A FAMILY OF TETRAVALENT ONE-REGULAR GRAPHS

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ABSTRACT. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper, 4-valent one-regular graphs of order $5p^2$, where p is a prime, are classified.

1. INTRODUCTION

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph X we use $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set, arc set and its full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X, and $N(u)$ is the neighborhood of u in X, that is, the set of vertices adjacent to u in X. A graph X is said to be *vertex-transitive* and *arc-transitive* (or *symmetric*) if $\text{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. In particular, if $\text{Aut}(X)$ acts regularly on $A(X)$, then X is said to be *one-regular.*

Clearly, a one-regular graph is connected, and it is of valency 2 if and only if it is a cycle. In this sense the first non-trivial case is that of cubic graphs. The first example of a cubic one-regular graph was constructed by Frucht [13] and later on lot of works have been done along this line (as part of the more general investigation of cubic arc-transitive graphs) see [9, 10, 11, 12]. 4-valent one-regular graphs have also received considerable attention. In [4], 4-valent one-regular graphs of prime order were constructed. In [21], an infinite family of 4-valent one-regular Cayley graphs on alternating groups is given. 4-valent one-regular circulant graphs were classified in [31] and 4-valent one-regular Cayley graphs on abelian groups were classified in [32]. Next, one may deduce a classification of 4-valent one-regular Cayley graphs on dihedral groups from [20, 26, 28]. Let p and q be primes. Then, clearly every 4-valent one-regular graph of order p is a circulant graph. Also, by [5, 24, 25, 27, 31, 32] every 4-valent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classifications of 4-valent one-regular graphs of order $3p^2$, $4p^2$, $6p^2$ and $2pq$ are given

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in [8, 14, 15, 34]. Along this line the aim of this paper is to classify 4-valent one-regular graphs of order $5p^2$, see Theorem 3.3.

2. Preliminaries

In this section, we introduce some notation and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph X, use $d(X)$ to represent the valency of X, and for any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. Let X be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(X)$ be vertex-transitive on X. For a G-invariant partition $\mathcal{B}$ of $V(X)$, the *quotient graph* $X_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(X)$ is a G-invariant partition of $V(X)$. In this case, the symbol $X_{\mathcal{B}}$ will be replaced by X_N.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. We call C_n an *n-cycle*.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. For any $g \in G$, g is said to be *semiregular* if $\langle g \rangle$ is semiregular.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $X = \mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. The permutation group $R(G) = \{R(g) \mid g \in G\}$ on G is called the *right regular representation* of G. It is easy to see that $R(G)$ is isomorphic to G, and it is a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Also it is easy to see that X is connected if and only if $G = \langle S \rangle$, that is, S is a connection set. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [33], proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$. Suppose that $\alpha \in \mathrm{Aut}(G)$. One may easily prove that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Cay}(G, S^\alpha)$ is normal.

For $u \in V(X)$, denote by $N_X(u)$ the *neighbourhood* of u in X, that is, the set of vertices adjacent to u in X. A graph $\widetilde{X}$ is called a *covering* of a graph

X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\widetilde{X}$ is called a *cyclic* or an *elementary abelian covering* of X, and if $\widetilde{X}$ is connected K becomes the covering transformation group. In this case we also say p is a *regular covering projection.* The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre on to itself. All of fibre-preserving automorphisms form a group called the *fibre-preserving group.*

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$ respectively.

For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$. Now we say Theorem 4.5 from [18, Chapter I].

Proposition 2.1. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of H.*

Proposition 2.2. [30, Chapter I, Proposition 4.4] *Every transitive abelian group G on a set Ω is regular.*

The following proposition is due to Praeger et al, refer to [16, Theorem 1.1].

Proposition 2.3. *Let X be a connected 4-valent $(G,1)$-arc-transitive graph. For each normal subgroup N of G, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N acts transitively on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} on X_N;*
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected 4-valent G/N-symmetric graph, and X is a G-normal cover of X_N.*

Moreover, if X is also $(G,2)$-arc-transitive, then case (3) *can not happen.*

The following classical result is due to Wielandt [30, Theorem 3.4]

Proposition 2.4. *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $\omega \in \Omega$. If p^m divides the length of the G-orbit containing ω, then p^m also divides the length of the P-orbit containing ω.*

To state the next result we need to introduce a family of 4-valent graphs that were first defined in [17]. The graph $C^{\pm 1}(p;5p,1)$ is defined to have the vertex set $\mathbb{Z}_p \times \mathbb{Z}_{5p}$ and edge set $\{(i,j)(i\pm 1, j\pm 1) | i \in \mathbb{Z}_p, j \in \mathbb{Z}_{5p}\}$. Also from [17, Definition 2.2], the graphs $C^{\pm 1}(p;5p,1)$ are Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{5p}$ with connection set $\{(1,1),(-1,1),(-1,-1),(1,-1)\}$. In the proof of Theorem 3.3, we will need $C^{\pm 1}(p;5p,1)$ with $p > 11$. It can be readily checked from [17, Definition 2.2] that for $p > 11$ these graphs are actually normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{5p}$.

Proposition 2.5. [17, Theorem 1.1] *Let X be a connected, G-symmetric, 4-valent graph of order $5p^2$, and let $N = \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p, where p is an odd prime. Let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_{5p}$ and $K_v \cong \mathbb{Z}_2$ then X is isomorphic to $C^{\pm 1}(p;5p,1)$.*

The graphs defined in [17, Lemma 8.4] are all one-regular (see [17, Section 8]) and therefore we refer to [17]
for an intrinsic description of these families.

Proposition 2.6. [17, Theorem 1.2] *Let X be a connected, G-symmetric, 4-valent graph of order $5p^2$, and let $N = \mathbb{Z}_p \times \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p^2, where p is an odd prime. Let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_5$ and $K_v \cong \mathbb{Z}_2$ then X is isomorphic to one of the graphs in* [17, Lemma 8.4].

Finally in the following example we introduce $G(5p;2,2,u)$, which first was defined in [24].

Example 2.7. *Let 2 be a divisor of $p-1$. Let $H(5,2) = \langle a \rangle$, let $t \in \mathbb{Z}_p^*$ be such that $t \in -H(p,2)$, and let u be the least common multiple of 2 and the order of t in $\mathbb{Z}_p^*$. Then $X = G(5p;2,2,u)$ is defined as the graph with vertex set*

$$V(X) = \mathbb{Z}_5 \times \mathbb{Z}_p = \{(i,x) | i \in \mathbb{Z}_5, x \in \mathbb{Z}_p\}$$

such that vertices (i,x) and (j,y) are adjacent if and only if there is an integer l such that $j - i = a^l$ and $y - x \in t^l H(p,2)$. Also X as defined above is independent of the choice of generator a of $H(5,2)$ up to isomorphism, and X is also independent of the choice of t, such that $lcm\{o(t),2\} = u$, up to isomorphism. Moreover, the above graph is circulant, that is, admits a cyclic group of automorphisms of order $5p$ acting regularly on vertices.

We may extract the following results from [3, pp. 76-80]

Proposition 2.8. *Let p be a prime and $p > 5$. Also let G be a non-abelian group of order $5p^2$.*

(i) *If G has a normal subgroup of order p, say N, such that G/N is cyclic, then G is isomorphic to $\langle x, y, z | x^p = y^5 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod p$ and $(i, p) = 1$;*

(ii) *If G has a normal subgroup of order p^2, say N, such that G/N is cyclic, then G is isomorphic to $\langle x, y | x^{p^2} = y^5 = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod{p^2}$.*

3. ONE-REGULAR GRAPHS OF ORDER $5p^2$

For proving the main theorem we need the following two lemmas.

Lemma 3.1. *Let p be a prime, $p > 5$ and $G = \langle x, y, z | x^p = y^5 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod p$ and $(i, p) = 1$. Then there is no 4-valent one-regular normal Cayley graph X of order $5p^2$ on G.*

Proof. Suppose to the contrary that X is a 4-valent one-regular normal Cayley graph Cay(G, S) on G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on S and so that elements in S are all of the same order. The elements of G of order 5 lie in $\langle x, y \rangle$ and the elements of G of order p lie in $\langle x, z \rangle$. Since X is connected, $G = \langle S \rangle$ and hence S consists of elements of order $5p$. Denote by S_{5p} the elements of G of order $5p$. Therefore

$$S \subseteq S_{5p} = \{x^s y^t z^j | s \in \mathbb{Z}_p, t \in \mathbb{Z}_5^*, j \in \mathbb{Z}_p^*\}.$$

Suppose that $x^s y^t z^j \in S$, where $s \neq 0$. Clearly $\sigma : x \mapsto x^s, y \mapsto y, z \mapsto z^j$ is an automorphism of G. Also if $y^t z^j \in S$, then $\sigma : x \mapsto x, y \mapsto y, z \mapsto z^j$ is an automorphism of G. Therefore we may suppose that either $S = \{xy^t z, y^{-t}x^{-1}z^{-1}, x^m y^n z^k, y^{-n}x^{-m}z^{-k}\}$ or $S = \{y^t z, y^{-t}z^{-1}, x^m y^n z^k, y^{-n}x^{-m}z^{-k}\}$, where $m \neq 0$. First suppose that $S = \{xy^t z, y^{-t}x^{-1}z^{-1}, x^m y^n z^k, y^{-n}x^{-m}z^{-k}\}$. Since $\mathrm{Aut}(G, S)$ acts transitively on S, it implies that there is $\alpha \in \mathrm{Aut}(G, S)$ such that $(xy^t z)^\alpha = y^{-t}x^{-1}z^{-1}$. Since $[x, z] = 1$, and $[y, z] = 1$, the element z^α needs to commute with x^α and y^α. Thus $(xz)^\alpha (y^t)^\alpha = y^{-t}x^{-1}z^{-1} = x^{-i^{-4t}} y^{-t} z^{-1}$. We may assume that $(y^t)^\alpha = x^{t_1} y^{t_2}$, where $t_1 \in \mathbb{Z}_p$, and $t_2 \in \mathbb{Z}_5^*$. Thus $(xz)^\alpha x^{t_1} y^{t_2} = x^{-i^{-4t}} y^{-t} z^{-1}$ and so

$$(xz)^\alpha = x^{-i^{-4t}} x^{-t_1 i^{4(-t-t_2)}} y^{-t-t_2} z^{-1}.$$

Since $o(xz) = p$, we have $t = -t_2$. Therefore $(xz)^\alpha = x^{-i^{-4t} - t_1} z^{-1}$. Also let $z^\alpha = x^{s_1} z^{s_2}$ where $s_1, s_2 \in \mathbb{Z}_p$. So $(x)^\alpha = x^{-i^{-4t} - s_1 - t_1} z^{-1-s_2}$. Since z^α commutes with $(y^t)^\alpha$, it follows that $s_1 = 0$ or $i^{4t_2} = 1$.

Since $t_2 \in \mathbb{Z}_5^*$ and $i^5 \equiv 1 \pmod p$, it follows that $i^{4t_2} \neq 1$. Thus we may suppose that $s_1 = 0$. Therefore $x^\alpha = x^{-i^{-4t}-t_1}z^{-1-s_2}$, $(y^t)^\alpha = x^{t_1}y^{-t}$, $z^\alpha = z^{s_2}$. Since $x^{y^t} = x^{i^t}$, we have $(x^\alpha)^{(y^t)^\alpha} = (x^\alpha)^{i^t}$ and so $s_2 = -1$ and $(-i^{-4t} - t_1)(i^{-4t_2} - i^t) = 0$. Since $t = -t_2$ and $t_2 \in \mathbb{Z}_5^*$, we have $(i^{-4t_2} - i^t) \neq 0$. Thus we may suppose that $(-i^{-4t} - t_1) = 0$. Therefore $x^\alpha = z^{-1-s_2} = z^0 = 1$, a contradiction. Now suppose that $S = \{y^t z, y^{-t}z^{-1}, x^m y^n z^k, y^{-n}x^{-m}z^{-k}\}$, where $m \neq 0$. Since $\mathrm{Aut}(G,S)$ acts transitively on S, it implies that there is α of $\mathrm{Aut}(G,S)$ such that $(y^t z)^\alpha = y^{-t}z^{-1}$. Thus $z^\alpha (y^t)^\alpha = y^{-t}z^{-1}$. We may assume that $(y^t)^\alpha = x^{t_1}y^{t_2}$, where $t_1 \in \mathbb{Z}_p$ and $t_2 \in \mathbb{Z}_5^*$. Thus $(z)^\alpha = y^{-t_2}x^{-t_1}y^{-t}z^{-1} = x^{-i^{-4t_2}}y^{-t_2-t}z^{-1}$. Since $o(z) = p$, it follows that $t = -t_2$ and so $z^\alpha = x^{-i^{-4t_2}}z^{-1}$. Also since z^α commutes with $(y^t)^\alpha$ it follows that $-t_1 i^{-4t_2}(i^{4t_2} - 1) = 0$, a contradiction.

Lemma 3.2. *Let p ba a prime, $p > 5$ and $G = \langle x, y | x^{p^2} = y^5 = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod{p^2}$. Then there is no 4-valent one-regular normal Cayley graph X of order $5p^2$ on G.*

Proof. Suppose to the contrary that X is a 4-valent one-regular normal Cayley graph $\mathrm{Cay}(G,S)$ on G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G,S)$ of the vertex $1 \in G$ is transitive on S and so that elements in S are all of the same order. Clearly x^p is the only element of order p. Also x^r where $r \in \mathbb{Z}_{p^2}^*$ are the only elements of order p^2. The elements of G of order 5 lie in $\langle x, y \rangle$. Since X is connected, $G = \langle S \rangle$ and hence S consists of elements of order 5. Denote by $\mathcal{S}_5$ the elements of G of order 5. Therefore

$$S \subseteq \mathcal{S}_5 = \{x^r y^s | r \in \mathbb{Z}_{p^2}, s \in \mathbb{Z}_5^*\}.$$

Clearly $\sigma : x \mapsto x^r, y \mapsto y$ is an automorphism of G, we may suppose that $S = \{xy^s, y^{-s}x^{-1}, x^u y^v, y^{-v}x^{-u}\}$. Since $\mathrm{Aut}(G,S)$ acts transitively on S, it implies that there is $\alpha \in \mathrm{Aut}(G,S)$ such that $(xy^s)^\alpha = y^{-s}x^{-1}$. We may assume that $y^\alpha = x^m y^n$, where $m \in \mathbb{Z}_{p^2}$, $n \in \mathbb{Z}_5^*$. Also let $x^\alpha = x^r$, where $r \in \mathbb{Z}_{p^2}^*$. Therefore $x^r (x^m y^n)^s = y^{-s}x^{-1}$, and so $ns = -s$. Thus $s = 0$ or $n = -1$. Clearly, $s \neq 0$, and so $n = -1$. Now $y^\alpha = x^m y^{-1}$. Since $x^y = x^i$, we have $(x^\alpha)^{y^\alpha} = (x^\alpha)^i$ and so $ri^4 - ri = 0$. Thus $i^3 = 1$, a contradiction.

The following result is the main result of this paper.

Let X be a tetravalent one-regular graph of order $5p^2$. If $p \leq 11$, then $|V(X)| = 20, 45, 125, 245$, or 605. Now, a complete census of the tetravalent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [22, 23]. Therefore, a quick inspection through this list (with the invaluable help of magma (see [2])) gives the number of tetravalent one-regular graphs in the case that $p \leq 11$. Thus we may suppose that $p > 11$.

Theorem 3.3. *Let p be a prime. A 4-valent graph X of order $5p^2$ is 1-regular if and only if one of the following holds:*

(i): *X is a Cayley graph over $\langle x, y | x^p = y^{5p} = [x, y] = 1\rangle$, with connection sets $\{y, y^{-1}, xy, x^{-1}y^{-1}\}$ and $\{y, y^{-2}, xy, x^{-2}y^{-2}\}$;*

(ii): *X is connected arc-transitive circulant graph with respect to every connection set S;*

(iii): *X is one of the graphs described in* [17, Lemma 8.4].

Proof. Let X be a 4-valent one-regular graph of order $5p^2$. If $p \leq 11$, then $|V(X)| = 20, 45$, 125, 245, or 605. Now, a complete census of the 4-valent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [22, 23]. Therefore, a quick inspection through this list (with the invaluable help of magma) gives the proof of the theorem in the case that $p \leq 11$.

Now, suppose that $p > 11$. Let $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer in A of the vertex $v \in V(X)$. Let P be a Sylow p-subgroup of A. Since A is one-regular, it follows that $|A| = 20p^2$. We show that P is normal in A. Since $|A| = 20p^2$, the Sylow's theorems show that the number of Sylow p-subgroups of A is equal to $|A : N_A(P)| = 1 + kp$, for some $k \geq 0$. If $k = 0$, then P is normal in A and thus we may assume that $k \geq 1$. Now, $1 + kp$ divides 20 and this is possible if and only if $k = 1$ and $p = 19$. Now $|A : N_A(P)| = 20$. So $N_A(P) = P$ and $C_A(P) = N_A(P)$. Therefore, by the Burnside's p-complement theorem [29, page 76], we see that A has a normal subgroup N of order 20. In particular, P acts by conjugation as a group of automorphisms on N. As a group of order 20 does not admit non-trivial automorphisms of order 19, we see that P centralizes A. Thus $A \cong N \times P$ and P is normal in A.

Assume first that P is cyclic. Let X_P be the quotient graph of X relative to the orbits of P and let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, the orbits of P are of length p^2. Thus $|V(X_P)| = 5$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 2.3, either $X_P \cong C_5$ and hence $A/K \cong D_{10}$ forcing that $|K| = 2p^2$, or P acts semiregularly on V(X), the quotient graph X_P is a tetravalent connected A/P-arc-transitive graph and X is a regular cover of X_P. First assume that $X_P \cong C_5$. If A/P is an abelian then, since A/K is a quotient group of A/P, also A/K is an abelian. But since A/K is vertex-transitive on X_P, Proposition 2.2, implies that it is regular on X_P, contradicting arc-transitivity of A/K on X_P. Thus A/P is non-abelian group. Clearly K is not semiregular on V(X). Then $K_v \cong \mathbb{Z}_2$, where $v \in V(X)$. By Proposition 2.1, $A/C \lesssim \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is not abelian we have that P is a proper subgroup of C. If $C \cap K \neq P$, then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus $C \cap K = P$ and $1 \neq C/P = C/C\cap K \cong CK/K \trianglelefteq A/K \cong D_{10}$. If $C/P \cong \mathbb{Z}_2$, then $|C| = 2p^2$. Clearly C_v is an abelian group. Clearly $|C_v| \in \{1, 2\}$. If $|C_v| = 2$, then C_v is a Sylow 2-subgroup of C, implying that C_v is characteristic in C. The normality of C in A implies that $C_v \trianglelefteq A$, forcing $C_v = 1$, a contradiction. If $C_v = 1$, then the orbits of C has length $2p^2$, a contradiction. It follows that $|C/P| \in \{5, 10\}$, and hence C/P has a characteristic subgroup of order 5, say H/P. Thus $|H| = 5p^2$ and $H/P \trianglelefteq A/P$, implies that $H \trianglelefteq A$. In addition since $H \leq C = C_A(P)$, we have that H is abelian. Clearly $|H_v| \in \{1, 5\}$. If $|H_v| = 5$, then H_v is a Sylow 5-subgroup of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing $H_v = 1$, a contradiction. If $H_v = 1$, then since $|H| = 5p^2$, H is regular on V(X). It follows that X is a Cayley graph on an abelian group with a cyclic Sylow p-subgroup P. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{5p^2}$, where $p > 11$, is the only abelian group with a cyclic Sylow p-subgroup. Also by [31, Theorem 7], X is one-regular.

Now assume that X_P is a tetravalent connected A/P-symmetric graph. Clearly, $X_P \cong K_5$ and by [1, Theorem 2.2], X_P is non-normal Cayley graph on $\mathbb{Z}_5$. On the other hand A/P is isomorphic to a subgroup of index 6 in Aut(K_5) $\cong S_5$. Thus A/P is isomorphic to affine group AGL$(1,5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Therefore A/P has a normal subgroup of order 5, say PM/P. Thus $PM \trianglelefteq A$ and PM is transitive on V(X). Since $|PM| = 5p^2$, PM is also regular on V(X), implying that X is a normal Cayley graph on PM. If PM is an abelian group, then PM is isomorphic to $\mathbb{Z}_{5p^2}$. Also if PM is not abelian, then by Proposition 2.8 part (ii), PM is isomorphic to $\langle x, y | x^{p^2} = y^5 = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod{p^2}$. If $PM \cong \mathbb{Z}_{5p^2}$, then by [31, Theorem 7] X is one-regular. In the latter case, X is not one-regular, by Lemma 3.2.

Now assume that P is an elementary abelian. Suppose first that P is a minimal normal subgroup of A, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on V(X_P). By Proposition 2.3, either $X_P \cong C_5$ and hence $A/K \cong D_{10}$ forcing that $|K| = 2p^2$, or P acts semiregularly on V(X), the quotient graph X_P is a tetravalent connected A/K-arc-transitive graph and X is a regular cover of X_P. First assume that $X_P \cong C_5$. Thus $K_v = \mathbb{Z}_2$. Proposition 2.6 implies that X is isomorphic to one of the graphs described in [17, Lemma 8.4].

Now assume that X_P is a tetravalent connected A/P-symmetric graph. So X is a $\mathbb{Z}_p \times \mathbb{Z}_p$-regular cover of K_5. By [19, Table 1], AGL$(1,5)$, lifts along p. Now we use the fact that the lift of an s-regular group that lifts along a regular covering projection is s-regular (see [6, 7]). Now by [19, Theorem 2.1, Propositions 3.4, 3.5], X is not one-regular.

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \le K$ and A/K is transitive on $V(X_N)$, moreover we have that $|V(X_N)| = 5p$. By Proposition 2.3, X_N is a cycle of length $5p$, or N acts semiregularly on $V(X)$, the quotient graph X_N is 4-valent connected A/N-arc-transitive graph and X is a regular cover of X_N. If $X_N \cong C_{5p}$, and hence $A/K \cong D_{10p}$, then $|K| = 2p$ and thus $K_v \cong \mathbb{Z}_2$. Applying Proposition 2.5, we get that X is isomorphic to $C^{\pm 1}(p; 5p, 1)$. If, however X_N is a 4-valent connected A/N-symmetric graph, then, by Proposition 2.3, X is a covering graph of a symmetric graph of order $5p$. By [24], $G(5p; 2, 2, u)$ is the just 4-valent symmetric graph of order $5p$ (see Example 2.7). Observe that in this case a one-regular subgroup of automorphism contains a normal regular subgroup isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_p$. Let H be a one-regular subgroup of automorphism of X_N. Since X is one-regular graph, A is the lift of H. Since H contains a normal regular subgroup isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_p$ also A contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $5p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_5 \times \mathbb{Z}_p \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_5$. If G is an abelian group, then G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{5p}$, or $\mathbb{Z}_{5p^2}$. Also if G is not abelian, then by Proposition 2.8 part (i), G is isomorphic to $\langle x, y, z | x^p = y^5 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i \rangle$, where $i^5 \equiv 1 \pmod{p}$ and $(i, p) = 1$. If $G \cong \mathbb{Z}_{5p^2}$, then by [31, Theorem 7] X is one-regular. Also if $G \cong \mathbb{Z}_p \times \mathbb{Z}_{5p}$ then by [32, Proposition 3.3, Example 3.2] X is isomorphic to either $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{5p}, \{a, a^{-1}, ab, a^{-1}b^{-1}\})$ or $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{5p}, \{a, a^{-2}, ab, a^{-2}b^{-2}\})$ which are 1-regular. These graphs are in Theorem 3.3 part (ii). Finally, in the latter case, X is not one-regular, by Lemma 3.1. This complete the proof.

REFERENCES

[1] Y.G. Baik, Y.-Q. Feng, H.S. Sim, M.Y. Xu, On the normality of Cayley graphs of abelian groups, Algebra Colloq. 5 (1998) 297–304.

[2] W. Bosma, C. Cannon, C. Playoust, The MAGMA algebra system I: the user language, J. Symbolic Comput. 24 (1997) 235–265.

[3] R. Carter, P. Fong, The Sylow s-subgroup of the finite classical groups, J. Algebra 1 (1964) 139–151.

[4] C.Y. Chao, On the classification of symmetric graphs with a prime number of vertices, Tran. Amer. Math. Soc. 158 (1971) 247–256.

[5] Y. Cheng, J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory B 42 (1987) 196–211.

[6] D.Ž. Djoković, Automorphisms of graphs and coverings, J. Combin. Theory Ser. B 16 (1974) 243–247.
[7] D.Ž. Djoković, G.L. Miller, Regular groups of automorphisms of cubic graphs, J. Combin. Theory Ser. B 29 (1980) 195–230.
[8] Y.-Q. Feng, K. Kutnar, D. Marušič, C. Zhang, 4-valent one-regular graphs of order $4p^2$, FILOMAT, to appear.
[9] Y.-Q. Feng, J.H. Kwak, Classifying cubic symmetric graphs of order $10p$ or $10p^2$, Science in China A 49 (2006) 300–319.
[10] Y.-Q. Feng, J.H. Kwak, Cubic symmetric graphs of order twice an odd prime power, J. Austral. Math. Soc. 81 (2006) 153–164.
[11] Y.-Q. Feng, J.H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, J. Combin. Theory B 97 (2007) 627–646.
[12] Y.-Q. Feng, J.H. Kwak, K.S. Wang, Classifying cubic symmetric graphs of order $8p$ or $8p^2$, European J. Combin. 26 (2005) 1033–1052.
[13] R. Frucht, A one-regular graph of degree three, Canad. J. Math. 4 (1952) 240–247.
[14] M. Ghasemi, A classification of tetravalent one-regular graphs of order $3p^2$, Colloquium Math, to appear.
[15] M. Ghasemi, P. Spiga, 4-valent one-regular graphs of order $6p^2$, submitted.
[16] A. Gardiner, C.E. Praeger, On 4-valent symmetric graphs, European. J. Combin. 15 (1994) 375–381.
[17] A. Gardiner, C.E. Praeger, A characterization of certain families of 4-valent symmetric graphs, European. J. Combin. 15 (1994) 383–397.
[18] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin, 1967.
[19] B. Kuzman, Arc-transitive elementary abelian covers of the complete graph K_5, Linear Algebra Appl. 433 (2010) 1909–1921.
[20] J.H. Kwak, J.M. Oh, One-regular normal Cayley graphs on dihedral groups of valency 4 or 6 with cyclic vertex stabilizer, Acta Math. Sin. Engl. Ser. 22 (2006) 1305–1320.
[21] D. Marušič, A family of one-regular graphs of valency 4, European J. Combin. 18 (1997) 59–64.
[22] P. Potočnik, P. Spiga, G. Verret, http://www.matapp.unimib.it/ spiga/
[23] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, Journal of Symbolic Computation 50 (2013) 465–477.
[24] C.E. Praeger, R.J. Wang, M.Y. Xu, Symmetric graphs of order a product of two distinct primes, J. Combin. Theory Ser. B 58 (1993) 299–318.
[25] C.E. Praeger, M.Y. Xu, Vertex-primitive graphs of order a product of two distinct primes, J. Combin. Theory Ser. B 59 (1993) 245–266.
[26] C.Q. Wang, M.Y. Xu, Non-normal one-regular and 4-valent Cayley graphs of dihedral groups D_{2n}, European J. Combin. 27 (2006) 750–766.
[27] R.J. Wang, M.Y. Xu, A classification of symmetric graphs of order $3p$, J. Combin. Theory B 58 (1993) 197–216.
[28] C.Q. Wang, Z.Y. Zhou, 4-valent one-regular normal Cayley graphs of dihedral groups, Acta Math. Sinica Chinese Ser. 49 (2006) 669–678.
[29] B.A.F. Wehrfritz, Finite Groups: A second course on Group Theory, World Scientific, 1999.
[30] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.
[31] M.Y. Xu, A note on one-regular graphs, Chin. Scin. Bull. 45 (2000) 2160–2162.
[32] J. Xu, M.Y. Xu, Arc-transitive Cayley graphs of valency at most four on abelian groups. Southeast Asian Bull. Math. 25 (2001) 355–363.
[33] M.Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, Discrete Math. 44 (2001) 1502–1508.

[34] J.-X. Zhou, Y.-Q. Feng, 4-valent one-regular graphs of order $2pq$, J. Algebraic Combin. 29 (2009) 457-471.

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