2}BB^t|$$

$$= (x-2)^{m-n}(x-1)^n |\tfrac{(x^2-2(r+1)x+3r)}{x-1}I_n - A|$$

$$= (x-2)^{m-n}(x-1)^n P_G(\tfrac{x^2-2(r+1)x+3r}{x-1}).$$

It follows that if $\lambda_1 = r, \cdots, \lambda_n$ are the eigenvalues of $A(G)$, then

$$P_{L^+(R(G))}(x) = (x-2)^{m-n}\Pi_{i=1}^n((x^2 - 2(r+1)x + 3r) - (x-1)\lambda_i).$$

Theorem 4 *Let G be an r-regular connected graph with n vertices and m edges. Then*

$$IE(R(G)) \le (m-n)\sqrt{2} + \sqrt{3r+2+4\sqrt{r}}$$

$$+\sqrt{2(n-1)^2(r+1) - (n-1)r + 2(n-1)\sqrt{3(n-1)^2r - (n-1)r}},$$

the equality holds if and only if $G \cong K_n$.

Proof. Let x_{i1} and x_{i2} are the roots of the equation $x^2 - 2(r+1)x + 3r - (x-1)\lambda_i = 0$, by Vita theorem,

$(\sqrt{x_{i1}} + \sqrt{x_{i2}})^2 = x_{i1} + x_{i2} + 2\sqrt{x_{i1}x_{i2}} = 2(r+1) + \lambda_i + 2\sqrt{3r+\lambda_i}.$

By Theorem 3, $L^+(R(G))$ has $m-n$ eigenvalues equal to 2 and $2n$ eigenvalues: x_{i1}, x_{i2} $(i = 1, 2, \cdots, n)$. Thus

$IE(R(G)) = (m-n)\sqrt{2} + \sqrt{3r+2+4\sqrt{r}} + \sum_{i=2}^n(\sqrt{x_{i1}} + \sqrt{x_{i2}})$

$= (m-n)\sqrt{2} + \sqrt{3r+2+4\sqrt{r}} + \sum_{i=2}^n \sqrt{2(r+1) + \lambda_i + 2\sqrt{3r+\lambda_i}}.$

Note that $\lambda_1 = r$, $\sum_{i=1}^n \lambda_i = 0$, by the Cauchy-Schwarz inequality, we have

$\sum_{i=2}^n(\sqrt{2(r+1) + \lambda_i + 2\sqrt{3r+\lambda_i}})$

$\le \sqrt{(n-1)\sum_{i=2}^n(2(r+1) + \lambda_i + 2\sqrt{3r+\lambda_i})}$

$\le \sqrt{2(n-1)^2(r+1) - (n-1)r + 2(n-1)\sqrt{3(n-1)^2r - (n-1)r}}.$

Thus, we have

$$IE(R(G)) \le (m-n)\sqrt{2} + \sqrt{3r+2+4\sqrt{r}}$$

$$+\sqrt{2(n-1)^2(r+1) - (n-1)r + 2(n-1)\sqrt{3(n-1)^2r - (n-1)r}}.$$

Equality in the above holds iff $\lambda_2 = \lambda_3 = \cdots = \lambda_n$. Then, the number of distinct eigenvalues of $A(G)$ is at most 2. From the result "the number of distinct eigenvalues of a connected graphs with diameter d is at least

$d+1$ [1]" , we know that the diameter of G is at most 1, then G must be K_n. Conversely, if $G \cong K_n$, by the above calculation, the equality holds. Thus, we complete the proof of Theorem 4.

Theorem 5 *Let G be an r-regular graph with n vertices and m edges. If the eigenvalues of $A(G)$ are $\lambda_1 = r, \lambda_2, \cdots, \lambda_n$, then*

$$P_{L^+(Q(G))}(x) = (x-2r+2)^{m-n}\Pi_{i=1}^{n}(x^2-(4r+\lambda_i-2)x+(3r+\lambda_i)(r-1))$$

$$= (x-2r+2)^{m-n}(x-r+1)^n P_G(\frac{x^2-4rx+2x+3r^2-3r}{x-r+1}).$$

Proof. Let B be the incident matrix of G, $L(G)$ be the line graph of G. By the definition of $Q(G)$, $A(Q(G)) = \begin{pmatrix} 0 & B \\ B^t & A(L(G)) \end{pmatrix}$,
$D(Q(G)) = \begin{pmatrix} rI_n & 0 \\ 0 & 2rI_m \end{pmatrix}$, we have

$$P_{L^+(Q(G))}(x) = \begin{vmatrix} (x-r)I_n & -B \\ -B^t & (x-2r)I_m - A(L(G)) \end{vmatrix}$$

$$= \begin{vmatrix} (x-r)I_n & 0 \\ -B^t & (x-2r)I_m - A(L(G)) - \frac{1}{x-r}B^tB \end{vmatrix}$$

$$= (x-r)^n|(x-2r)I_m - A(L(G)) - \tfrac{2}{x-r}I_m - \tfrac{1}{x-r}A(L(G))|.$$

It follows that if $\lambda_1 = r, \cdots, \lambda_n$ are the eigenvalues of $A(G)$, then $\lambda_1+r-2, \cdots, \lambda_n+r-2, -2, \cdots, -2$ are the eigenvalues of $A(L(G))$, thus

$$P_{L^+(Q(G))}(x) = (x-2r+2)^{m-n}\Pi_{i=1}^{n}(x^2-(4r+\lambda_i-2)x+(3r+\lambda_i)(r-1))$$
$$= (x-2r+2)^{m-n}(x-r+1)^n P_G(\frac{x^2-4rx+2x+3r^2-3r}{x-r+1}).$$

Theorem 6 *Let G be an r-regular connected graph with n vertices and m edges. Then*

$$IE(Q(G)) \leq (m-n)\sqrt{2} + \sqrt{3r+2+4\sqrt{r}}$$

$$+\sqrt{2(n-1)^2(r+1)-(n-1)r+2(n-1)\sqrt{3(n-1)^2r-(n-1)r}}.$$

the equality holds if and only if $G \cong K_n$.

Proof. Let x_{i1} and x_{i2} are the roots of the equation $x^2 - (4r + \lambda_i - 2)x + (3r + \lambda_i)(r - 1) = 0$, by Vita theorem,
$(\sqrt{x_{i1}} + \sqrt{x_{i2}})^2 = x_{i1} + x_{i2} + 2\sqrt{x_{i1}x_{i2}} = 4r + \lambda_i - 2 + 2\sqrt{(3r + \lambda_i)(r - 1)}$.

By Theorem 5, $L^+(Q(G))$ has $m-n$ eigenvalues equal to $2r-2$ and $2n$ eigenvalues: x_{i1}, x_{i2} $(i = 1, 2, \cdots, n)$. Thus, $IE(Q(G)) = (m-n)\sqrt{2r-2} + \sqrt{5r - 2 + 4\sqrt{r(r-1)}} + \sum_{i=2}^{n} \sqrt{4r + \lambda_i - 2 + 2\sqrt{(3r + \lambda_i)(r - 1)}}$.

Note that $\lambda_1 = r$, $\sum_{i=1}^{n} \lambda_i = 0$, by the Cauchy-Schwarz inequality, we have

$$\sum_{i=2}^{n} \sqrt{4r + \lambda_i - 2 + 2\sqrt{(3r + \lambda_i)(r - 1)}}$$

$$\leq \sqrt{(n-1)\sum_{i=2}^{n}((4r + \lambda_i - 2) + 2\sqrt{(3r + \lambda_i)(r - 1)})}$$

$$\leq \sqrt{2(n-1)^2(2r-1) - (n-1)r + 2(n-1)\sqrt{(3n-4)(n-1)(r-1)r}}.$$

Thus, we have

$$IE(Q(G)) \leq (m-n)\sqrt{2r-2} + \sqrt{5r - 2 + 4\sqrt{r(r-1)}}$$

$$+\sqrt{2(n-1)^2(2r-1) - (n-1)r + 2(n-1)\sqrt{(3n-4)(n-1)(r-1)r}}.$$

Similar to the argument in Theorem 4, equality in the above holds iff $G \cong K_n$.

Theorem 7 *Let G be an r-regular graph with n vertices and m edges. If the eigenvalues of $A(G)$ are $\lambda_1 = r$, $\lambda_2, \cdots, \lambda_n$, then* $P_{L^+(T(G))}(x)$

$$= (x - 2r + 2)^{m-n}\Pi_{i=1}^{n}((x^2 + (2 - 2\lambda_i - 5r)x + 6r^2 - 5r + 5r\lambda_i - 3\lambda_i + {\lambda_i}^2)).$$

Proof. Let B be the incident matrix of G, $L(G)$ be the line graph of G. By the definition of $T(G)$, $A(T(G)) = \begin{pmatrix} A(G) & B \\ B^t & A(L(G)) \end{pmatrix}$, $D(T(G)) = 2rI_{m+n}$, we have

$$P_{L^+(T(G))} = \begin{vmatrix} (x-2r)I_n - A(G) & -B \\ -B^t & (x - 2r + 2)I_m - B^tB \end{vmatrix}$$

$$= \begin{vmatrix} (x-2r)I_n - A(G) & -B(G) \\ -(x-2r+1)B^t + B^t A(G) & (x-2r+2)I_m \end{vmatrix}$$

$$= (x-2r+2)^{m-n} |((x-2r)(x-2r+2) - r(x-2r+1))I_n + (5r-2x-3)A + A^2|.$$

Thus if $\lambda_1 = r, \cdots, \lambda_n$ are the eigenvalues of $A(G)$, then $P_{L^+(T(G))}(x)$

$$= (x-2r+2)^{m-n} \Pi_{i=1}^n ((x^2 + (2-2\lambda_i - 5r)x + 6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2)).$$

Theorem 8 *Let G be an r-regular connected graph with n vertices and m edges. Then $IE(T(G)) \leq (m-n)\sqrt{2r-2} + \sqrt{7r - 2 + 2\sqrt{12r^2 - 8r}} +$*

$$\sqrt{(n-1)(5rn - 2n - 7r + 2) + 2\sqrt{(n-1)^3((6n-12)r^2 - (5n-8)r + 2m)}}.$$

the equality holds if and only if $G \cong K_n$.

Proof. Let x_{i1} and x_{i2} are the roots of the equation $x^2 + (2 - 2\lambda_i - 5r)x + 6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2 = 0$, by Vita theorem, $(\sqrt{x_{i1}} + \sqrt{x_{i2}})^2$

$$= x_{i1} + x_{i2} + 2\sqrt{x_{i1}x_{i2}} = -2 + 2\lambda_i + 5r + 2\sqrt{6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2}.$$

By Theorem 7, $L^+(T(G))$ has $m-n$ eigenvalues equal to $2r-2$ and $2n$ eigenvalues: x_{i1}, x_{i2} $(i = 1, 2, \cdots, n)$. Thus $IE(T(G)) = (m-n)\sqrt{2r-2} + \sqrt{7r - 2 + 2\sqrt{12r^2 - 8r}}$

$$+ \sum_{i=2}^n \sqrt{-2 + 2\lambda_i + 5r + 2\sqrt{6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2}}.$$

Note that $\lambda_1 = r$, $\sum_{i=1}^n \lambda_i = 0$, $\sum_{i=1}^n \lambda_i^2 = 2m = nr$ [16], by the Cauchy-Schwarz inequality, we have

$$\sum_{i=2}^n \sqrt{-2 + 2\lambda_i + 5r + 2\sqrt{6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2}}$$

$$\leq \sqrt{(n-1)\sum_{i=2}^n ((5r + 2\lambda_i - 2) + 2\sqrt{6r^2 - 5r + 5r\lambda_i - 3\lambda_i + \lambda_i{}^2})} \leq$$

$$\sqrt{(n-1)(5rn - 2n - 7r + 2) + 2\sqrt{(n-1)^3((6n-12)r^2 - (5n-8)r + 2m)}}.$$

Thus, we have

$$IE(T(G)) \leq (m-n)\sqrt{2r-2} + \sqrt{7r - 2 + 2\sqrt{12r^2 - 8r}} +$$

$$\sqrt{(n-1)(5rn - 2n - 7r + 2) + 2\sqrt{(n-1)^3((6n-12)r^2 - (5n-8)r + 2m)}}.$$

Similar to the argument in Theorem 4, equality in the above holds iff $G \cong K_n$.

A bipartite graph G with a bipartition $V(G) = U \cup W$ is called a semiregular graph if all vertices in U have degree r_1 and all vertices in W have degree r_2.

Theorem 9 *Let G be a semiregular graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 \geq n_2$. If the eigenvalues of $A(G)$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then*

$$P_{L^+(R(G))}(x) = (x-2)^{m-n}(x^2-2(r_1+1)x+3r_1)^{n_1-n_2}$$
$$\Pi_{i=1}^{n_2}((x^2-2(r_1+1)x+3r_1)(x^2-2(r_2+1)x+3r_2)-(x-1)^2\lambda_i^2)$$
$$= (x-2)^{m-n}(x-1)^n$$
$$\left(\tfrac{x^2-2(r_1+1)x+3r_1}{x^2-2(r_2+1)x+3r_2}\right)^{(n_1-n_2)/2} P_G\left(\sqrt{\tfrac{(x^2-2(r_1+1)x+3r_1)(x^2-2(r_2+1)x+3r_2)}{(x-1)^2}}\right),$$

where $n = n_1 + n_2$ and m is the number of edges of G.

Proof. Let B be the incident matrix of G. By the definition of $R(G)$, Since $A(R(G)) = \begin{pmatrix} 0 & B^t \\ B & A \end{pmatrix}$, $D(R(G)) = \begin{pmatrix} 2I_m & 0 \\ 0 & 2D(G) \end{pmatrix}$, we have

$$P_{L^+(R(G))} = \begin{vmatrix} (x-2)I_m & -B^t \\ -B & xI_n - A - 2D(G) \end{vmatrix}$$
$$= \begin{vmatrix} (x-2)I_m & 0 \\ -B & xI_n - A - \frac{1}{x-2}BB^t - 2D(G) \end{vmatrix}$$

$$= (x-2)^m |xI_n - \tfrac{x-1}{x-2}A - \tfrac{2x-3}{x-2}D(G)|$$

$$= (x-2)^{m-n}|(x^2-2x)I_n - (x-1)A - (2x-3)D(G)|.$$

Since $D(G) = \begin{pmatrix} r_1 I_{n_1} & 0 \\ 0 & r_2 I_{n_2} \end{pmatrix}$, $A(G) = \begin{pmatrix} 0 & K^t \\ K & 0 \end{pmatrix}$, then

$$|(x^2-2x)I_n - (x-1)A - (2x-3)D(G)|$$

$$= \begin{vmatrix} (x(x-2)-(2x-3)r_1)I_{n_1} & -(x-1)K^t \\ -(x-1)K & (x(x-2)-(2x-3)r_2)I_{n_2} \end{vmatrix}$$

$$= (x(x-2)-(2x-3)r_1)^{n_1}|(x(x-2)-(2x-3)r_2)I_{n_2} - \tfrac{(x-1)^2}{x(x-2)-(2x-3)r_1}KK^t|$$

$$= (x(x-2)-(2x-3)r_1)^{n_1-n_2}|(x(x-2)-(2x-3)r_1)(x(x-2)-(2x-3)r_2)I_{n_2}$$
$$-(x-1)^2KK^t|.$$

Since $A^2 = \begin{pmatrix} K^tK & 0 \\ 0 & KK^t \end{pmatrix}$, KK^t and K^tK have the same non-zero eigenvalues, it follows that if $\lambda_1, \cdots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of $A(G)$, then $\lambda_1^2, \lambda_2^2, \cdots, \lambda_{n_2}^2$ are the eigenvalues of KK^t, thus

$$P_{L^+(R(G))}(x) = (x-2)^{m-n}(x^2 - 2(r_1+1)x + 3r_1)^{n_1-n_2}$$

$$\Pi_{i=1}^{n_2}((x^2-2(r_1+1)x+3r_1)(x^2-2(r_2+1)x+3r_2)-(x-1)^2\lambda_i^2)$$

$$= (x-2)^{m-n}(x-1)^n\left(\frac{x^2-2(r_1+1)x+3r_1}{x^2-2(r_2+1)x+3r_2}\right)^{(n_1-n_2)/2}$$
$$P_G\left(\sqrt{\frac{(x^2-2(r_1+1)x+3r_1)(x^2-2(r_2+1)x+3r_2)}{(x-1)^2}}\right).$$

Theorem 10 *Let G be a semiregular connected graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 > n_2$. Then*

$$IE(R(G)) < (m-n)\sqrt{2}+(n_1-n_2)\sqrt{2(r_1+1)+2\sqrt{3r_1}}+2n_2\sqrt{2(r_1+r_2+2)}.$$

where $n = n_1 + n_2$ and m is the number of edges of G.

Proof. Let x_{i1}, x_{i2}, x_{i3} and x_{i4} are the roots of the equation $(x^2 - 2(r_1 + 1)x + 3r_1)(x^2 - 2(r_2+1)x + 3r_2) - (x-1)^2\lambda_i^2 = 0$, by the Cauchy-Schwarz inequality and Vita theorem,

$$\sqrt{x_{i1}}+\sqrt{x_{i2}}+\sqrt{x_{i3}}+\sqrt{x_{i4}} \leq \sqrt{4(x_{i1}+x_{i2}+x_{i3}+x_{i4})} = 2\sqrt{2(r_1+r_2+2)}.$$

Let y_1 and y_2 are the roots of the equation $x^2 - 2(r_1+1)x + 3r_1 = 0$, by Vita theorem,

$$(\sqrt{y_1}+\sqrt{y_2})^2 = y_1 + y_2 + 2\sqrt{y_1y_2} = 2(r_1+1) + 2\sqrt{3r_1}.$$

By Theorem 9, $L^+(R(G))$ has $m-n$ eigenvalues equal to 2, $n_1 - n_2$ eigenvalues equal to y_1, n_1-n_2 eigenvalues equal to y_2 and $4n_2$ eigenvalues: $x_{i1}, x_{i2}, x_{i3}, x_{i4}$ $(i = 1, 2, \cdots, n_2)$. Thus

$$IE(R(G)) \leq (m-n)\sqrt{2}+(n_1-n_2)\sqrt{2(r_1+1)+2\sqrt{3r_1}}+2n_2\sqrt{2(r_1+r_2+2)}.$$

If equality in the above holds then $x_{i1} = x_{i2} = x_{i3} = x_{i4} = \frac{r_1+r_2+2}{2}$ $(i = 1, 2, \cdots, n_2)$ and then $\lambda_1 = \lambda_2 = \cdots = \lambda_{n_2}$. Thus the number of distinct eigenvalues of G is at most 3. Then the diameter of G is at most 2. If the diameter of G is 1, then G must be K_2. By some simple calculation, the equality would not be hold. If the diameter of G is 2, then G must be K_{n_1,n_2}. It is known that the eigenvalues of K_{n_1,n_2} is $0^{n_1-n_2}$, $\sqrt{n_1n_2}^{n_2}$, $-\sqrt{n_1n_2}^{n_2}$. By some simple calculation, $(x^2-2(n_1+1)x+3n_1)(x^2-2(n_2+$

$1)x + 3n_2) - (x-1)^2 n_1 n_2$ is not equal to $(x - \frac{n_1+n_2+2}{2})^4$, thus the equality also would not be hold. So

$$IE(R(G)) < (m-n)\sqrt{2}+(n_1-n_2)\sqrt{2(r_1+1)+2\sqrt{3r_1}}+2n_2\sqrt{2(r_1+r_2+2)}.$$

Theorem 11 *Let G be a semiregular graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 > n_2$. If the eigenvalues of $A(G)$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then*

$$P_{L^+(Q(G))} = (x - r_1 - r_2 + 2)^{m-n}(x^2 - (3r_1 + r_2 - 2)x + r_1(2r_1 + r_2 - 3))^{n_1-n_2} \prod_{i=1}^{n_2}((x^2 - (3r_1 + r_2 - 2)x + r_1(2r_1 + r_2 - 3))(x^2 - (3r_2 + r_1 - 2)x + r_2(2r_2 + r_1 - 3)) - (x + 1 - r_1)(x + 1 - r_2)\lambda_i^2)$$

$$= (x - r_1 - r_2 + 2)^{m-n}(x + 1 - r_1)^{\frac{n}{2}}(x + 1 - r_2)^{\frac{n}{2}}$$

$$\left(\frac{x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3)}{x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3)}\right)^{(n_1-n_2)/2}$$

$$P_G\left(\sqrt{\frac{(x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))(x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3))}{(x+1-r_1)(x+1-r_2)}}\right),$$

where m is the number of edges of G.

Proof. Since $A(Q(G)) = \begin{pmatrix} 0 & B \\ B^t & A(L(G)) \end{pmatrix}$, $D(Q(G)) = \begin{pmatrix} D(G) & 0 \\ 0 & (r_1 + r_2)I_m \end{pmatrix}$, we have

$$P_{L^+(Q(G))} = \begin{vmatrix} xI_n - D(G) & -B \\ -B^t & (x - r_1 - r_2 + 2)I_m - B^tB \end{vmatrix}$$

$$= \begin{vmatrix} xI_n - D(G) & -B \\ -B^t((x+1)I_n - D(G)) & (x - r_1 - r_2 + 2)I_m \end{vmatrix} =$$

$$\begin{vmatrix} xI_n - D(G) - \frac{1}{x-r_1-r_2+2}BB^t((x+1)I_n - D(G)) & 0 \\ -B^t((x+1)I_n - D(G)) & (x - r_1 - r_2 + 2)I_m \end{vmatrix}$$

$$= (x - r_1 - r_2 + 2)^{m-n}|x(x - r_1 - r_2 + 2)I_n - (2x - r_1 - r_2 + 3)D(G) - (x+1)A + D(G)^2 + A(G)D(G)|.$$

Since $D(G) = \begin{pmatrix} r_1I_{n_1} & 0 \\ 0 & r_2I_{n_2} \end{pmatrix}$, $A(G) = \begin{pmatrix} 0 & K^t \\ K & 0 \end{pmatrix}$, then $AD = \begin{pmatrix} 0 & r_2K^t \\ r_1K & 0 \end{pmatrix}$, thus $|x(x-r_1-r_2+2)I_n-(2x-r_1-r_2+3)D(G)-(x+1)A+D(G)^2+A(G)D(G)|=$

$= (x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))^{n_1-n_2}|(x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))(x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3))I_{n_2}$

$-(x+1-r_1)(x+1-r_2)K^tK|$.

Similar to the above theorem, it follows that if $\lambda_1, \cdots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of $A(G)$, then $\lambda_1^2, \lambda_2^2, \cdots, \lambda_{n_2}^2$ are the eigenvalues of KK^t, thus

$P_{L^+(Q(G))} = (x-r_1-r_2+2)^{m-n}(x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))^{n_1-n_2}$

$\Pi_{i=1}^{n_2}((x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))(x^2-(3r_2+r_1-2)x+$

$r_2(2r_2+r_1-3))-(x+1-r_1)(x+1-r_2)\lambda_i^2)$

$= (x-r_1-r_2+2)^{m-n}(x+1-r_1)^{\frac{n}{2}}(x+1-r_2)^{\frac{n}{2}}$

$(\frac{x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3)}{x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3)})^{(n_1-n_2)/2}$

$P_G(\sqrt{\frac{(x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))(x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3)}{(x+1-r_1)(x+1-r_2)}})$.

Theorem 12 *Let G be a semiregular connected graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 > n_2$. Then*

$$IE(Q(G)) < (m-n)\sqrt{r_1+r_2-2}$$
$$+(n_1-n_2)\sqrt{3r_1+r_2-2+2\sqrt{r_1(2r_1+r_2-3)}}+4n_2\sqrt{r_1+r_2-1}.$$

where m is the number of edges of G.

Proof. Let x_{i1}, x_{i2}, x_{i3} and x_{i4} are the roots of the equation $(x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3))(x^2-(3r_2+r_1-2)x+r_2(2r_2+r_1-3))-(x+1-r_1)(x+1-r_2)\lambda_i^2=0$, by the Cauchy-Schwarz inequality and Vita theorem,

$$\sqrt{x_{i1}}+\sqrt{x_{i2}}+\sqrt{x_{i3}}+\sqrt{x_{i4}} \le \sqrt{4(x_{i1}+x_{i2}+x_{i3}+x_{i4})} = 4\sqrt{r_1+r_2-1}.$$

Let y_1 and y_2 are the roots of the equation $x^2-(3r_1+r_2-2)x+r_1(2r_1+r_2-3)=0$, by Vita theorem,

$$(\sqrt{y_1}+\sqrt{y_2})^2 = y_1+y_2+2\sqrt{y_1y_2} = 3r_1+r_2-2+2\sqrt{r_1(2r_1+r_2-3)}.$$

By Theorem 11, $L^+(Q(G))$ has $m-n$ eigenvalues equal to r_1+r_2-2, n_1-n_2 eigenvalues equal to y_1, n_1-n_2 eigenvalues equal to y_2 and $4n_2$ eigenvalues: x_{i1}, x_{i2}, x_{i3}, x_{i4} $(i=1,2,\cdots,n_2)$. Thus

$$IE(Q(G)) \leq (m-n)\sqrt{r_1+r_2-2}$$
$$+(n_1-n_2)\sqrt{3r_1+r_2-2+2\sqrt{r_1(2r_1+r_2-3)}}+4n_2\sqrt{r_1+r_2-1}.$$

Similar to the argument in Theorem 10, the equality would not be hold. So

$$IE(Q(G)) < (m-n)\sqrt{r_1+r_2-2}+(n_1-n_2)$$
$$\sqrt{3r_1+r_2-2+2\sqrt{r_1(2r_1+r_2-3)}}+4n_2\sqrt{r_1+r_2-1}.$$

Hence, we complete the proof of Theorem 12.

Theorem 13 *Let G be a semiregular graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 > n_2$. If the eigenvalues of $A(G)$ are λ_1, λ_2, $\cdots$, λ_n, then*

$$P_{L^+(T(G))} = (x-r_1-r_2+2)^{m-n}((x-r_2+2)(x-2r_1)+r_1)^{n_1-n_2}$$

$$\Pi_{i=1}^{n_2}((x-r_2+2)(x-2r_1)+r_1)((x-r_1+2)(x-2r_2)+r_2)$$

$$-((x-r_2+2)(x-2r_1)+r_1)\lambda_i^2-(r_1+2)^2\lambda_i^2-((x-r_1+2)(x-2r_2)+r_2)\lambda_i^2+\lambda_i^4).$$

where $n=n_1+n_2$ and m is the number of edges of G.

Proof. Since $A(T(G)) = \begin{pmatrix} A(G) & B \\ B^t & A(L(G)) \end{pmatrix}$,

$D(T(G)) = \begin{pmatrix} 2D(G) & 0 \\ 0 & (r_1+r_2)I_m \end{pmatrix}$, we have

$$P_{L^+(T(G))} = \begin{vmatrix} xI_n - A(G) - 2D(G) & -B \\ -B^t & (x-r_1-r_2+2)I_m - B^tB \end{vmatrix}$$

$$= \begin{vmatrix} xI_n - A(G) - 2D(G) & -B \\ -B^t((x+1)I_n - A - 2D) & (x-r_1-r_2+2)I_m \end{vmatrix}$$

$$= (x-r_1-r_2+2)^{m-n}|((x-r_1-r_2+2)I_n-D-A)(xI_n-A-2D)-A-D|.$$

Since $D(G) = \begin{pmatrix} r_1I_{n_1} & 0 \\ 0 & r_2I_{n_2} \end{pmatrix}$, $A(G) = \begin{pmatrix} 0 & K^t \\ K & 0 \end{pmatrix}$, then $A+D=\begin{pmatrix} r_1 I_{n_1} & K^t \\ K & r_2 I_{n_2} \end{pmatrix}$,

thus $|((x-r_1-r_2+2)I_n-A-D)(xI_n-A-2D)-A-D|=$

$=((x-2r_1-r_2+2)(x-2r_1))^{n_1-n_2}|(x-2r_1-r_2+2)(x-2r_1)(x-r_1-2r_2+2)(x-2r_2)I_{n_2}-(2x-3r_1-2r_2+2)(2x-2r_1-3r_2+2)KK^t$

$+(x-2r_1-r_2+2)(x-2r_1)KK^t+(x-r_1-2r_2+2)(x-2r_2)KK^t+KK^tKK^t|$

It follows that if $\lambda_1, \cdots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of $A(G)$, then $\lambda_1^2, \lambda_2^2, \cdots, \lambda_{n_2}^2$ are the eigenvalues of KK^t, thus

$P_{L^+(T(G))}=(x-r_1-r_2+2)^{m-n}((x-2r_1-r_2+2)(x-2r_1))^{n_1-n_2}$

$\Pi_{i=1}^{n_2}((x-2r_1-r_2+2)(x-2r_1)(x-r_1-2r_2+2)(x-2r_2)+$

$(x-2r_1-r_2+2)(x-2r_1)\lambda_i^2-(2x-3r_1-2r_2+2)(2x-2r_1-3r_2+2)\lambda_i^2+$

$(x-r_1-2r_2+2)(x-2r_2)\lambda_i^2+\lambda_i^4)$

Theorem 14 *Let G be a semiregular connected graph with n_1 indendent vertices of degree r_1 and n_2 indendent vertices of degree r_2, where $n_1 > n_2$. Then*

$$IE(T(G)) < (m-n)\sqrt{r_1+r_2-2}$$
$$+(n_1-n_2)\sqrt{2r_1+r_2-2+2\sqrt{2r_1r_2-3r_1}}+2n_2\sqrt{3r_1+3r_2-4}.$$

where $n=n_1+n_2$ and m is the number of edges of G.

Proof. Let x_{i1}, x_{i2}, x_{i3} and x_{i4} are the roots of the equation
$(x-2r_1-r_2+2)(x-2r_1)(x-r_1-2r_2+2)(x-2r_2)+(x-2r_1-r_2$
$+2)(x-2r_1)\lambda_i^2-(2x-3r_1-2r_2+2)(2x-2r_1-3r_2+2)\lambda_i^2$
$+(x-r_1-2r_2+2)(x-2r_2)\lambda_i^2+\lambda_i^4=0$, by the Cauchy-Schwarz inequality and Vita theorem,

$$\sqrt{x_{i1}}+\sqrt{x_{i2}}+\sqrt{x_{i3}}+\sqrt{x_{i4}} \le \sqrt{4(x_{i1}+x_{i2}+x_{i3}+x_{i4})}=2\sqrt{5r_1+5r_2-4}.$$

By Theorem 13, $L^+(T(G))$ has $m-n$ eigenvalues equal to r_1+r_2-2, n_1-n_2 eigenvalues equal to $2r_1$, n_1-n_2 eigenvalues equal to $2r_1+r_2-2$ and $4n_2$ eigenvalues: x_{i1}, x_{i2}, x_{i3}, x_{i4} $(i=1,2,\cdots,n_2)$. Thus

$$IE(T(G)) < (m-n)\sqrt{r_1+r_2-2}+(n_1-n_2)(\sqrt{2r_1}+\sqrt{2r_1+r_2-2})$$

$+2n_2\sqrt{5r_1+5r_2-4}$.

Similar to the argument in Theorem 10, the equality also would not be hold. So

$$IE(T(G)) < (m-n)\sqrt{r_1+r_2-2} + (n_1-n_2)(\sqrt{2r_1} + \sqrt{2r_1+r_2-2})$$
$$+2n_2\sqrt{5r_1+5r_2-4}.$$

The proof of Theorem 14 is completed.

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